S^2 Partition Function and Applications

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Recent Developments

Sphere-Partition Function	Superconformal Index
S^5 : See Kim's talk	
S^4 : [Pestun]	**S^3 x S^1**: [Romelsberger]
S^3 : [Kapustin,Willet,Yaakov]	**S^2 x S^1**: [Kim]
[Jafferis]	[Imamura,Yokohama]
[Hama,Hosomichi,**S.L**]	

We learned AGT correspondence, F-theorem, Test of Dualities and so on

S^2 : **This is what I want to discuss today !**

Attack some basic questions in 2D (SUSY) theories

S^2 Partition Function

based on: Benini,Cremonesi, **arXiv: 1206.2356**
Doroud,Gomis,Le Floch, **S.L.**, **arXiv:1206.2609**

N=(2,2) SUSY on S^2

SUSY on Two-Sphere: **SU(2|1)**

- Subalgebra of N=(2,2) SCA

- Bosonic subalgebra:

  - SU(2): rotational symmetry of S^2
  - U(1): vector U(1) R-symmetry

NB: axial U(1) R-symmetry is broken unless the theory is conformal

- Parametrized by Killing spinors $(\epsilon, \bar{\epsilon})$ satisfying

$$\nabla_i \epsilon = +\frac{1}{2l}\gamma_i \gamma^3 \epsilon \qquad \nabla_i \bar{\epsilon} = -\frac{1}{2l}\gamma_i \gamma^3 \bar{\epsilon}$$

N=(2,2) SUSY on S²

SU(2|1) Representation on Supermultiplets: 1/l corrections (**l** : radius of S^2)

e.g. Vector multiplet $\left(A_i, \sigma_1, \sigma_2, \lambda, D\right)$

$$\delta\lambda = \cdots + i\left(F_{12} + i[\sigma_1, \sigma_2] + \frac{1}{l}\sigma_1\right)\gamma^3\epsilon$$

SUSY Lagrangian on S^2: up to $(1/l)^2$ corrections

e.g. Kinetic Lagrangian for vector multiplet

$$\mathcal{L}_{\text{v.m.}} = \frac{1}{2g^2}\text{Tr}\Big[\left(F_{12} + \frac{\sigma_1}{l}\right)^2 + (D_i\sigma_1)^2 + (D_i\sigma_2)^2 - [\sigma_1, \sigma_2]^2 + D^2 + \cdots\Big]$$

N=(2,2) SUSY on S^2

SUSY Interactions on S^2

- Superpotential $\mathcal{W}(\phi)$: possible if **q[W] = 2** (q : $U(1)_R$ charge)

$$\mathcal{L}_{\mathcal{W}} = \frac{\partial \mathcal{W}}{\partial \phi^i} F^i - \frac{1}{2} \frac{\partial^2 \mathcal{W}}{\partial \phi^i \partial \phi^j} \psi^i \psi^j + \mathrm{c.c.}$$

- Twisted Superpotential $W(Y)$ (twisted chiral multiplet (Y, χ, G))

$$\mathcal{L}_W = -iW'(Y)G - W''(Y)\bar{\chi}\left(\frac{1-\gamma^3}{2}\chi\right) + \frac{i}{l}W(Y)$$

Localization

Start with a following path-integral

$$Z[t] = \int \mathcal{D}\Phi \ e^{-S[\Phi]-tQ.V[\Phi]} \qquad \begin{aligned} Q.S[\phi] &= 0 \\ Q^2 &= J \end{aligned}$$

- $S[\Phi]$: action of a theory we want to study
- The term V is invariant under J, $J.V[\Phi] = 0$

Supersymmetry tells us

$Z[0]$	=	$Z[\infty]$
$S[\Phi]$		$S_{\text{def}} = Q.V[\Phi]$
Quantum		**Semi-Classical**
Hard to evaluate		Easy to evaluate (**Gaussian Integral**)

Localization

RESULT

$$Z[0] = \sum_{\phi_*} e^{-S[\phi_*]} \left(\frac{\det \Delta_f^{\text{def}}}{\det \Delta_b^{\text{def}}} \right) \Bigg|_{\phi_*}$$

where ϕ_* satisfy (1) equation of motion of the deformed theory $\left.\frac{\delta S_{\text{def}}}{\delta \phi}\right|_{\phi=\phi_*} = 0$

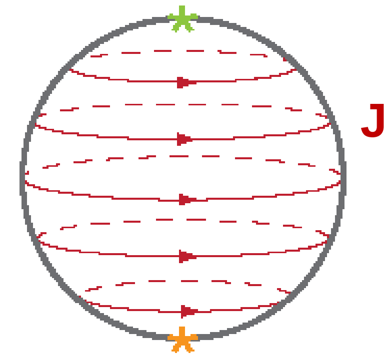
(2) supersymmetric condition

For BPS operators $Q.\mathcal{O}_{\text{BPS}} = 0$,

$$\langle \mathcal{O}_{\text{BPS}} \rangle = \sum_{\phi_*} e^{-S[\phi_*]} \left(\frac{\det \Delta_f^{\text{def}}}{\det \Delta_b^{\text{def}}} \right) \Bigg|_{\phi_*} \mathcal{O}_{\text{BPS}}(\phi_*)$$

Exact Partition Function

Localization Scheme



- Choice of supercharge : $\mathcal{Q}^2 = J + \frac{R}{2}$

- *Q*-exact deformation : Given the above choice,

$$\underbrace{\mathcal{L}_{\text{v.m.}} = \mathcal{Q}V_{\text{v.m.}} \quad \mathcal{L}_{\text{c.m.}} = \mathcal{Q}V_{\text{c.m.}} \quad \mathcal{L}_{\text{t.c.m.}} = \mathcal{Q}V_{\text{t.c.m.}}}_{\text{Kinetic Lagrangians: } Q\text{-exact deformations}} \qquad \underbrace{\mathcal{L}_{\mathcal{W}} = \mathcal{Q}V_{\mathcal{W}}}_{\text{Superpotential}}$$

Decoupling Theorem: S^2 partition function is independent of

(1) gauge coupling constant

(2) parameters in superpotential $\mathcal{W}(\phi)$

Exact Partition Function

Gauge Linear Sigma Model (GLSM) N=(2,2) gauge theory with gauge group G and chiral multiplets of $U(1)_R$ charge q in rep. **R**

- SUSY saddle point configurations:

$$\sigma_2 = \sigma = const. \qquad F_{12} + \frac{\sigma_1}{l} = 0$$

$$[\sigma_1, \sigma_2] = 0 \qquad \int_{S^2} F_{12} = 2\pi \underbrace{B}_{\text{GNO Quantized}}$$

GNO

Quantized

and all other fields vanish

Exact Partition Function

Gauge Linear Sigma Model (GLSM)

Gauge group G with chiral multiplets of $U(1)_R$ charge **q** in rep. **R**

- **One-loop determinant** : some of divergent terms can not be cured by any local counter terms

e.g. Chiral multiplet of weight $\boldsymbol{\rho}$:

$$Z_{\text{1-loop}} = \prod_{J=0}^{\infty} \frac{J + 1 - \frac{q}{2} + il\rho\cdot\sigma - \frac{\rho\cdot B}{2}}{J + \frac{q}{2} - il\rho\cdot\sigma - \frac{\rho\cdot B}{2}}$$

$$\log Z_{\text{1-loop}} = \underbrace{(1 - q}_{[1]} + \underbrace{2il\rho\cdot\sigma}_{[2]})\log[l\Lambda] + \cdots$$

Λ : cut-off scale

[1] Central charge

[2] One-loop correction to FI parameter

Exact Partition Function

Gauge Linear Sigma Model (GLSM)

ξ : FI parameter θ : theta angle

W : Weyl group **r** : rank of G

- Result:

$$Z^{\text{GLSM}}_{S^2} = \frac{(l\Lambda)^{c/3}}{|W|} \sum_B \int_{\mathfrak{t}} d^r\sigma \;\; e^{-4\pi i \xi_{\text{ren}}\sigma + i\theta B} \times Z^{\text{reg}}_{\text{1-loop}}(\sigma)$$

- (regularized) One-loop determinant :

$$Z_{\text{v.m.}} = \prod_{\alpha\in\Delta^+} \left[\left(\frac{\alpha\cdot B}{2} \right)^2 + (\alpha\cdot l\sigma)^2 \right] \qquad Z_{\text{c.m.}} = \prod_{\rho\in\mathbf{R}} \frac{\Gamma\left(\frac{q}{2} - il(\rho\cdot\sigma + m) - \frac{\rho\cdot B}{2}\right)}{\Gamma\left(1 - \frac{q}{2} + il(\rho\cdot\sigma + m) - \frac{\rho\cdot B}{2}\right)}$$

- Central charge :

$$\frac{c}{3} = \underbrace{\sum_i \dim[\mathbf{R}_i](1 - q_i)}_{\text{chiral multiplets}} - \underbrace{\dim[G]}_{\text{vector multiplet}} = \text{Tr}_f[R]$$

[Silverstein,Witten]
[Hori,Tong]

Exact Partition Function

Landau-Ginzburg (LG) Model, which involves

Twisted chiral multiplets Y coupled by twisted superpotential W(Y)

- SUSY saddle points : Y = const. over S^2 and all other fields vanish

- One-loop determinant : trivial in a sense that it is independent of Y

- **Result**

$$Z_{S^2}^{\mathrm{LG}} = \int dY d\bar{Y} \;\; e^{-4\pi i l W(Y) - 4\pi i l \bar{W}(\bar{Y})}$$

Applications

based on: Jockers,Kumar,Lapan,Morrison,Romo, **arXiv: 1208.6244**
Gomis, **S.L.**, **arXiv:1210.6022**

World-Sheet Instanton

CY_3 sigma model plays a key role in perturbative string theory

e.g. type II string in $R^{1,3}$ x CY_3

2D SUSY gauge theories (GLSM) flowing to CY_3 sigma models are useful

e.g. Space of marginal couplings

- Kahler moduli of CY_3: complexified FI parameters

- Complex structure moduli of CY_3: parameters in superpotential W

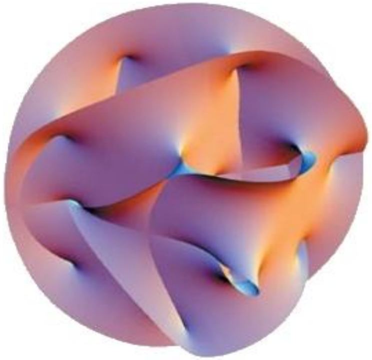
Quantum correction (world-sheet instanton) to Kahler moduli space

NB: No quantum correction in complex structure moduli space

Compute quantum corrections to the metric on Kahler moduli space of CY_3

Mirror Symmetry

A solution to this problem is the celebrated **Mirror Symmetry**



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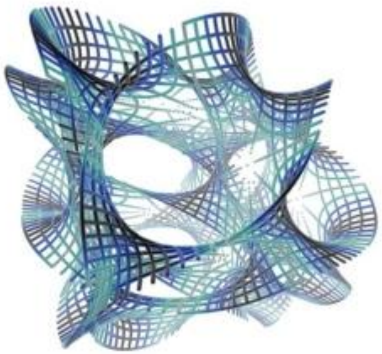


Kahler Moduli

Quantum (Hard)

Complex Structure Moduli

Classical (Easy)

However, known examples for mirror symmetry are limited

NB: Complete intersection in toric variety = 2d abelian GLSMs

New Solution

The exact **S^2 partition function** provides a direct and powerful method to compute such quantum corrections without use of mirror symmetry

$$Z_{S^2}(\tau,\bar{\tau}) = e^{-K(\tau,\bar{\tau})}$$

τ: Kahler moduli

- $K(\tau,\bar{\tau})$: exact (in α') Kahler potential for the quantum Kahler moduli space
- Conjectured by [Jockers,Kumar,Lapan,Morrison,Romo]
- Works for many known examples, and predicts new results for CY_3 whose mirror descriptions are unknown yet! For details, see Morrison's talk

Why does this formula work ?

New Solution

Proof I (Warm-up) [Gomis,**S.L**]

LG theories with twisted superpotential W(Y), which describe N=(2,2) SCFTs

$$Z_{S^2}[\mathrm{LG}] = \int dY\, d\bar{Y}\; e^{-4\pi i l W(Y) - 4\pi i l \bar{W}(\bar{Y})}$$

|| [Cecotti]

$$e^{-K(\tau,\bar{\tau})}$$

New Solution

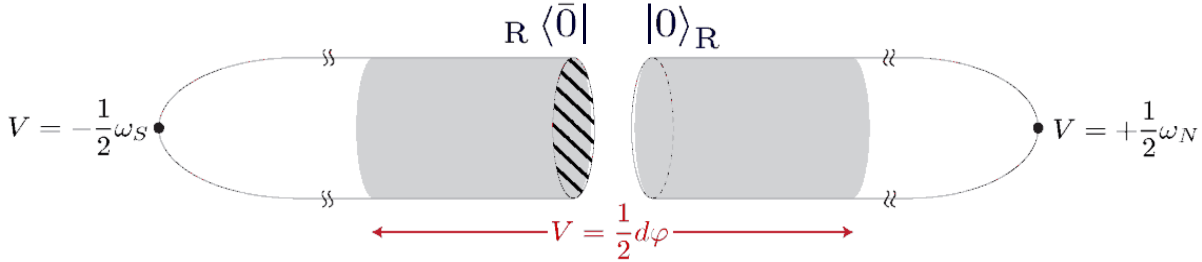
Sketch of Proof II [Gomis,**S.L**]

$$Z_{S^2} \underset{[1]}{=} Z_{S_b^2} \overset{b=l/\tilde{l}}{\underset{[2]}{\overset{b\to 0}{=}}} {}_{\rm R}\langle\bar{0}|0\rangle_{\rm R} \underset{\text{tt* eqn}}{=} e^{-K(\vec{\tau},\vec{\bar{\tau}})}$$

[1] SUSY theory on squashed two-sphere $S_b^2 : \frac{x_1^2+x_2^2}{l^2} + \frac{x_3^2}{\tilde{l}^2} = 1$

- Need a background gauge field V for $U(1)_R$ Symmetry
- Partition function on S_b^2 : independent of squashing parameter **b**

[2] Infinite squashing limit



Mirror Symmetry Revisited

Non-compact toric CY$_3$

GLSM: G=U(1), n chiral multiplets of charge Q_a (a=1,2,..,n) with $\sum_a Q_a = 0$

LG: n twisted chiral multiplets Y=Y+2πi with $W = -\frac{1}{4\pi}\left[\Sigma\left(\sum_{a=1}^{n} Q_a Y^a + 2\pi i\tau\right) + i\sum_{a=1}^{n} e^{-Y^a}\right]$

$$Z_{S^2}^{\mathrm{GLSM}} = \sum_{B\in\mathbb{Z}} e^{+iB\vartheta}\int d\sigma\; e^{-4\pi i r\sigma\xi}\prod_{a=1}^{n}(-1)^{\frac{|BQ_a|+BQ_a}{2}}\frac{\Gamma\left(\frac{1}{2}|BQ_a| - irQ_a\sigma\right)}{\Gamma\left(1+\frac{1}{2}|BQ_a| + irQ_a\sigma\right)}$$

$$\| \quad \int_{-\infty}^{+\infty} dx\; e^{-qx} J_\alpha(2e^{-x}) = (-1)^{\frac{|\alpha|-\alpha}{2}}\frac{1}{2}\frac{\Gamma\left(\frac{q}{2}+\frac{1}{2}|\alpha|\right)}{\Gamma\left(1-\frac{q}{2}+\frac{1}{2}|\alpha|\right)}$$

$$Z_{S^2}^{\mathrm{LG}} = \sum_{B\in\mathbb{Z}}\int_{-\infty}^{\infty} d\sigma\int_{-\infty}^{\infty} dx^a\int_{-\pi}^{+\pi} dy^a\; e^{+2ir\sigma(Q_a x^a - 2\pi\xi) + iB(Q_a y^a + \vartheta)}\cdot e^{2ie^{-x^a}\sin y^a}$$
$$= \int d\Sigma\, d\bar{\Sigma} dY_a d\overline{Y}_a\; e^{-4\pi i r W - 4\pi i r\overline{W}}$$

Mirror Symmetry Revisited

Compact CY_3 : Complete intersection in toric variety

GLSM: G=U(1), n chiral multiplets of charge Q_a and $U(1)_R$ charge **q_a** with

superpotential *W*

LG: n twisted chiral multiplets $Y=Y+2\pi i$ with the **same** twisted superpotential W

$$W = -\frac{1}{4\pi}\left[\Sigma\Big(\sum_{a=1}^{n} Q_a Y^a + 2\pi i\tau\Big) + i\sum_{a=1}^{n} e^{-Y^a}\right]$$

Subtlety in choosing fundamental variables of mirror LG models

However, S^2 partition function can resolve the subtlety automatically !

Mirror Symmetry Revisited

GLSM: G=U(1), n chiral multiplets of charge Q_a and $U(1)_R$ charge q_a

LG: n twisted chiral multiplets with the **same** twisted superpotential W

$$Z_{S^2}^{\text{GLSM}} = \sum_{B\in\mathbb{Z}} e^{+iB\vartheta} \int d\sigma\, e^{-4\pi i r\sigma\xi} \prod_{a=1}^{n} (-1)^{\frac{|BQ_a|+BQ_a}{2}} \frac{\Gamma\left(\frac{q_a}{2}+\frac{1}{2}|BQ_a| - irQ_a\sigma\right)}{\Gamma\left(1-\frac{q_a}{2}+\frac{1}{2}|BQ_a| + irQ_a\sigma\right)}$$

$$\| \qquad \int_{-\infty}^{+\infty} dx\, e^{-qx} J_\alpha(2e^{-x}) = (-1)^{\frac{|\alpha|-\alpha}{2}} \frac{1}{2} \frac{\Gamma\left(\frac{q}{2}+\frac{1}{2}|\alpha|\right)}{\Gamma\left(1-\frac{q}{2}+\frac{1}{2}|\alpha|\right)}$$

$$Z_{S^2}^{\text{GLSM}} = \int d\Sigma\, d\bar{\Sigma} \int \left[\prod_{a=1}^{n} dY_a d\overline{Y}_a e^{-\frac{q_a}{2}(Y^a+\overline{Y}^a)}\right] e^{-4\pi i W(Y) - 4\pi i \overline{W}(Y)}$$

Fundamental LG variables $X_a = e^{-\frac{q_a}{2}Y^a}$

New Idea in Mirror Symmetry

Mirror beyond toric ? [Hori,Vafa] method (due to T-duality) cannot extend to 2D non-abelian GLSM describing CY_3 beyond toric variety

e.g. G=U(N) with chiral multiplets in rep. **R**

Complete-intersection in Grassmannian mfd.

S^2 partition function of 2d non-abelian GLSM can be computed exactly

Same as the S^2 partition function of a following Landau-Ginzburg model

$$Z_{S^2}^{\mathrm{GLSM}} = \frac{1}{|\mathcal{W}(G)|} \int \Big[\prod_{j=1}^{\mathrm{rk}(G)} d\Sigma_j d\bar{\Sigma}_j \Big] \Big[\prod_{\rho \in \mathbf{R}} dY^\rho d\overline{Y}^\rho \Big] \prod_{j<k} |\Sigma_j - \Sigma_k|^2 \, e^{-4\pi i W - 4\pi i \overline{W}}$$

$$W = -\frac{1}{4\pi} \left[\sum_{i=1}^{\mathrm{rk}(G)} \Sigma_i \left(\sum_{\rho \in \mathbf{R}} \rho^i Y^\rho + 2\pi i \tau \right) + i \sum_{\rho} e^{-Y^\rho} \right] \quad \Leftarrow$$

Nontrivial evidence of Hori-Vafa conjecture !

Summary and Outlook

S^2 partition function provides a new method to study 2d SUSY (gauge) theories

B-model data : [Doroud,Gomis] (to appear)

Choose a different SU(2|1) inside N=(2,2) SCA containing axial $U(1)_R$ symmetry

Then, S^2 partition function depends on parameters in superpotential of GLSM

Superconformal index (elliptic genera) :

For LG, done by [Witten]

For GLSM, done RECENTLY by [Gadde,Gukov] [Benini,Eager,Hori,Tachikawa]

Hemi-sphere partition function : D-brane in CY_3 [Hori,Romo] (to appear)

A lot more to be explored !

Thank You Very Much

Appendix

Restoration of Axial $U(1)_R$ Symmetry at IR fixed point

There are a one-parameter family of SUSY theories on S^2, $T[\theta]$, related by the axial $U(1)_R$ rotation

However, S^2 partition functions of $T[\theta]$ are all the same !

This result confirms in the S^2 partition function framework that the axial $U(1)_R$ symmetry is **restored at IR**!