Some tutorial problems for GHD lectures

Bangalore School on Statistical Physics-XVI

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1 Problems on classical systems

The main source for this part is [1, Sec 2.1, 2.2, 2.3], and, especially for how GHD applies to free theories, [2]. This covers parts of the first three lectures. However, the free classical particle example was not really done, so I develop it a bit more in these problems.

Generic Galilean systems and ideal gas

A Galiglean invariant gas typically has three conserved quantities

$$Q_0 = N = \sum_{n=1}^{N} 1, \quad Q_1 = P = \sum_{n=1}^{N} p_n, \quad Q_2 = H = \sum_{n=1}^{N} \frac{p_n^2}{2} + \frac{1}{2}\sum_{n,m=1}^{N} V(x_n - x_m) \tag{1.1}$$

where $V(x)$ is the pair-interaction potential. The associated densities $Q_i = \int \mathrm{d}x\, q_i(x)$ are

$$q_0(x) = \sum_n \delta(x-x_n), \quad q_1(x) = \sum_n p_n\delta(x-x_n), \quad q_2(x) = \sum_n \Big(\frac{p_n^2}{2}+\frac{1}{2}\sum_m V(x_n-x_m)\Big)\delta(x-x_n). \tag{1.2}$$

Likewise $q_i(x,t)$ is obtained by replacing x_n by $x_n(t)$ and p_n by $p_n(t)$. The Gibbs states have Boltzmann weight

$$e^{-\beta(H-\mu N - vP)}. \tag{1.3}$$

That is, the grand-canonical partition function is

$$Z = \sum_{N=0}^{\infty} \frac{1}{N!} \int_{x_n\in[0,L]\,\forall n} \prod_n \mathrm{d}x_n \mathrm{d}p_n\, e^{-\beta(H-\mu N-vP)}. \tag{1.4}$$

We denote average observables, evaluated at position 0 and time 0, in this state, as

$$\mathtt{q}_0, \quad \mathtt{q}_1, \quad \mathtt{q}_2, \quad \text{etc.} \tag{1.5}$$

1. Show that the conservation laws

$$\partial_t q_i(x,t) + \partial_x j_i(x,t) = 0, \quad i = 0,1,2 \tag{1.6}$$

hold, with

$$j_0(x,t) = q_1(x,t). \tag{1.7}$$

Calculate $j_1(x,t)$, $j_2(x,t)$.

2. Consider the "ideal gas" case $V = 0$. Show that averages take the form

$$\mathsf{q}_0 = \int \mathrm{d}p\, f(p), \quad \mathsf{q}_1 = \int \mathrm{d}p\, p f(p), \quad \mathsf{q}_2 = \int \mathrm{d}p\, \frac{p^2}{2} f(p) \tag{1.8}$$

and

$$\mathsf{j}_0 = \int \mathrm{d}p\, p f(p), \quad \mathsf{j}_1 = \int \mathrm{d}p\, p^2 f(p), \quad \mathsf{j}_2 = \int \mathrm{d}p\, \frac{p^3}{2} f(p) \tag{1.9}$$

where $f(p)$ has the Gaussian form

$$f(p) = C e^{-\beta(p^2/2-\mu-vp)}. \tag{1.10}$$

for some C (calculate it).

3. Galilean invariance implies, in general,

$$\mathsf{j}_0 = \mathsf{q}_1, \quad \mathsf{j}_1 = P(\mathsf{q}_0,\mathsf{q}_2) + \frac{\mathsf{q}_1^2}{\mathsf{q}_0}, \quad \mathsf{j}_2 = \frac{3\mathsf{q}_1}{2\mathsf{q}_0} P(\mathsf{q}_0,\mathsf{q}_2) + \frac{\mathsf{q}_1^2}{\mathsf{q}_0^2}. \tag{1.11}$$

a. Using the statement of question 2, verify these relations in the case of the ideal gas $V = 0$. In this case, determine the pressure function $P(\mathsf{q}_0,\mathsf{q}_2)$.
b. In general, check that these relations from Galilean invariance imply in particular the two usual hydrodynamic equations

$$\partial_t \rho + \partial_x(v\rho) = 0, \quad \partial_t v + v\partial_x v = -\frac{1}{\rho}\partial_x P(\rho, e) \tag{1.12}$$

where $\rho = \mathsf{q}_0$, $v\rho = \mathsf{q}_1$ and $e = \mathsf{q}_2$. Write down the third equation, for e.

Free gas

The free gas is not the ideal gas. In the free gas, we take into consideration that relaxation is towards the states which account for *all conserved quantities.* There is a one-parameter family of conserved quantities

$$Q_p = \sum_n \delta(p - p_n) \tag{1.13}$$

and the Boltzmann weight are

$$e^{-\int \mathrm{d}p\, w(p) Q_p}. \tag{1.14}$$

The grand-canonical partition function is

$$Z = \sum_{N=0}^{\infty} \frac{1}{N!} \int_{x_n \in [0,L]\, \forall n} \prod_n \mathrm{d}x_n \mathrm{d}p_n\, e^{-\sum_n w(p_n)}. \tag{1.15}$$

We define

$$q_p(x,t) = \sum_n \delta(x - x_n)\delta(p - p_n). \tag{1.16}$$

1. Show that the conservation laws

$$\partial_t q_p(x,t) + \partial_x j_p(x,t) = 0, \tag{1.17}$$

hold, and calculate $j_p(x,t)$.

2. Evaluate the averages

$$\mathsf{q}_p, \quad \mathsf{j}_p \tag{1.18}$$

as *functionals of the function* $p \mapsto w(p)$. You do not need to perform the integral $\int \mathrm{d}p\, e^{-w(p)}$. Verify that

$$\mathsf{j}_p = p\mathsf{q}_p. \tag{1.19}$$

Once we have this, the hydrodynamic equation is obtained by *making the stationary states space-time dependent*, that is $w(p)$ becomes $w(p,x,t)$, so that q_p becomes $\rho(p,x,t)$ and j_p becomes $p\rho(p,x,t)$. The equation is simply the conservation law, for these space-time dependent stationary states

$$\partial_t \rho(p,x,t) + p\partial_x \rho(p,x,t) = 0. \tag{1.20}$$

This is the Liouville equation for conservation of phase-space element, or the collisionless Boltzmann equation.

The ideal gas assumes that *there is a little bit of interaction*, just enough to destroy the higher conserved quantities and to keep only N, P, H, while the expressions of densities and currents are still well approximated by their free-particle form. In higher dimensions, this is a good approximation. But in one dimension, the dyamics, even with a bit of interaction that destroys higher conserved quantities, is still well described by accounting for these conserved quantities. Their decay over time is slow enough. The full theory for that is, of course, Boltzmann equation, which is very special in one dimension, making decay of higher conserved quantities very slow.

2 Problems on general structure of Euler-scale hydrodynamics

Again, this is from [1, Sec 2.1, 2.2, 2.3], except question 3.

The general Euler-scale hydrodynamic equation, in its quasi-linear forrm, is

$$\partial_t \mathsf{q}_i + \mathsf{A}_i^{\ j} \partial_x \mathsf{q}_j = 0 \tag{2.1}$$

where

$$\mathsf{A}_i^{\ j} = \frac{\partial \mathsf{j}_i}{\partial \mathsf{q}_j} \tag{2.2}$$

is the flux Jacobian. Another important matrix is the covariance matrix

$$\mathsf{C}_{ij} = \langle Q_i q_j \rangle^{\mathrm{c}}_{\underline{\beta}} = -\frac{\partial \mathsf{q}_i}{\partial \beta^j}. \tag{2.3}$$

We showed in class that the matrix C is symmetric, and that $\mathsf{B} = \mathsf{AC}$ is also symmetric.

1. Using the fact that C is symmetric, argue, using the Poincaré lemma, that there must exist a function $\mathsf{f}(\underline{\beta})$ such that

$$\mathsf{q}_i = \frac{\partial \mathsf{f}}{\partial \beta^i}. \tag{2.4}$$

Of course, this function is the specific free energy,

$$\mathsf{f} = -\lim_{L\to\infty} L^{-1}\log Z. \tag{2.5}$$

2. Using the fact that B is symmetric, argue, again using the Poincaré lemma, that there must exist a function $\mathtt{g}(\underline{\beta})$ such that

$$\mathtt{j}_i = \frac{\partial \mathtt{g}}{\partial \beta^i}. \tag{2.6}$$

The function $\mathtt{g}$ is called the *free energy flux*. Here, we don't have an explicit expression of $\mathtt{g}$ in terms of traces etc.; but we know it must exist.

3. For the ideal gas, we have from (1.4)

$$Z = \sum_{N=0}^{\infty} \frac{1}{N!} L^N \Big(\int \mathrm{d}p\, e^{-w(p)}\Big)^N = \exp\Big[L\int \mathrm{d}p\, e^{-w(p)}\Big] \tag{2.7}$$

where $w(p) = \beta(p^2/2 - \mu - vp) = \beta^2 p^2/2 + \beta^1 p + \beta^0$, identifying $\beta^0 = -\mu\beta$, $\beta^1 = -v\beta$, $\beta^2 = \beta$ using the numbering (1.1) (recall in β^i, the i is an index, not a power). Therefore

$$\mathsf{f} = -\int \mathrm{d}p\, e^{-w(p)}. \tag{2.8}$$

Verify that (2.4) reproduces (1.8). Using (1.9), evaluate $\mathtt{g}$.

4. Using the Euler-scale equations (2.1), verify that the *entropy density and currents*, defined as

$$\mathtt{s} = \sum_i \beta^i \mathtt{q}_i - \mathsf{f}, \quad \mathtt{j} = \sum_i \beta^i \mathtt{j}_i - \mathtt{g}, \tag{2.9}$$

satisfy the conservation law

$$\partial_t \mathtt{s} + \partial_x \mathtt{j} = 0. \tag{2.10}$$

Thus, entropy conservation *is a consequence of the general properties of the equations of state.*

3 Problems on GHD, part 1

We have derived in class the following formula for the effective velocity

$$v^{\mathrm{eff}}(p) = p - \int \mathrm{d}p'\, \rho(p')\varphi(p,p')(v^{\mathrm{eff}}(p) - v^{\mathrm{eff}}(p')). \tag{3.1}$$

where $\varphi(p,p')$ represents the shift incurred by a particle with “quasi-momentum” p upon meeting a particle with quasi-momentum p'. The case of the hard rods is

$$\varphi(p,p') = -a \tag{3.2}$$

where a is the rod length – that is, the shift is independent of the velocity of the rod. The general formula could be seen to arise from a dynamical equation for quasi-particles (here I use θ_n instead of p_n)

$$y_n = x_n + \frac{1}{2}\sum_{m\neq n} \varphi(\theta_n - \theta_m)\,\mathrm{sgn}(x_n - x_m), \quad \dot{y}_n = \theta_n. \tag{3.3}$$

1. Solve explicitly for $v^{\mathrm{eff}}(p)$ in terms of p, of the total paritcle density $\bar\rho = \int \mathrm{d}p\,\rho(p)$, and of the average velocity $\bar u = \bar\rho^{-1}\int \mathrm{d}p\,p\rho(p)$. You should get

$$v^{\mathrm{eff}}(p) = \frac{p - \bar u a\bar\rho}{1 - a\bar\rho} \tag{3.4}$$

2. Consider the problem of a single free particle with Hamiltonian

$$H = -\frac{1}{2}\frac{\partial^2}{\partial x^2} \tag{3.5}$$

and initial wave function

$$\psi(x) = e^{\mathrm{i}px}A(x/\ell). \tag{3.6}$$

Show that

$$(e^{-\mathrm{i}H\ell t}\psi)(\ell x) = z_t e^{\mathrm{i}p\ell x}A(x - pt) + O(\ell^{-1}) \tag{3.7}$$

where z_t is a pure phase, $|z_t| = 1$. That is, the large-scale time evolution corresponds to a simple shift of the slowly-varying amplitude modulation (wave packet). A good way is to Fourier transform the amplitude $A(x/\ell)$, and use the fact that

$$He^{\mathrm{i}px} = \frac{p^2}{2}e^{\mathrm{i}px}. \tag{3.8}$$

3. Show that the transformation of coordinates

$$y_n = x_n + \frac{1}{2}\sum_{m\neq n}\varphi(\theta_n - \theta_m)\,\mathrm{sgn}(x_n - x_m) \tag{3.9}$$

$$p_n = \theta_n + \sum_{m\neq n}\phi(\theta_n - \theta_m)\delta(x_n - x_m), \tag{3.10}$$

where $\varphi(\theta) = \mathrm{d}\phi(\theta)/\mathrm{d}\theta$, is a canonical transformation in the sense of classical dynamical systems, from canonical coordinates x_n, p_n to canonical coordinates y_n, θ_n. Conclude that the dynamical system of quasi-particles (3.3) has classical Hamiltonian

$$H = \sum_n \frac{\theta_n^2}{2}. \tag{3.11}$$

Hint: use a clever generating function for canonical transformation (related to the phase of the Bethe ansatz wave function!).

References

[1] Benjamin Doyon. Lecture notes on generalised hydrodynamics. *SciPost Physics Lecture Notes*, 018, 2020.

[2] Fabian H.L. Essler. A short introduction to generalized hydrodynamics. *Physica A: Statistical Mechanics and its Applications*, 631:127572, 2023. Lecture Notes of the 15th International Summer School of Fundamental Problems in Statistical Physics.