# World-sheet Scattering in $AdS_5xS^5$

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T.Klose, T.McLoughlin, R.Roiban, K.Z., hep-th/0611169

T.Klose, K.Z., hep-th/0701240

T.Klose, T.McLoughlin, J.Minahan, K.Z., 0704.3891

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# AdS/CFT correspondence

Maldacena'97
Gubser,Klebanov,Polyakov'98
Witten'98

| $\mathcal{N}=4$ SYM | Strings on $AdS_5 \times S^5$ |
| --- | --- |
| 't Hooft coupling: $\lambda = g_{YM}^2 N$ | String tension: $T = \frac{\sqrt{\lambda}}{2\pi}$ |
| Number of colors: $N$ | String coupling: $g_s = \frac{\lambda}{4\pi N}$ |
| Large-N limit | Free strings |
| Strong coupling | Classical strings |
| Local operators<br>Scaling dimension: $\Delta$ | String states<br>Energy: $E$ |

# Anti-de-Sitter space (AdS$_5$)

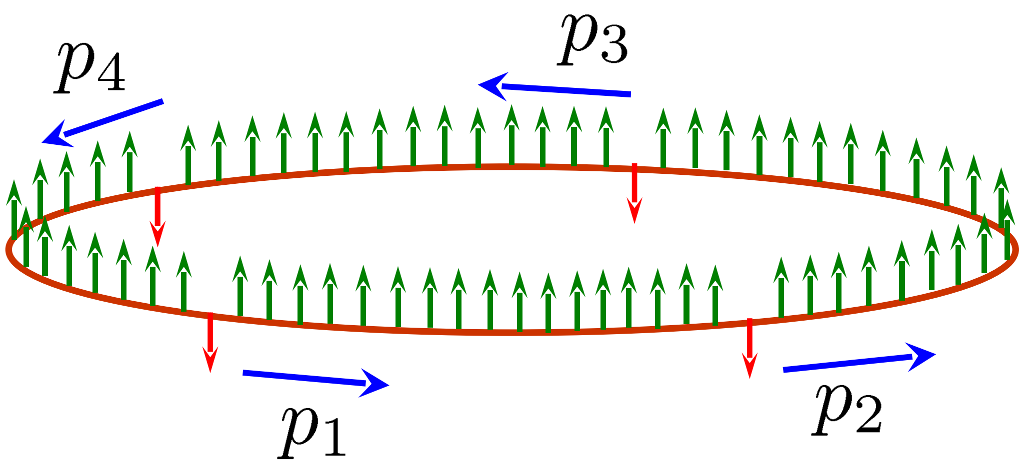
$$ds^2 = \frac{dx^\mu dx_\mu + dz^2}{z^2}$$

SYM "rings of glue"

# Spectral problem in large-N SYM:

- can be reformulated in terms of an integrable spin chain…

Minahan,Z.'02
Beisert,Kristjansen,Staudacher'03
Beisert,Staudacher'03




$$\mathrm{tr}\, ZZZZWZZZZ\Psi ZZZD_\mu ZZZZZZF_{\nu\lambda}ZZ\ldots$$

Berenstein,Maldacena,Nastase'02

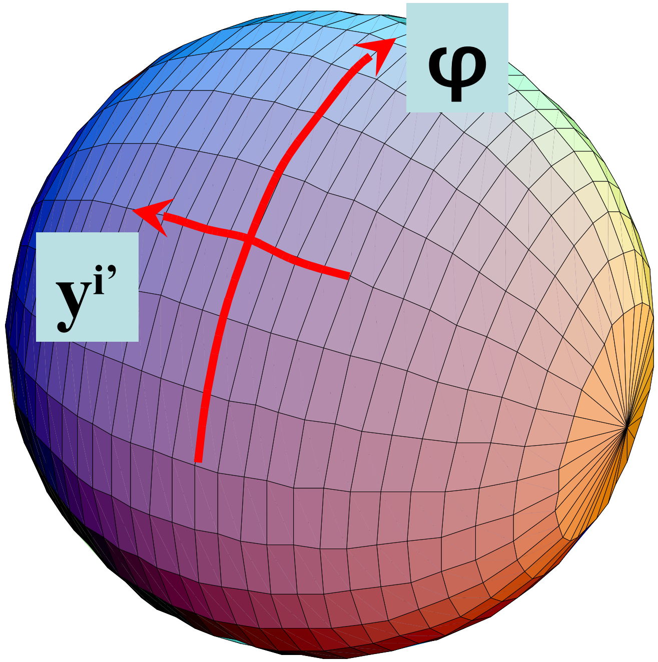
- … with unknown Hamiltonian (beyond O(λ))
- but with exactly known S-matrix!

Beisert'05
Beisert,Eden,Staudacher'06
Beisert,Hernandez,Lopez'06

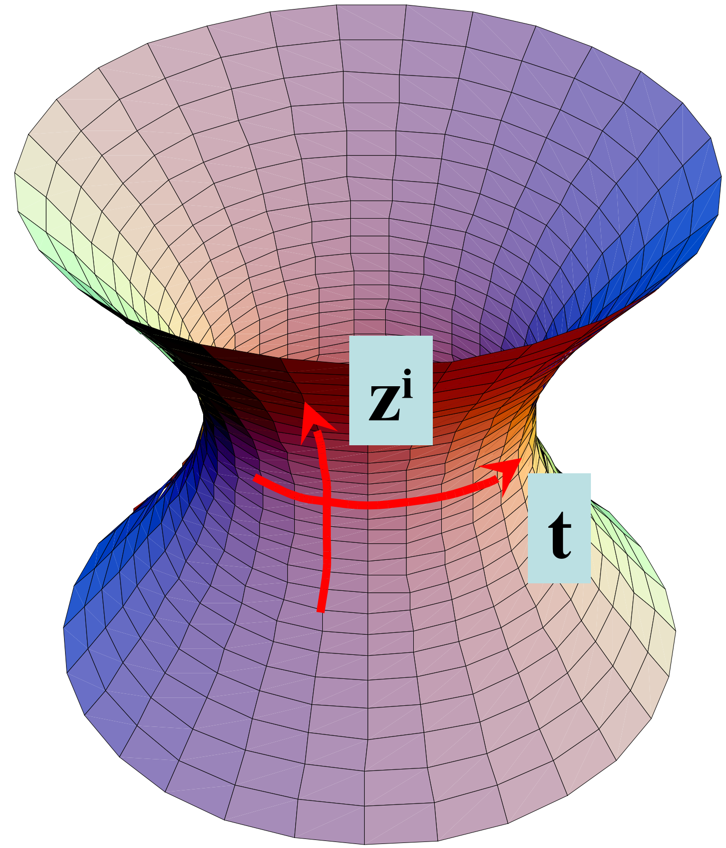
# Sigma-model in $AdS_5 x S^5$:

- well-defined theory with known (although complicated) Lagrangian
- also integrable Bena,Polchinski,Roiban'03
- How does the (discrete) spin chain arise from the (continuous) world-sheet?

Goal: *compute* the S-matrix on the world-sheet explicitly to few first orders in perturbation theory




$S^5$




$AdS_5$

# The metric

$$ds^2 = \underbrace{-\left(\frac{1+\frac{z^2}{4}}{1-\frac{z^2}{4}}\right)^2 dt^2 + \frac{dz^2}{\left(1-\frac{z^2}{4}\right)^2}}_{\mathrm{AdS}_5} + \underbrace{\left(\frac{1-\frac{y^2}{4}}{1+\frac{y^2}{4}}\right)^2 d\varphi^2 + \frac{dy^2}{\left(1+\frac{y^2}{4}\right)^2}}_{\mathrm{S}^5}$$

Transverse coordinates:

$$z^\mu, \qquad \mu = 1 \ldots 4$$

$$y^m, \qquad m = 1 \ldots 4$$

+ RR flux

# Sigma-model

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \sqrt{-h}\, h^{ab} G_{MN}(X) \partial_a X^M \partial_b X^N + \text{fermions}$$

Green-Schwarz-type coset PSU(2,2|4)/SO(4,1)xSO(5)

Metsaev,Tseytlin'98

Sigma-model coupling constant: $\frac{2\pi}{\sqrt{\lambda}}$

Classical limit

is $\lambda \gg 1$

RR flux requires manifest space-time supersymmetry

Fermion part of the sigma-model is
of Green-Schwarz type.

## Conformal gauge is problematic:

- no kinetic term for fermions
- no holomorphic factorization for currents, …

Standard CFT does not work…

# Kinetic term for fermions:

$$\mathcal{L} = i\left(\sqrt{-h}h^{\mathbf{ab}}\delta^{IJ} - \varepsilon^{\mathbf{ab}}s^{IJ}\right)\bar{\theta}^I\rho_{\mathbf{a}}\mathcal{D}_{\mathbf{b}}^{IK}\theta^K$$

$$\mathcal{D}_{\mathbf{a}}^{JK} = \delta^{JK}\left(\partial_{\mathbf{a}} + \frac{1}{4}\,\partial_{\mathbf{a}}X^M\omega_M^{AB}\Gamma_{AB}\right) + \frac{i}{2}\,\epsilon^{JK}\Gamma_{01234}\rho_{\mathbf{a}}$$

$$s^{IJ} = \mathrm{diag}\,(1,-1)$$

$$\rho_{\mathbf{a}} = \partial_{\mathbf{a}}X^M E_M^A\Gamma_A$$

If X=const, $\rho_a$=0 and the kinetic term vanishes.

Need to fix the light-cone gauge: $X^+=\tau$

(then $\rho_0=\Gamma_+$ and the kin. term is not degenerate)

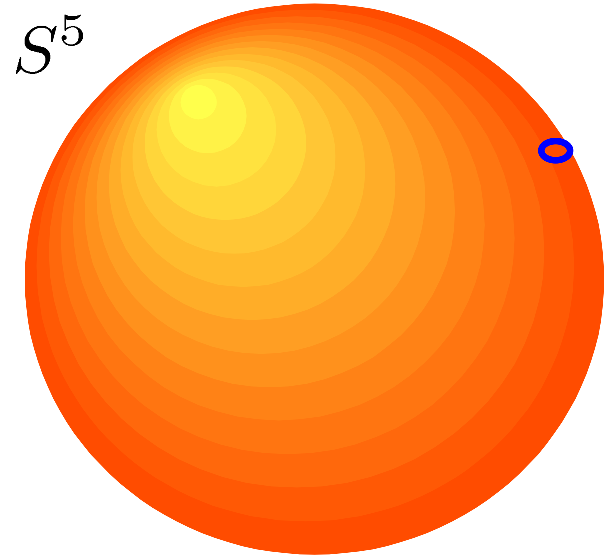
# Gauge fixing

## Light-like geodesics:

$$\varphi = Jt$$

angular momentum

$$x^{\pm} = t \pm \varphi$$




Berenstein,Maldacen,Nastase'02; Gubser,Klebanov,Polyakov'02
Parnachev,Ryzhov'02
Callan,Lee,McLoughlin,Schwarz,Swanson,Wu'03
Arutyunov,Frolov,Plefka,Zamaklar'06

$$SO(6) \longrightarrow SO(4) = SU(2) \times SU(2)/Z_2$$
$$SO(4,2) \longrightarrow SO(4) = SU(2) \times SU(2)/Z_2$$

Unbroken bosonic subgroup: $SU(2)^4$

Taking into account supersymmetry:

$$\boxed{PSU(2,2|4) \longrightarrow P(SU(2|2) \times SU(2|2))}$$

This also gets central extension with

$$\text{central charge} = \frac{\sqrt{\lambda}}{\pi} \sin \frac{\pi p}{\sqrt{\lambda}}$$

Beisert'05
Arutyunov,Frolov,Plefka,Zamaklar'06

$p$ – world-sheet momentum

# World-sheet fields

World-sheet fields (embedding coordinates in $AdS_5xS^5$):

$$T, \ \Phi, \ Y^m, \ Z^\mu, \ \text{fermions}$$

global time in $AdS_5$ and angle on $S^5$ – eliminated by gauge fixing

$$Y_{a\dot{a}} = (\sigma_m)_{a\dot{a}} Y^m, \qquad Z_{\alpha\dot{\alpha}} = (\sigma_\mu)_{\alpha\dot{\alpha}} Z^\mu$$

Fields form (2|2)x(2|2) multiplet of $PSU(2|2)^2$:

$$\begin{tikzcd}
& Y_{a\dot{a}} \arrow[dl, leftrightarrow, "PSU(2|2)_L"'] \arrow[dr, leftrightarrow, "PSU(2|2)_R"] & \\
\Upsilon_{\alpha\dot{a}} \arrow[dr, leftrightarrow, "PSU(2|2)_R"'] & & \Psi_{a\dot{\alpha}} \arrow[dl, leftrightarrow, "PSU(2|2)_L"] \\
& Z_{\alpha\dot{\alpha}} &
\end{tikzcd}$$

$$\Phi_{A\dot{A}} \qquad A = (a, \alpha)$$

# How to fix the gauge?

## Step 1: T-duality

$$X^+ = (1-a)T + a\,\Phi$$

$$\partial_{\mathbf{a}}\tilde{\Phi} = \frac{\sqrt{\lambda}}{2\pi}\,\sqrt{-h}\,\varepsilon_{\mathbf{ab}}h^{\mathbf{bc}}\left[(1-a)G_{\varphi\varphi}\,\partial_{\mathbf{c}}\Phi + aG_{tt}\,\partial_{\mathbf{c}}T\right]$$

$a$ - gauge parameter

Interpolates between

temporal gauge: $a=0$

and light-cone gauge: $a=1/2$

Arutyunov,Frolov,Zamaklar'06

$$\tilde{\Phi}(\sigma+2\pi,\tau) = \tilde{\Phi}(\sigma,\tau) + J_+$$

$$J_+ = (1-a)J + aE$$

## Step2: *Nambu-Goto* Lagrangian:

$$L_{NG} = -\sqrt{-\det_{\mathbf{ab}} G_{MN}\,\partial_{\mathbf{a}}\tilde{\mathbb{X}}^M \partial_{\mathbf{b}}\tilde{\mathbb{X}}^N} - \frac{1}{2}\,\varepsilon^{\mathbf{ab}} B_{MN}\partial_{\mathbf{a}}\tilde{\mathbb{X}}^M \partial_{\mathbf{b}}\tilde{\mathbb{X}}^N$$

## Step3: Gauge condition

$$X^+ = \tau \qquad \tilde{\Phi} = \frac{J_+\sigma}{2\pi}$$

$$J_+ = (1-a)J + aE$$

## Step 4: Rescaling $\sigma \to \dfrac{J_+\sigma}{\sqrt{\lambda}}$

# Gauge-fixed Lagrangian (the bosonic part):

$$
\begin{aligned}
\mathcal{L} \quad = \quad & -\frac{\sqrt{G_{\varphi\varphi}G_{tt}}}{(1-a)^2 G_{\varphi\varphi} - a^2 G_{tt}} \, \Bigg\{ 1 - \frac{(1-a)^2 G_{\varphi\varphi} - a^2 G_{tt}}{2} \\
& \times \left[ \left( 1 + \frac{1}{G_{\varphi\varphi}G_{tt}} \right) \partial_{\mathbf{a}} X \cdot \partial^{\mathbf{a}} X - \left( 1 - \frac{1}{G_{\varphi\varphi}G_{tt}} \right) \left( \dot{X} \cdot \dot{X} + \acute{X} \cdot \acute{X} \right) \right] \\
& + \frac{\left[ (1-a)^2 G_{\varphi\varphi} - a^2 G_{tt} \right]^2}{2 G_{\varphi\varphi} G_{tt}} \left[ \left( \partial_{\mathbf{a}} X \cdot \partial^{\mathbf{a}} X \right)^2 - \left( \partial_{\mathbf{a}} X \cdot \partial_{\mathbf{b}} X \right)^2 \right] \Bigg\}^{1/2} \\
& + \frac{a}{1-a} \; \frac{G_{tt}}{(1-a)^2 G_{\varphi\varphi} - a^2 G_{tt}} \; .
\end{aligned}
$$

$$
G_{tt} = \left( \frac{1 + \frac{Z^2}{4}}{1 - \frac{Z^2}{4}} \right)^2 \qquad\qquad G_{\varphi\varphi} = \left( \frac{1 - \frac{Y^2}{4}}{1 + \frac{Y^2}{4}} \right)^2
$$

# Gauge-fixed action

$$S = \frac{\sqrt{\lambda}}{2\pi} \int_{-\frac{\pi J_+}{\sqrt{\lambda}}}^{+\frac{\pi J_+}{\sqrt{\lambda}}} \mathcal{L}[X, \Theta]$$

Loop counting parameter: $\frac{2\pi}{\sqrt{\lambda}}$

Internal string length: $\boxed{L = \frac{\pi J_+}{\sqrt{\lambda}}}$

Decompatification (infinite string) limit: $L \to \infty$

## Bosonic Lagrangian in AdS$_5$xS$^5$ (up to quartic order in fields):

$$\mathcal{L}_{\rm AdS} = \frac{1}{2}\left(\partial_{\mathbf{a}} X\right)^2 - \frac{1}{2} X^2 + \frac{1}{4} Z^2 \left(\partial_{\mathbf{a}} Z\right)^2 - \frac{1}{4} Y^2 \left(\partial_{\mathbf{a}} Y\right)^2 + \frac{1}{4}\left(Y^2 - Z^2\right)\left(\dot{X}^2 + \acute{X}^2\right) - \frac{1-2a}{8}\left(X^2\right)^2 + \frac{1-2a}{4}\left(\partial_{\mathbf{a}} X \cdot \partial_{\mathbf{b}} X\right)^2 - \frac{1-2a}{8}\left[\left(\partial_{\mathbf{a}} X\right)^2\right]^2 + O(X^6)$$

$$X = (Y^m, Z^\mu)$$

- Massive, integrable 2d field theory
- Lorentz-invariant kinetic terms
- Lorentz invariance is broken by interactions
- Gauge-dependent: *a=1/2* (pure l.c. gauge) is the simplest case

# Spectrum

## (8|8) massive states:

$$\begin{tikzcd}
 & Y_{a\dot{a}} \arrow[dl, leftrightarrow, "PSU(2|2)_L"'] \arrow[dr, leftrightarrow, "PSU(2|2)_R"] & \\
\Upsilon_{\alpha\dot{a}} \arrow[dr, leftrightarrow, "PSU(2|2)_R"'] & & \Psi_{a\dot{\alpha}} \arrow[dl, leftrightarrow, "PSU(2|2)_L"] \\
 & Z_{\alpha\dot{\alpha}} &
\end{tikzcd}$$

$$\varepsilon = \sqrt{p^2+1} + \text{loop corrections}$$

## Loop corrections should give:

$$\varepsilon = \sqrt{\frac{\lambda}{\pi^2}\,\sin^2\frac{\pi p}{\sqrt{\lambda}} + 1}$$

exact

Beisert,Dippel,Staudacher'04;Beisert'05

two-loop correction!

# World-sheet scattering

## S-matrix:

$$\mathbb{S}\,|\Phi_{A\dot{A}}(p)\Phi_{B\dot{B}}(p')\rangle = |\Phi_{C\dot{C}}(p)\Phi_{D\dot{D}}(p')\rangle\,\mathbb{S}^{C\dot{C}D\dot{D}}_{A\dot{A}B\dot{B}}(p,p')$$

Integrability (consistency with Yang-Baxter equations) requires:

$$\mathbb{S} = \mathbf{S}\otimes\mathbf{S} \qquad\qquad \mathbb{S}^{C\dot{C}D\dot{D}}_{A\dot{A}B\dot{B}}(p,p') = \mathbf{S}^{CD}_{AB}(p,p')\mathbf{S}^{\dot{C}\dot{D}}_{\dot{A}\dot{B}}(p,p')$$

## SU(2)⁴ – invariace:

$$\mathbf{S}^{cd}_{ab} = \mathbf{A}\,\delta^c_a\delta^d_b + \mathbf{B}\,\delta^d_a\delta^c_b \qquad\qquad \mathbf{S}^{\gamma\delta}_{ab} = \mathbf{C}\,\epsilon_{ab}\epsilon^{\gamma\delta}$$

$$\mathbf{S}^{\gamma\delta}_{\alpha\beta} = \mathbf{D}\,\delta^\gamma_\alpha\delta^\delta_\beta + \mathbf{E}\,\delta^\delta_\alpha\delta^\gamma_\beta \qquad\qquad \mathbf{S}^{cd}_{\alpha\beta} = \mathbf{F}\,\epsilon_{\alpha\beta}\epsilon^{cd}$$

$$\mathbf{S}^{c\delta}_{a\beta} = \mathbf{G}\,\delta^c_a\delta^\delta_\beta \qquad\qquad \mathbf{S}^{\gamma d}_{\alpha b} = \mathbf{L}\,\delta^\gamma_\alpha\delta^d_b$$

$$\mathbf{S}^{\gamma d}_{a\beta} = \mathbf{H}\,\delta^d_a\delta^\gamma_\beta \qquad\qquad \mathbf{S}^{c\delta}_{\alpha b} = \mathbf{K}\,\delta^\delta_\alpha\delta^c_b$$

Beisert'05

$$\mathbf{A} = \frac{x'_- - x_-}{x'_- - x_+}\,\frac{1-\frac{1}{x'_- x_+}}{1-\frac{1}{x'_+ x_+}}$$

$$\mathbf{B} = \frac{x'_+ - x_-}{x'_- - x_+}\left(1 - \frac{x'_- - x_-}{x'_+ - x_-}\,\frac{1-\frac{1}{x'_- x_+}}{1-\frac{1}{x'_+ x_+}}\right)$$

$$\mathbf{C} = \frac{i\eta\eta'}{x_+ x'_+}\,\frac{1}{1-\frac{1}{x'_+ x_+}}\,\frac{x'_- - x_-}{x'_- - x_+}\,\mathrm{e}^{\frac{i\pi p'}{\sqrt{\lambda}}}$$

$$x_\pm = \frac{\pi e^{\pm i\pi p/\sqrt{\lambda}}}{\sqrt{\lambda}\sin\frac{\pi p}{\sqrt{\lambda}}}\left(1+\sqrt{1+\frac{\lambda}{\pi^2}\sin^2\frac{\pi p}{\sqrt{\lambda}}}\right)$$

$$\mathbf{D} = \frac{x'_+ - x_+}{x'_- - x_+}\,\frac{1-\frac{1}{x'_+ x_-}}{1-\frac{1}{x'_- x_-}}\,\mathrm{e}^{\frac{i\pi(p'-p)}{\sqrt{\lambda}}}$$

$$\eta = \left|x_p^- - x_p^+\right|^{1/2}$$

$$\mathbf{E} = 1 - \frac{x'_+ - x_+}{x'_- - x_+}\,\frac{1-\frac{1}{x'_+ x_-}}{1-\frac{1}{x'_- x_-}}\,\mathrm{e}^{\frac{i\pi(p'-p)}{\sqrt{\lambda}}}$$

$$\mathbf{F} = -\frac{i(x_+ - x_-)(x'_+ - x'_-)}{\eta\eta' x_- x'_-}\,\frac{1}{1-\frac{1}{x'_- x_-}}\,\frac{x'_+ - x_+}{x'_- - x_+}\,\mathrm{e}^{-\frac{i\pi p}{\sqrt{\lambda}}}$$

$$\mathbf{G} = \frac{x'_+ - x_+}{x'_- - x_+}\,\mathrm{e}^{-\frac{i\pi p}{\sqrt{\lambda}}} \qquad \mathbf{H} = \frac{\eta}{\eta'}\,\frac{x'_+ - x'_-}{x'_- - x_+}\,\mathrm{e}^{\frac{i\pi(p'-p)}{\sqrt{\lambda}}}$$

$$\mathbf{L} = \frac{x'_- - x_-}{x'_- - x_+}\,\mathrm{e}^{\frac{i\pi p'}{\sqrt{\lambda}}} \qquad \mathbf{K} = \frac{\eta'}{\eta}\,\frac{x_+ - x_-}{x'_- - x_+}$$

up to an overall phase

# Symmetries of the S-matrix

Bosonic (su(4)$^4$) generators:

$$[K \otimes 1 + 1 \otimes K, \mathbb{S}] = 0$$

Supersymmetries:

$$\left(1 \otimes Q^A{}_B + Q^A{}_B \otimes \mathrm{e}^{-\frac{i\pi\sigma_{AB}}{\sqrt{\lambda}} p'}\right) \mathbb{S}(p,p') = \mathbb{S}(p,p') \left(Q^A{}_B \otimes 1 + \mathrm{e}^{-\frac{i\pi\sigma_{AB}}{\sqrt{\lambda}} p} \otimes Q^A{}_B\right)$$

$$\sigma_{AB} = [A] - [B] \qquad [a] = [\dot{a}] = 0, \ [\alpha] = [\dot{\alpha}] = 1$$

Spin chain explanation: the chain is dynamical, its length may decrease on increase in the process of scattering

Beisert'05

# Hopf algebra

Co-product:

$$\Delta(Q^A{}_B) = Q^A{}_B \otimes 1 + \mathrm{e}^{-\frac{i\sigma_{AB}p}{\sqrt{\lambda}}} \otimes Q^A{}_B$$

promotes symmetries of the S-matrix to non-trivial Hopf algebra:

Gomez,Hernandez'06; Plefka,Spill,Torrielli'06

$$\Delta^{\mathrm{op}}(Q^A{}_B)\,\mathbb{S} = \mathbb{S}\,\Delta(Q^A{}_B)$$

$$\Delta^{\mathrm{op}} = \Delta \cdot \Pi$$

graded permutation

# String explanation 1: non-locality of supercharges

## Non-locality of supercurrents:

local operator

$$Q^A{}_B = \int_{-\infty}^{+\infty} d\sigma \; \mathrm{e}^{\,i\sigma_{AB} X^-/2} \, \widetilde{J}^A{}_B$$

Arutyunov,Frolov,Plefka,Zamaklar'06

non-local

$$[\acute{X}^-(\sigma),\, \mathrm{AnyField}(\xi)] = i\,\frac{2\pi}{\sqrt{\lambda}}\,\delta(\sigma-\xi)\,\mathrm{AnyField}'(\xi)$$

## By contour manipulations:

$$J^A{}_B(\sigma)\Phi(\xi) = \left(\mathrm{e}^{-\frac{\pi\sigma_{AB}}{\sqrt{\lambda}}\partial_\xi}\Phi(\xi)\right) J^A{}_B(\sigma) \quad \text{for} \quad \sigma > \xi$$

$\Downarrow$

Klose,McLoughlin,Roiban,Z.'06

non-trivial co-product

# String explanation 2: fractional statistics

If $p \sim \sqrt{\lambda}$ the world-sheet scattering states become solitons (giant magnons) Hofman,Maldacena'06

Fermion number:

$$[F, Q^A{}_B] = \sigma_{AB} Q^A{}_B$$

Conjecture: giant magnon with momentum $p$ carries fractional fermion number $F = -\dfrac{p}{\sqrt{\lambda}}$

In principle, can be checked by summing over known fermion modes Minahan'06; Papathanasiou,Spradlin'07 of the giant magnon

Jackiw,Rebbi'76; Goldstone,Wilczek'81; Niemi,Semenoff'86

$$\Delta(Q) = Q \otimes 1 + (-1)^F \otimes Q$$ (Fermi statistics)

$$(-1)^F \equiv \mathrm{e}^{i\pi\sigma_{AB}F} \longrightarrow \mathrm{e}^{-\frac{i\sigma_{AB}p}{\sqrt{\lambda}}}$$

Bernard,Leclair'90; Fendley,Intriligator'91

and then:

$$\Delta(Q^A{}_B) = Q^A{}_B \otimes 1 + \mathrm{e}^{-\frac{i\sigma_{AB}p}{\sqrt{\lambda}}} \otimes Q^A{}_B$$

# Tree level

$$\mathbb{S} = 1 + \frac{2\pi i}{\sqrt{\lambda}}\,\mathbb{T} + \ldots$$

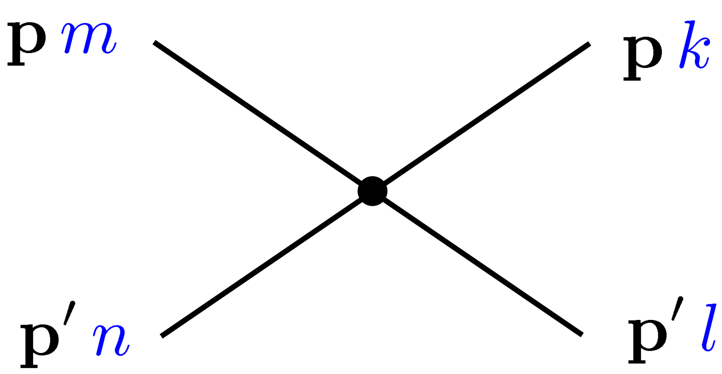
$$\mathbf{S}\otimes\mathbf{S} = \left(1 + \frac{2\pi i}{\sqrt{\lambda}}\,\mathbf{T} + \ldots\right)\otimes\left(1 + \frac{2\pi i}{\sqrt{\lambda}}\,\mathbf{T} + \ldots\right) = 1 + \frac{2\pi i}{\sqrt{\lambda}}\,(1\otimes\mathbf{T} + \mathbf{T}\otimes 1) + \ldots$$

Tree-level factorization:

$$\mathbb{T} = 1\otimes\mathbf{T} + \mathbf{T}\otimes 1$$

# Scattering of bosons at tree level

$Y^m(p)Y^n(p') \rightarrow Y^k(p)Y^l(p')$




$$\mathbb{T}_{kl}^{mn} = \frac{1}{2}\left[(1-2a)(\varepsilon' p - \varepsilon p') + \frac{p^2 + p'^2}{\varepsilon' p - \varepsilon p'}\right]\delta_k^m \delta_l^n + \frac{pp'}{\varepsilon' p - \varepsilon p'}\left(\delta_l^m \delta_k^n - \delta^{mn}\delta_{kl}\right)$$

# SO(4) → SU(2)xSU(2)

$$T^{ab\dot{a}\dot{b}}_{cd\dot{c}\dot{d}} = \sigma^{a\dot{a}}_{m}\sigma^{b\dot{b}}_{n}\sigma^{k}_{c\dot{c}}\sigma^{l}_{d\dot{d}}T^{mn}_{kl}$$

$$P^{ab}_{cd} = \delta^{a}_{d}\delta^{b}_{c} \qquad 1^{ab}_{cd} = \delta^{a}_{c}\delta^{b}_{d}$$

$$\mathbb{T} = \frac{1}{2}\left[(1-2a)(\varepsilon' p - \varepsilon p') + \frac{(p^2 - p')^2}{\varepsilon' p - \varepsilon p'}\right] \underline{1 \otimes 1} + \frac{pp'}{\varepsilon' p - \varepsilon p'} \underline{(1 \otimes P + P \otimes 1)}$$

Factorization works!

No $P \otimes P$ terms, allowed by group theory

Explicit calculation yields:

$$T_{ab}^{cd} = A\,\delta_a^c\delta_b^d + B\,\delta_a^d\delta_b^c \qquad T_{ab}^{\gamma\delta} = C\,\epsilon_{ab}\epsilon^{\gamma\delta}$$

$$T_{\alpha\beta}^{\gamma\delta} = D\,\delta_\alpha^\gamma\delta_\beta^\delta + E\,\delta_\alpha^\delta\delta_\beta^\gamma \qquad T_{\alpha\beta}^{cd} = F\,\epsilon_{\alpha\beta}\epsilon^{cd}$$

$$T_{a\beta}^{c\delta} = G\,\delta_a^c\delta_\beta^\delta \qquad T_{\alpha b}^{\gamma d} = L\,\delta_\alpha^\gamma\delta_b^d$$

$$T_{a\beta}^{\gamma d} = H\,\delta_a^d\delta_\beta^\gamma \qquad T_{\alpha b}^{c\delta} = K\,\delta_\alpha^\delta\delta_b^c$$

$$\mathrm{A}(p,p') = \frac{1}{4}\left[(1-2a)\left(\varepsilon' p - \varepsilon p'\right) + \frac{\left(p-p'\right)^2}{\varepsilon' p - \varepsilon p'}\right]\,,$$

$$\mathrm{B}(p,p') = -\mathrm{E}(p,p') = \frac{pp'}{\varepsilon' p - \varepsilon p'}\,,$$

$$\mathrm{C}(p,p') = \mathrm{F}(p,p') = \frac{1}{2}\frac{\sqrt{(\varepsilon+1)\left(\varepsilon'+1\right)}\left(\varepsilon' p - \varepsilon p' + p' - p\right)}{\varepsilon' p - \varepsilon p'}\,,$$

$$\mathrm{D}(p,p') = \frac{1}{4}\left[(1-2a)\left(\varepsilon' p - \varepsilon p'\right) - \frac{\left(p-p'\right)^2}{\varepsilon' p - \varepsilon p'}\right]\,,$$

$$\mathrm{G}(p,p') = -\mathrm{L}(p',p) = \frac{1}{4}\left[(1-2a)\left(\varepsilon' p - \varepsilon p'\right) - \frac{p^2 - p'^2}{\varepsilon' p - \varepsilon p'}\right]\,,$$

$$\mathrm{H}(p,p') = \mathrm{K}(p,p') = \frac{1}{2}\,\frac{pp'}{\varepsilon' p - \varepsilon p'}\,\frac{(\varepsilon+1)\left(\varepsilon'+1\right) - pp'}{\sqrt{(\varepsilon+1)\left(\varepsilon'+1\right)}}\,.$$

$$\varepsilon = \sqrt{p^2+1}$$

# Dressing phase

Beisert,Hernandez,Lopez'06

Beisert,Eden,Staudacher'06

$$\mathbb{S} = \mathrm{e}^{\,i\theta(p,p')}\,\frac{x_{p'}^{-} - x_{p}^{+}}{x_{p'}^{+} - x_{p}^{-}}\;\frac{1 - \frac{1}{x_{p'}^{+}x_{p}^{-}}}{1 - \frac{1}{x_{p'}^{-}x_{p}^{+}}}\,\mathbf{S}\otimes\mathbf{S}$$

$$\theta(p,p') = \sum_{r,s=\pm} rs\,\chi(x_r, x'_s)$$

$$\chi(x,y) = \frac{i}{8\pi^2}\oint_{|z|=1=|w|}\frac{dz}{z}\,\frac{dw}{w}\;\frac{1}{xz-1}\;\frac{1}{yw-1}\;\ln\frac{\Gamma\left(1+\frac{i\sqrt{\lambda}}{4\pi}\left(z+\frac{1}{z}-w-\frac{1}{w}\right)\right)}{\Gamma\left(1-\frac{i\sqrt{\lambda}}{4\pi}\left(z+\frac{1}{z}-w-\frac{1}{w}\right)\right)}$$

Dorey,Hofman,Maldacena'07

## To leading order in $1/\sqrt{\lambda}$ :

$$\theta(p,p') = \frac{\pi}{\sqrt{\lambda}}\,\frac{\left[(\varepsilon'-1)p - (\varepsilon-1)\,p'\right]^2}{\varepsilon' p - \varepsilon p'}$$

agrees with tree-level

scattering amplitudes in the sigma-model

# Near-flat limit

Maldacena,Swanson'06

$p \sim 1 \qquad \varepsilon \sim 1$ perturbative string modes

$p \sim \sqrt{\lambda} \qquad \varepsilon \sim \sqrt{\lambda}$ giant magnons Hofman,Maldacena'06

$$p_\pm = \varepsilon \pm p$$

"Left-moving" sector:

$$p_- \sim \lambda^{1/4} \qquad p_+ \sim \lambda^{-1/4} \qquad p_- \gg p_+$$

# Reduced Sigma Model

Consistent truncation of the full sigma-model:

$$\begin{aligned}\mathcal{L} \;=\; & \frac{1}{2}\left(\partial Y\right)^2 - \frac{m^2}{2}\,Y^2 + \frac{1}{2}\left(\partial Z\right)^2 - \frac{m^2}{2}\,Z^2 + \frac{i}{2}\,\psi\,\frac{\partial^2 + m^2}{\partial_-}\,\psi \\ & + \gamma\left(Y^2 - Z^2\right)\left[\left(\partial_- Y\right)^2 + \left(\partial_- Z\right)^2\right] + i\gamma\left(Y^2 - Z^2\right)\psi\partial_-\psi \\ & + i\gamma\psi\left(\partial_- Y^{i'}\Gamma^{i'} + \partial_- Z^i\Gamma^i\right)\left(Y^{i'}\Gamma^{i'} - Z^i\Gamma^i\right)\psi \\ & - \frac{\gamma}{24}\left(\psi\Gamma^{i'j'}\psi\,\psi\Gamma^{i'j'}\psi - \psi\Gamma^{ij}\psi\,\psi\Gamma^{ij}\psi\right)\end{aligned}$$

Maldacena,Swanson'06

$m = 1 \qquad \gamma = \dfrac{\pi}{\sqrt{\lambda}}$

$\psi$ - chiral SO(8) spinor

# Dispersion relation

Exact dispersion relation:

$$\varepsilon = \sqrt{1 + \frac{\lambda}{\pi^2}\,\sin^2\frac{\pi\,p}{\sqrt{\lambda}}}$$

Beisert,Dippel,Staudacher'04

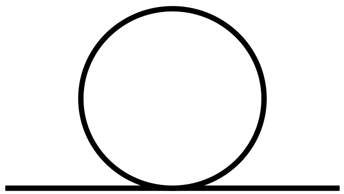
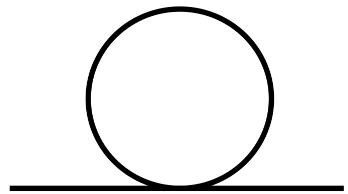
At λ→∞: $4p_+p_- = m^2$

In the near-flat limit:

$$p_+ = \frac{m^2}{4p_-} - \frac{\gamma^2 p_-^3}{12m^2}$$

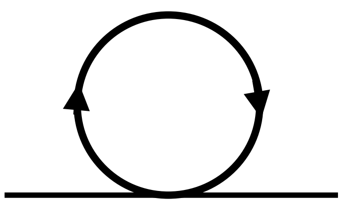
(exact) $\left(\gamma = \pi/\sqrt{\lambda}\right)$

# Tadpole cancellation

 +  = 0

$Y^{i'}$ ($S^5$)

$Z^i$ ($AdS_5$)

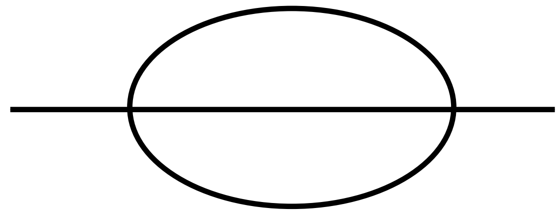
 = 0

(fermions)

The same mechanism renders the model finite to any order in perturbation theory

# Mass renormalization

$$i\Sigma(p_+, p_-) \quad = $$



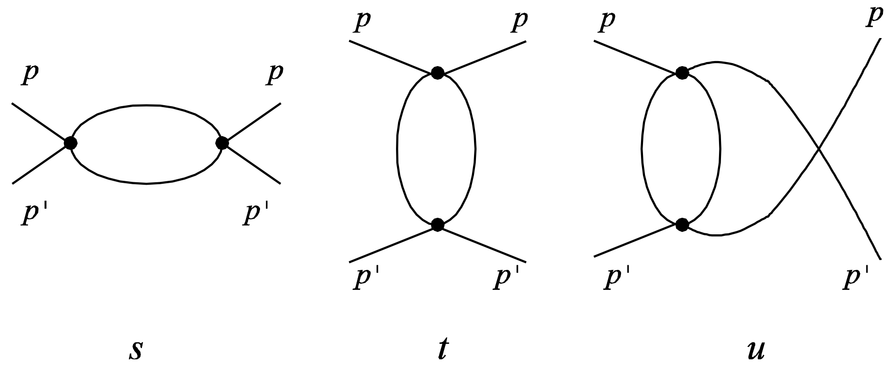
$$\Sigma\Big|_{\mathrm{on-shell}} = -\frac{p_-^4}{3m^2}$$

Loop-corrected mass-shell condition:

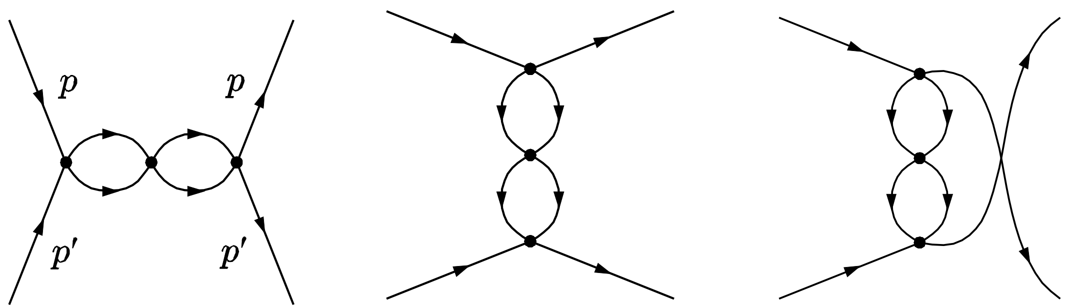
$$4p_+p_- = m^2 - \frac{\gamma^2 p_-^4}{3m^2}$$

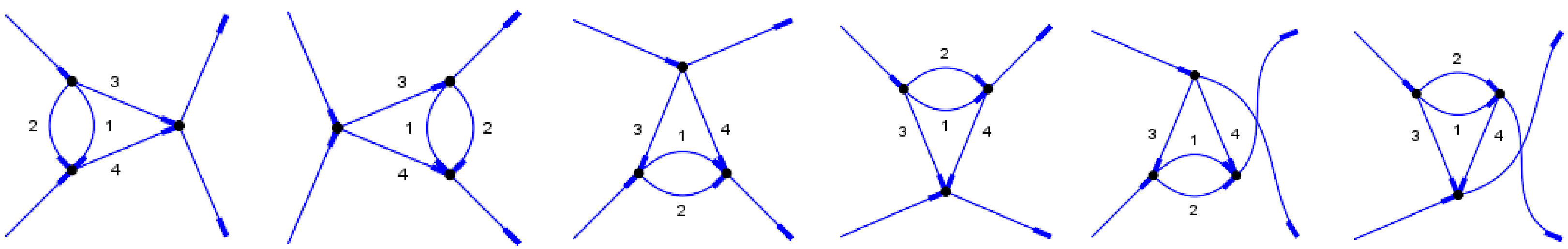
# S-matrix

One loop:




Two loops:




+ mass renormalization

+ two-loop corrections to Z-factors

# Tree-level amplitude:

$$S^{ZY\to ZY} = 1 - 2i\gamma p_- p'_- + \mathcal{O}(\gamma^2)$$

# One-loop correction:

$$S^{ZY\to ZY}_{1-\text{loop}} = \frac{8i\gamma^2 p_-^3 p'^3_-}{\pi\left(p'^2_- - p_-^2\right)}\left(1 - \frac{p'^2_- + p_-^2}{p'^2_- - p_-^2}\ln\frac{p'_-}{p_-}\right) - \frac{2\gamma^2 p_-^2 p'^2_-\left(p'^2_- + p_-^2\right)}{\left(p'_- - p_-\right)^2}$$

Klose,Z.'07

# Two loops:

$$S^{ZY\to ZY}_{2-\text{loops}} = 2i\gamma^3 p_-^3 p'^3_-\left(\frac{p'_- + p_-}{p'_- - p_-}\right)^2 + \frac{16\gamma^3}{\pi}\,\frac{p_-^4 p'^4_-}{p'^2_- - p_-^2}\left(1 - \frac{p'^2_- + p_-^2}{p'^2_- - p_-^2}\ln\frac{p'_-}{p_-}\right)$$

Klose,McLoughlin,Minahan,Z.'07

Agrees with BHL/BES!

# "Lorentz-invariant" form of the S-matrix

All Lorentz non-invariance of amplitudes in the reduced model can be hidden in the momentum-dependent redefinition of the coupling:

$$\boxed{\tilde{\gamma} = \gamma\, p_- p'_-}$$

$$\mathbb{S} = \frac{\mathrm{e}^{\frac{4i\tilde{\gamma}^2}{\pi}\,\frac{1-\theta\coth\theta}{\sinh\theta}}}{1+\tilde{\gamma}^2\,\coth^2\frac{\theta}{2}}\,\mathbf{S}\otimes\mathbf{S}$$

$$\mathbf{S}_{ab}^{cd} = \left(1+i\tilde{\gamma}\tanh\tfrac{\theta}{2}\right)\delta_a^c\delta_b^d + 2i\tilde{\gamma}\,\mathrm{csch}\,\tfrac{\theta}{2}\,\delta_a^d\delta_b^c \qquad \mathbf{S}_{ab}^{\gamma\delta} = i\tilde{\gamma}\,\mathrm{sech}\,\tfrac{\theta}{2}\,\epsilon_{ab}\epsilon^{\gamma\delta}$$

$$\mathbf{S}_{\alpha\beta}^{\gamma\delta} = \left(1-i\tilde{\gamma}\tanh\tfrac{\theta}{2}\right)\delta_\alpha^\gamma\delta_\beta^\delta - 2i\tilde{\gamma}\,\mathrm{csch}\,\tfrac{\theta}{2}\,\delta_\alpha^\delta\delta_\beta^\gamma \qquad \mathbf{S}_{\alpha\beta}^{cd} = i\tilde{\gamma}\,\mathrm{sech}\,\tfrac{\theta}{2}\,\epsilon_{\alpha\beta}\epsilon^{cd}$$

$$\mathbf{S}_{a\beta}^{c\delta} = (1+i\tilde{\gamma})\,\delta_a^c\delta_\beta^\delta \qquad \mathbf{S}_{\alpha b}^{\gamma d} = (1-i\tilde{\gamma})\,\delta_\alpha^\gamma\delta_b^d$$

$$\mathbf{S}_{a\beta}^{\gamma d} = i\tilde{\gamma}\,\mathrm{csch}\,\tfrac{\theta}{2}\,\delta_a^d\delta_\beta^\gamma \qquad \mathbf{S}_{\alpha b}^{c\delta} = i\tilde{\gamma}\,\mathrm{csch}\,\tfrac{\theta}{2}\,\delta_\alpha^\delta\delta_b^c$$

In the full σ-model/spin chain, LI is possibly q-deformed rather than violated

Gomez,Hernandez'06; Young'07

# Conclusions

- At strong coupling, the SU(2|2)xSU(2|2) S-matrix describes the scattering of the string modes in the physical (e.g. light-cone) gauge
- Perturbative calculations of scattering amplitudes agree with the conjectured BHL/BES phase up to two loops at strong coupling (and to four loops at week coupling Bern,Czakon,Dixon,Kosower,Smirnov'06)
- Ultimately we want to quantize the σ-model with *periodic bounary conditions*
- S-matrix → asymptotic Bethe ansatz
- Exact Bethe ansatz?