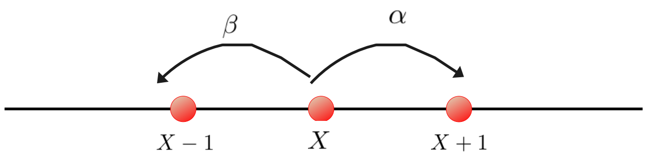
BSSP 2025

---

1. Let x_i be drwan from some distribution $g(x)$, such that $\langle x \rangle = \bar{x}$ and $\langle x^2 \rangle_c = \sigma^2$ are finite. Let $X = \sum_{i=1}^{n} x_i$. Show that
   (a)$\langle X \rangle = n\bar{x}$, $\langle X^2 \rangle = n\sigma^2$
   (b) Show that for $X \lesssim \sqrt{n}\sigma$, $P(x) = \frac{\exp\left(-\frac{X-n\bar{x}}{2n\sigma^2}\right)}{\sqrt{2\pi n\sigma^2}}$
   Hint: Find the generating function $Z(k) = \langle e^{ikX} \rangle$
   (C) *Large deviation:* For $X \gg \sqrt{n}\sigma$, the Gaussian form is not a good approximation. Show that we then have the form $P(X) \sim e^{nF\left(\frac{X}{n}\right)}$, where $F(\zeta)$ is the large deviation function. Also show that for random walk, $F(\zeta) = -\frac{1}{2}(1+\zeta)\ln(1+\zeta) - \frac{1}{2}(1-\zeta)\ln(1-\zeta)$.

2. For the continuous time random walk, the equation for the probability $P(X,t)$ follows



$$\partial_t P(X,t) = \alpha P(X-1,t) + \beta P(X+1,t) - (\alpha+\beta)P(X,t)$$

   (a) For the case $\alpha = 1 = \beta$ and $X \in \mathbb{Z}$, find the solution of this equation (closed form) $P(X,t)$.
   (b) Compare with the solution of the quantum walk

$$\iota\partial_t \psi(X,t) = \psi(X-1,t) + \psi(X+1,t) - 2\psi(X,t)$$

   (c) Plot the evolution with time (i) $P(X,t)$ vs X for different times, (ii) $P_Q(X,t) = |\psi(X,t)|^2$ vs X for different times.
   (d) Find the analytical solution for $\alpha \neq \beta$ and again plot the above quantities at different times.
   (e) Simulate the continuous time random walk on your computer and (i) Plot the individual tra-

jectories and (ii) Plot $P(X,t)$ vs X and compare with the analytical results.

(f) Is it possible to write a biased quantum walk?

3. The continuous time random walk time evolution equation can be written in this form

$$\partial_t |P\rangle = W|P\rangle \tag{1}$$

where $|P\rangle$ is the probability vector and W is the transition rate matrix.

(a) For a periodic lattice with L sites, find the spectrum of W, *i.e.* the eigenvalues λ_s, right eigenvectors $\langle X|\phi_s\rangle$ and left eigenvectors $\langle \chi_s|X\rangle$.

(b) Write the general solution $\langle X|P(t)\rangle = P(X,t)$ as an expansion of the eigenvectors.

(c) Show that the steady state is

$$P(X, t\to\infty) = P_{ss}(X) = \frac{1}{L}$$

and that the time to reach the steady state is $\tau \sim L^2$.

4. For a random walk starting from the origin in a d dimensional hypercubic lattice, the mean number of visits to the origin is given by

$$M_d = \frac{1}{(2\pi)^d}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} d^d\theta \frac{1}{1-\frac{1}{d}\sum_{j=1}^{d}\cos\theta_j}$$

(a) Show that $M_d = \infty$ for $d = 1,2$ and $M_d =$ finite for $d = 3$.

(b) Can you relate the mean number of visits to the origin to the probability to return to the origin. NOTE that the probability to return to the origin at time t is NOT given by $P(X=0,t)$.

****