The quantum structure of the type II effective action and UV behavior of maximal supergravity

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[Based on work in coll. with **M.B. Green** and **P. Vanhove**, hep-th/0610299, hep-th/0611273, and in progress]

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WHAT IS STRING THEORY IN THE DEEP QUANTUM REGIME?

- Explicit string theory calculations like graviton scattering amplitudes have never carried out beyond genus 2.
- Such calculations involve extremely complicated integrals over vertex operator positions and integration over the supermoduli space of Riemann surfaces.
- How to get information about string perturbation theory beyond genus 2 ?
- Strategy: USE THE CONNECTION WITH D=11 SUPERGRAVITY ON S^1 AND T^2.
- L-loop scattering amplitudes depend on the radius $\mathbf{R_{11}}$ of $\mathbf{S^1}$.

One has $A_4^{(L)}$ (S,T,R_{11}). BUT $R_{11} = g_s$

Expanding this at small S,T, one gets a Laurent expansion in powers of $(R_{11}^2\ S)$, $(R_{11}^2\ T)$ or $(g_s^2\ s)$, $(g_s^2\ t)$.

- In this way the expansion generates higher genus string-theory coefficients to arbitrary order !! One gets explicit results at genus 3, 4, 5, 6, 7,…etc.

… no chance to compute them using string perturbation theory with the present knowledge.

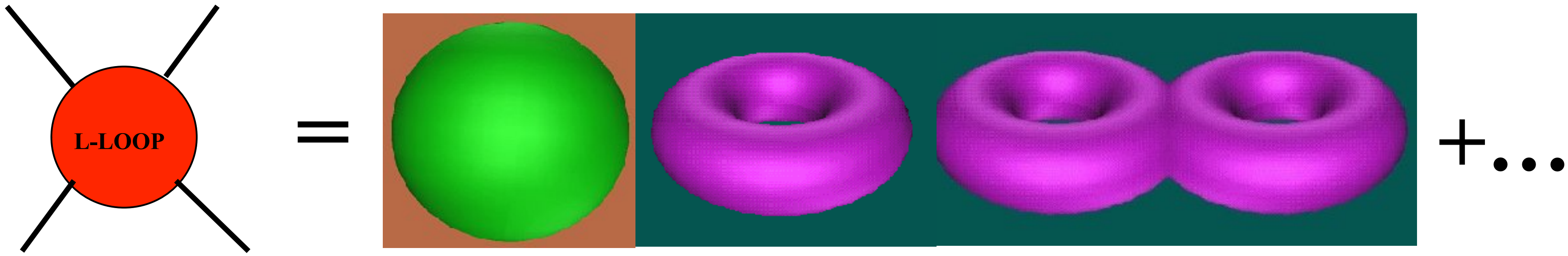
The **tree-level four-graviton effective action**:

$$S = \int d^{10}x \sqrt{-g}\, e^{-2\phi} \big(R + \alpha'^3\, 2\zeta(3) R^4 + \alpha'^5\, \zeta(5) D^4 R^4 + \alpha'^6\, \tfrac{2}{3}\zeta(3)^2 D^6 R^4$$
$$+ \alpha'^7\, \tfrac{1}{2}\zeta(7) D^8 R^4 + \alpha'^8\, \tfrac{2}{3}\zeta(3)\zeta(5) D^{10} R^4$$
$$+ \alpha'^9 \big[\tfrac{1}{4}\zeta(9) D^{12} R^4 + \tfrac{2}{27}(2\zeta(3)^3 + \zeta(9)) \tilde{D}^{12} R^4\big] + \cdots\big)$$

$$D^{12} R^4 = (s^2 + t^2 + u^2)^3 R^4\,, \qquad \tilde{D}^{12} R^4 = (s^3 + t^3 + u^3)^2 R^4$$

In **type IIB**, each **$D^{2k}R^4$ term** in the exact effective action will contain, in addition, perturbative and non-perturbative corrections. The zeta functions will be replaced by a modular function of the SL(2,Z) duality group.

THE MAGIC OF L LOOP 11D SUPERGRAVITY



EXAMPLE: ONE LOOP IN 11D SUPERGRAVITY ON S^1

- [Green,Gutperle,Vanhove]
- [Russo,Tseytlin]
- [D'Hoker,Gutperle,Phong]
- [Green,Vanhove]

$$= (e^{-2\phi}\zeta(3) + \Lambda^3) R^4 + \sum_{k=2}^{\infty} e^{2(k-1)\phi} \frac{\zeta(2k)}{k(k-1)} D^{2k} R^4$$

$$= \left(\frac{1}{g_A^2}\zeta(3) + \Lambda^3\right) R^4 + g_A^2 \frac{\zeta(4)}{2} D^4 R^4 + g_A^4 \frac{\zeta(6)}{6} D^6 R^4 + \ldots$$

This L=1 amplitude determines the genus k coefficient of all $D^{2k}R^4$ terms in type IIA superstring theory.

ONE LOOP IN 11D SUPERGRAVITY ON T^2

$$= \int dx^9 \sqrt{-g}\, (\mathcal{V}\, \Lambda^3 R^4 + \mathcal{V}^{-1/2}\, Z_{3/2} R^4 + \sum_{k=2}^{\infty} c_k \mathcal{V}^{k-1/2} Z_{k-1/2}\, D^{2k} R^4$$

$$= \int dx^9 r_B \sqrt{-g_{IIB}} (r_B^{-2} \Lambda^3 R^4 + (\zeta(3) g_B^{-2} + 2\zeta(2) + O(e^{-2\pi/g_B})) R^4$$

$$+ \sum_{k=2}^{\infty} c_k\, r_B^{-2k} (\zeta(2k-1) + \zeta(2k-2) g_B^{2k-2} + O(e^{-2\pi/g_B})) D^{2k} R^4$$

Note that it is finite in the ten dimensional limit

$$i.e. \quad \sum_{k=2}^{\infty} r_B^{-2k} (genus\, 1 + genus\, k) D^{2k} R^4$$

[Green, Russo, Vanhove, to appear]

The modular functions Z_s are the so-called non-holomorphic Eisenstein series

$$Z_r(\Omega,\overline{\Omega}) = \sum_{(m,n)\neq(0,0)} \frac{\Omega_2^r}{|m+n\Omega|^{2r}}$$

$$= 2\zeta(2r)\Omega_2^r + \gamma_k \Omega_2^{1-r} + \sum_{n,w=1}^{\infty} c_{nw}^{(k)} \cos(2\pi w n\Omega_1) K_k(2\pi w n \Omega_2)$$

$$\Rightarrow \quad \Omega_2^{2-r} Z_r(\Omega,\overline{\Omega}) = genus\, 0 + genus\, (r-\tfrac{1}{2}) \quad + O(\exp(-1/g_s)\,,$$

$$\Omega = \chi + i\exp(-\phi)$$

$$\Omega_2 = \exp(-\phi) = g_s^{-1}$$

$$(\Delta_\Omega - r(r-1))Z_r = 0\,, \qquad \Delta_\Omega = \Omega_2^2(\partial_{\Omega_1}{}^2 + \partial_{\Omega_2}{}^2)$$

The Z_r are unique modular function satisfying this differential equation (having at most polynomial growth at infinity)

TWO LOOPS IN 11D SUPERGRAVITY ON S^1

$$= \frac{\zeta(5)}{g_s^2} D^4 R^4 + \frac{2\zeta(3)^2}{3g_s^2} D^6 R^4$$

$$- \frac{16}{15}\zeta(3)\zeta(2)\log g_s D^8 R^4 + \frac{\zeta(3)\zeta(4)}{27\cdot 15} g_s^2 D^{10} R^4$$

$$+ \zeta(3)\zeta(6) g_s^4 (n_1 D^{12} R^4 + n_2 \tilde{D}^{12} R^4)$$

$$- \frac{2\zeta(3)}{45} g_s^4 \log g_s (11 D^{12} R^4 + 14 \tilde{D}^{12} R^4) + \ldots$$

- [Green,Kwon,Vanhove]
- [Green,Vanhove]
- [Green,Russo,Vanhove, to appear]

TWO LOOPS IN 11D SUPERGRAVITY ON T^2

$$= r_B \Big(Z_{5/2}(\Omega,\bar{\Omega})\; D^4R^4 + E_{(3/2,3/2)}(\Omega,\bar{\Omega})\, D^6R^4$$
$$+ r_B^{-2} E(\Omega,\bar{\Omega})\; D^8R^4 + r_B^{-4} F(\Omega,\bar{\Omega})\; D^{10}R^4$$
$$+ r_B^{-6}\big(G_1(\Omega,\bar{\Omega})\, D^{12}R^4 + G_2(\Omega,\bar{\Omega})\, \tilde{D}^{12}R^4\big)$$
$$+ \ldots \Big)$$

- [Green,Kwon,Vanhove,2000]
- [Green,Vanhove, 2005]
- [Green, Russo,Vanhove, to appear]

THE COUPLING D^6R^4

$$E_{(\frac{3}{2},\frac{3}{2})}\, D^6 R^4 :$$

$$(\Delta_\Omega - 12) E_{(\frac{3}{2},\frac{3}{2})} = -6 Z_{\frac{3}{2}}^{\;2} , \qquad Z_{\frac{3}{2}} = 2\zeta(3) g_s^{-3/2} + 4\zeta(2) g_s^{1/2} + O(\exp(-1/g_s))$$

$$g_s E_{(\frac{3}{2},\frac{3}{2})} = \alpha \frac{1}{g_s^3} + \zeta(3)^2 \frac{1}{g_s^2} + \ldots + \beta\, g_s^4 + O(\exp(-1/g_s))$$

The modular function is uniquely determined by the differential equation with the simple boundary condition that alpha = 0.

Thus one finds [Green,Vanhove, 2005]

D^6R^4 : genus 0 + genus 1 + genus 2 + genus 3 + non-pert.

E, F, G^(i), G^(ii) (multiplying D^8R^4, $D^{10}R^4$, $D^{12}R^4$, $D'^{12}R^4$) **are new modular functions** exhibiting a novel structure: they are sums of modular functions satisfying Poisson equations with the same source term but different eigenvalues [GRV, to appear]

$$E = E_0 + E_1 + E_2 + E_3$$
$$(\Delta_\Omega - 6)E_1 = Z_{\frac{3}{2}}\, Z_{\frac{1}{2}}$$
$$(\Delta_\Omega - 20)E_2 = Z_{\frac{3}{2}}\, Z_{\frac{1}{2}}$$
$$(\Delta_\Omega - 42)E_3 = Z_{\frac{3}{2}}\, Z_{\frac{1}{2}}$$

$$F = F_0 + F_1 + F_2 + F_3 + F_4$$
$$(\Delta_\Omega - \lambda_j)F_j = Z_{\frac{3}{2}}\, Z_{\frac{3}{2}} + w_j Z_{\frac{1}{2}}\, Z_{\frac{1}{2}}$$
$$\lambda_j = 2, 12, 30, 56, 90$$

$$G^{x,y} = G_0 + G_1 + G_2 + G_3 + G_4 + G_5 + G_6$$
$$(\Delta_\Omega - \lambda_j)G_j^x = f_j^x Z_{\frac{3}{2}}\, Z_{\frac{5}{2}} + c_j^x Z_{\frac{1}{2}}\, Z_{\frac{1}{2}}$$
$$\lambda_j = 2j(2j+1) = 0, 6, 20, 42, 72, 110, 156$$

Conjecture: all modular functions in the type IIB effective action are determined by Poisson equations.

$$g_s^2 E = \frac{118}{75}\zeta(3) + \boxed{\frac{16}{5}\zeta(3)\log g_s} + \frac{1114}{2205}\zeta(2) g_s^2 + \frac{4}{9}\zeta(4) g_s^4 + \frac{4}{945}\zeta(6) g_s^6 + \frac{512}{496125}\zeta(8) g_s^8 + \text{nonpert}$$

$$g_s^3 F \boxed{= 180\zeta(3)^2} \boxed{+\frac{7168}{15}\zeta(2)\zeta(3) g_s^2} \boxed{-\zeta(2) g_s^2 \left(\frac{91721}{135} + \frac{1456}{3}\log g_s\right)} + \frac{3248}{3}\zeta(4) g_s^4 + \frac{98}{9}\zeta(6) g_s^6$$
$$+ \frac{896}{405}\zeta(8) g_s^8 + \frac{304}{1875}\zeta(10) g_s^{10} + \beta\,\zeta(12) g_s^{12} + \text{nonpert}$$

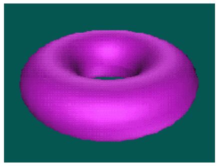
$$g_s^4 G^x \boxed{= 96\zeta(3)\zeta(5)} + \frac{8828}{77}\zeta(2)\zeta(5) g_s^2 - \frac{28096}{77}\zeta(3)\zeta(2) g_s^2 \boxed{+\frac{357856}{5005}\zeta(3)\zeta(4) g_s^4}$$
$$+ g_s^4 (c_1 \boxed{-1760\zeta(4)\log g_s)} + \frac{280}{3}\zeta(6) g_s^6 + c\zeta(8) g_s^8 + c\zeta(10) g_s^{10} + c\,\zeta(12) g_s^{12}$$
$$+ c\zeta(14) g_s^{14} + c\zeta(16) g_s^{16} + \text{nonpert}$$

$$\boxed{g_s^4 G^x = \text{(similar)}}$$

Strikingly, the term proportional to 1760 zeta(4) exactly matches with a genus 3 string theory result determined by the genus one amplitude and unitarity:

the term $Z_{5/2}\, s^6 \log s$

Genus 1 from D=11 supergravity on $\mathbf{T^2}$

String theory  $= \sum_{k=2}^{\infty} (a_k\, r_{IIA}^{2k-1} + b_k r_{IIA}^{2k-7} + \ldots + d_k r_{IIA} \log r_{IIA} + \ldots + c_k r_{IIA}^{-2k+1} + O(e^{-r_{IIA}}))\, D^{2k} R^4$

1) The L = 1 11d supergravity amplitude on a 2-torus contains **genus one** terms

$$(L=1\ \ sugra) \rightarrow \sum_{k=2}^{\infty} c_k \zeta(2k-1)\, r_{IIA}^{2k-1}\, D^{2k} R^4$$

They exactly match with the corresponding terms in the 9d genus one amplitude computed in string theory.

In a **large r expansion** for each given k, they represent the **leading term**.

2) The L = 2 11d supergravity amplitude on a 2-torus gives the **subleading term**

$$(L=2\ sugra) \rightarrow \sum_{k=2}^{\infty} b_k\, r_{IIA}^{2k-7}\, D^{2k} R^4$$

where b_k is a product of zeta functions. **We have checked that they exactly match with the corresponding terms in the 9d genus one amplitude up to k=6.**

Log r_A is obtained from Log g_B term after using $g_B=g_A/r_A$

D=10 TYPE IIB EFFECTIVE ACTION

So far the terms that have been determined are:

$$S = \int d^{10}x \sqrt{-g}\; (\Omega_2^{1/2} Z_{3/2}(\Omega,\overline{\Omega})\, R^4 + \Omega_2^{-1/2} Z_{5/2}(\Omega,\overline{\Omega}) D^4 R^4 + \Omega_2^{-1/2} E_{(\frac{3}{2},\frac{3}{2})}(\Omega,\overline{\Omega}) D^6 R^4)$$

R^4 : genus 0 + genus 1 + non-pert.

D^4R^4 : genus 0 + genus 2 + non-pert.

D^6R^4 : genus 0 + genus 1 + genus 2 + genus 3 + non-pert.

It suggests

$D^{2k}R^4$: genus 0 + genus 1 + ...+ genus k + non-pert.

In other words, that the genus h amplitude has the form

$$A_4^{(h)} = s^h \left(1 + O(\alpha' s)\right) R^4$$

What would be the expected contributions from M-theory?

String theory analogy: integrating out string excitations gives rise to an effective action of the general form:

$$S = \sum_{n=3}^{\infty} c_n \, T^{-n} \int d^{10}x \sqrt{-g}\, D^{2(n-3)} R^4 \quad , \qquad T = \frac{1}{2\pi\alpha'}$$

Similarly, in d=11 on $\mathbf{S^1}$ one would expect that membrane excitations should give contributions of the form

$$S = \sum_{n,k\geq 0}^{\infty} c_{nk} \, T_2^{-n} R_{11}^{-m} \int d^{10}x \, R_{11} \sqrt{-g}\, D^{2k} R^4 \, , \qquad [T_2] = (l_P)^{-3}$$

$$8 + 2k + m = 3n + 11$$

Convert to type IIA variables: $ds^2 = R_{11}^{-1} ds_{IIA}^2 + R_{11}^2 (dx^{11} - C_\mu dx^\mu)^2 \, , \quad R_{11}^3 = g_A^2 \, l_P^3$

$$\Rightarrow S = \sum_{k=0}^{\infty} \sum_{n=0}^{k} c_{nk} \, g_s^{2(k-n-1)} \int d^{10}x \, (\sqrt{-g}\, D^{2k} R^4)_{IIA}$$

$$\Rightarrow \quad h = k - n \leq k \quad !!$$

For M5-branes, $[T_5] = L^{-6}$, leading to corrections **$h = k - 2n$.**

TYPE IIA EFFECTIVE ACTION

An infinite sequence of non-renormalization conditions

$$A_4^L = const\, s^{\beta_L}(1+O(s))R^4$$

$$S = \kappa_{11}^{2(L-1)}\Lambda^n R_{11}^{-m}\int d^{11}x\sqrt{-g}\, D^{2\beta_L+2r}R^4 \quad , \qquad r = 0,1,2,\ldots$$

$$0 \le n \le 9L-6-2\beta_L$$

Convert to type IIA variables:

$$S = \Lambda^n g_A^{2(k-3L+\frac{n}{3}+2)}\int d^{10}x\,(\sqrt{-g}\, D^{2k}R^4)_A$$

$$\rightarrow \quad genus\ h = k+\frac{n}{3}-3(L-1) \le k+1-\frac{2\beta_L}{3}, \quad if\ n>0$$

$$genus\ h = k-3(L-1) \le k \quad if \quad n=0$$

[Berkovits, 2006]:

Direct string theory proof based on zero mode counting that $D^{2k}R^4$ are not renormalized beyond genus k, for k=2,3,4,5.

Using $\beta_L \ge 2 \quad \longrightarrow \quad h < k-\frac{1}{3} \quad for \quad n>0$

$so\ in\ general \quad h \le k$

Thus we find a maximum genus for every value of k. This is irrespective of the value of L. The highest genus contribution is ***h = k*** *and comes from* ***L = 1***.

Is N=8 Supergravity a finite theory?

Since the discovery of the theory, it was clear from various superspace arguments that the high degree of symmetry of the theory would significantly improve the UV behavior

- Kallosh, 1981
- Howe, Lindstrom, 1981
- Howe, Stelle and Townsend, 1984
- Howe, Stelle, 1989

In the last few years, a number of results have appeared which suggest that N=8 might even be finite.

- Bern, Dixon, Dunbar, Perelstein, Rozowsky, 1998.
- Howe, Stelle, 2002
- Bern, Bjerrum-Bohr and Dunbar [0501137]
- Bjerrum-Bohr, Dunbar, Ita, Perkins and Risager, 0610043.
- Bern, Dixon and Roiban [0611086]
- Bern, Carrasco, Dixon,, Johansson, Kosower and Roiban, 0702112.

h-
Th

Th … -loop
Fe … a

- Explicit calculations show $\mathbf{beta_1 = 0}$ [Green et al], $\mathbf{beta_2 = 2}$ [Bern et al] and, more recently, $\mathbf{beta_3 = 3}$ [0702112]
- Berkovits (2006): also $\mathbf{beta_4 = 4}$ and $\mathbf{beta_5 = 5}$
- Our arguments based on the use of M-theory/IIA duality indicate $\mathbf{beta_h = h}$ **for all h.**

Maximal supergravity in lower dimensions:

Compactify string loop amplitude on a 10-d torus. Consider low energy limit alpha' -> 0 with the radii of the torus proportional to sqrt(alpha'), so that all massive KK states, winding numbers and excited string states decouple.

(Subtle limit for non-perturbative states [see Green, Ooguri, Schwarz])

In terms of the d-dimensional gravitational constant kappa$_d^2$,

$$A_{4,d}^h = \kappa_{(d)}^{2(h-1)} \Lambda^{(d-2)h-6-2\beta_h} \quad s^{\beta_h} (1+O(\alpha' s))R^4,$$

$$\kappa_d^2 = \alpha'^{(d-2)/2} e^{2\phi} r^{d-10}, \quad radii = r\sqrt{\alpha'}$$

Therefore UV divergences are absent in dimensions satisfying the bound

$$d < 2 + \frac{2\beta_h + 6}{h}$$

Now using IIA/M theory duality[GRV]: beta$_h$ = h for all h > 1

Hence there would be no UVdivergences if

$$d < 4 + \frac{6}{h} \quad , \quad h > 1$$

d = 4: UV FINITE FOR ALL h !!

Remarkably, this is the same condition as maximally supersymmetric Yang-Mils in d dimensions [Bern, Dixon, Dunbar, Perelstein, Rozowsky, 1998], [Howe,Stelle, 2002]

One can be more conservative and strictly use the non-renormalization theorems that are obvious from superstring perturbation theory in the pure spinor formalism [Berkovits, 2006].

One finds

$beta_h$ = h for h =2, 3, 4, 5

$beta_h$ > or = 6 for h > 5.

Hence there would be no UVdivergences if

$$d < 2 + 18/h \quad , \quad h > 5$$
$$d < 4 + 6/h \quad , \quad h = 2,\ldots,5$$

d = 4: UV finite at least up to h = 8 !!

M-graviton amplitudes

Our non-renormalization theorems equally apply to terms $D^{2k} R^M$

They state that these terms do not get any genus corrections beyond h = k + M – 4

In other words, the M-graviton amplitude at genus h should have the form

$$A_M^h = s^{h+4-M} R^M (1 + O(\alpha' s))$$

A simple generalization of the previous analysis shows that there is no UV divergence in the M-graviton supergravity amplitude for

$$d < 4 + 6/h \quad , \quad h > 1$$

According to this, the d=4 N=8 supergravity theory would be completely finite in the UV.

The Berkovits non-renormalization theorems can also be generalized to the M-graviton amplitude. This leads to the conditions

$$\begin{aligned} d &< 4 + 6/h \quad , \quad 1 < h < 4 + M/2 \\ d &< 2 + (14 + M)/2 \quad , \quad h \geq 4 + M/2 \end{aligned}$$

In d = 4, this implies finiteness if h < 7 + M/2 (hence M > 4 amplitudes are even smoother in the UV)

SUMMARY

KNOWN CASES:

h = 1: UV finite for d < 8 ($beta_1$=0)

h = 2: UV finite for d < 7 ($beta_2$=2)

h = 3: UV finite for d < 6 ($beta_3$=3)

h > 3: expected $beta_h$ > or = 3. This gives **d < 2 + 12/h. Thus d = 4 finite up to h < 6.**

$beta_h$ = h, for all h>1

- **d = 4: UV finite for all h !!**
- d = 5 : UV finite for h = 1,2,3,4,5
- d = 6 : UV finite for h = 1,2
- d = 7: UV finite for h = 1
- d = 8,9,10: UV divergent at h = 1 already

$beta_h$ = h, up to h=5
and $beta_h$ > 5 for h>5

- d = 4: UV finite for h < 9 (i.e. up to h = 8)
- d = 5 : UV finite for h = 1,2,3,4,5
- d = 6 : UV finite for h = 1,2
- d = 7: UV finite for h = 1
- d = 8,9,10: UV divergent at h = 1 already

Some final remarks:

-There is an intriguing structure underlying the superstring effective action and thus underlying the quantum structure of string theory.

Much of this structure can be understood by combining L loops in eleven dimensions, supersymmetry and duality.

-We have seen that non-renormalization theorems for $D^{2k}R^4$ couplings indicate finiteness of N=8 supergravity.

Conversely, note that finiteness of N=8 d=4 supergravity would imply that all terms $D^{2k}R^4$ in the 10d type II effective action are not renormalized.

$$A_4^{(h)} \approx S^{\beta_h} \quad \Rightarrow \quad d < 2 + \frac{2\beta_h + 6}{h} \quad \Rightarrow \quad \beta_h > h - 3$$