SPECIAL HOLONOMY AND THE CONIFOLD

or

PHENOMENOLOGY OF EXCEPTIONAL HOLONOMY

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Calabi-Yau Compactification of Strings

- The moduli of the CY 6-manifold acquire the interpretation as massless scalars in the 4-dimensional theory.

- At special values of the moduli the CY develops singularities. Near a singularity the CY metric is approximated by the conifold — cone over $T^{1,1} = \frac{S^3 \times S^3}{U(1)}$

$$\boxed{ds_6^2 = dt^2 + t^2 ds_5^2(T^{1,1})}$$

$$\boxed{ds_5^2(T^{1,1}) = \frac{1}{6}(\Sigma_1^2 + \Sigma_2^2 + \sigma_1^2 + \sigma_2^2) + \frac{1}{9}(\Sigma_3 - \sigma_3)^2}$$

$\sigma_1 + i\sigma_2 = e^{i\psi}(d\theta + \sin\theta\, d\phi)$, $\sigma_3 = d\psi + \cos\theta\, d\phi$ a

Left-invariant 1-forms of $SU(2) \times SU(2)$

- For modulus values near to those giving the singularity, the Conifold metric is replaced by a smooth non-compact metric of $SU(3)$ holonomy — a "Resolution" of the conifold.

Resolved Metrics of Cohomogeneity One

$$\boxed{ds_6^2 = dt^2 + a^2\left[(\Sigma_1+g\sigma_1)^2+(\Sigma_2+g\sigma_2)^2\right] + b^2\left[(\Sigma_1-g\sigma_1)^2+(\Sigma_2-g\sigma_2)^2\right] + c^2(\Sigma_3-\sigma_3)^2}$$

a, b, c, g Functions of t

SU(3) Holonomy $\Leftrightarrow$ Ricci-flat, Kähler $\Leftrightarrow$

$$\boxed{dJ = 0\,, \quad d\Psi_3 = 0}$$

$\Rightarrow$ First-order equations plus $a=b$ or $g=1$

$\underline{a=b}$: "Small Resolution"

$$\dot a = -\frac{c}{4b}\,, \quad \dot c = -1 + \frac{c^2(g^2+1)}{4b^2g^2}\,, \quad \dot g = \frac{c(g^2-1)}{4b^2 g}$$

Integrable. $\rightarrow$

$$\boxed{ds_6^2 = \frac{3(r+1)}{r(2r+3)}dr^2 + r(\sigma_1^2+\sigma_2^2) + \frac{r(2r+3)}{3(r+1)}(\Sigma_3-\sigma_3)^2 + \underbrace{(r+1)(\Sigma_1^2+\Sigma_2^2)}_{\text{Minimal } S^2}}$$

Candelas de la Ossa

$r \geq 0$

R^4 bundle over S^2

$\underline{g=1}$: "Deformed Conifold"

$$\dot a = \frac{a^2-b^2-c^2}{4bc}\,, \quad \dot b = \frac{b^2-a^2-c^2}{4ac}\,, \quad \dot c = \frac{c^2-a^2-b^2}{2ab}$$

Integrable $\rightarrow$

$$\boxed{ds_6^2 = \frac{1}{3}R^{-2/3}\sinh^2 r\left[dr^2+(\Sigma_3-\sigma_3)^2\right] + R^{1/3}\left[\coth r\left[(\Sigma_1+\sigma_1)^2+(\Sigma_2+\sigma_2)^2\right] + \tanh r\left[(\Sigma_1-\sigma_1)^2+(\Sigma_2-\sigma_2)^2\right]\right]}$$

$R \equiv \frac{1}{8}(\sinh 4r - 4r)$ $\qquad r \geq 0$

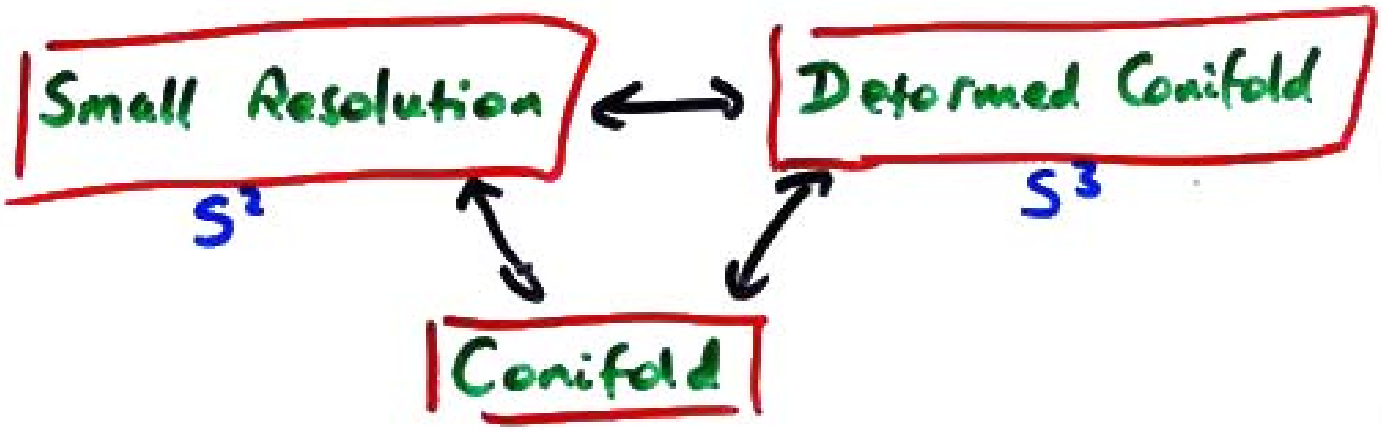
Candelas de la Ossa Stenzel.

R^3 bundle over S^3

Both metrics are Asymptotically Conical

CONIFOLD TRANSITION

- AS THE SCALAR MODULI VARY, THE TOPOLOGY OF THE CALABI-YAU MANIFOLD CAN CHANGE, VIA THE SINGULAR CONIFOLD



- COLLAPSING CYCLES → EXTRA MASSLESS STATES

- MORE GENERALLY IN STRINGS AND M-THEORY, WE CAN STUDY NON-COMPACT METRICS THAT MAY DESCRIBE COMACT SPACE NEAR A SINGULARITY.

SOLUTION → Ricci-flat

SUPERSYMMETRY → Special Holonomy

BERGER CLASSIFICATION

RICCI-FLAT SPECIAL HOLONOMIES:

$D=2n$	Ricci-flat Kähler	$SU(n) \subset SO(2n)$
$D=4n$	Hyper-Kähler	$Sp(n) \subset SO(4n)$
$D=7$	G_2 Holonomy	$G_2 \subset SO(7)$
$D=8$	Spin(7) Holonomy	$Spin(7) \subset SO(8)$

- THE EXCEPTIONAL CASES OF G_2 AND SPIN(7) HOLONOMY ARE OF PARTICULAR INTEREST IN STRINGS & M-THEORY.

- THEY CAN BE CHARACTERISED BY HAVING 1 COVARIANTLY-CONSTANT SPINOR η

$\underline{D=7}$:

$$\frac{SO(7)}{\underline{8}} \longrightarrow \frac{G_2}{\underline{7}+\underline{1}}$$

$\underline{D=8}$:

$$\frac{SO(8)}{\begin{matrix}8_+\\ 8_-\end{matrix}} \longrightarrow \frac{Spin(7)}{\begin{matrix}7+1\\ 8\end{matrix}}$$

$\boxed{\underline{D=7}: \quad \Phi_{abc} = \bar\eta \Gamma_{abc} \eta}$

3-form

$\boxed{\underline{D=8}: \quad \Phi_{abcd} = \bar\eta \Gamma_{abcd} \eta}$

self-dual 4-form

• MORE CONVENTIONALLY, G_2 HOLONOMY IS DEFINED BY THE REQUIREMENT THAT AN ASSOCIATIVE 3-FORM Φ_3 SATISFIES

$$\boxed{d\Phi_3 = 0 \qquad d*\Phi_3 = 0}$$

wh. $O_a O_b = -\delta_{ab} + \Phi_{abc} O_c$ $\quad O_a$ imaginary octonions

$$\boxed{\Phi_{123} = \Phi_{145} = \Phi_{176} = \Phi_{246} = \Phi_{257} = \Phi_{347} = \Phi_{365} = 1}$$

• Spin(7) holonomy $\Leftrightarrow$ CAYLEY 4-FORM $\Phi_4 = *\Phi_4$ IS CLOSED

$$\boxed{d\Phi_4 = 0}$$

$$\Phi_4 \sim \Phi_3 \wedge e^8 + *(\Phi_3 \wedge e^8)$$

• BY THIS MEANS, WE GET 1'ST ORDER EQUATIONS FOR G_2 or SPIN(7) HOLONOMY

• A Spin(7) EXAMPLE ILLUSTRATES THE GENERAL FEATURES . . .

$\mathbb{R}^4$ BUNDLE OVER S^4 WITH SPIN(7) HOLONOMY

- COHOMOGENEITY ONE, WITH $S^7 = \frac{SO(5)}{SO(3)}$

ORBITS:

$$\boxed{ds_8^2 = dt^2 + \frac{a^2}{4}(\sigma_i - A^i)^2 + c^2 d\Omega_4^2}$$

S^3 $N=1$ YM INSTANTON S^4

SPIN(7) HOLONOMY $\Rightarrow$

$$\boxed{\dot{a} = \frac{1}{2} - \frac{a^2}{2c^2}} \qquad \boxed{\dot{c} = \frac{3a}{2c}}$$

INTEGRABLE

$$\boxed{\begin{aligned} ds_8^2 = {} & \left(1 - \frac{\ell^{10/3}}{r^{10/3}}\right)^{-1} dr^2 + \frac{9}{100} r^2 \left(1 - \frac{\ell^{10/3}}{r^{10/3}}\right)(\sigma_i - A^i)^2 \\ & + \frac{9}{20} r^2 d\Omega_4^2 \end{aligned}}$$

$r \geq \ell$

Bryant & Salamon
Gibbons, Page & Pope

- S^4 Bolt at $r = \ell$ $\mathbb{R}^4$ BUNDLE
- METRIC IS ASYMPTOTICALLY CONICAL (AC) — APPROACHES CONE OVER SQUASHED EINSTEIN METRIC ON S^7

MORE GENERAL SPIN(7) METRICS

- NOW ALLOW THE S^3 FIBRES TO SQUASH – BERGER SPHERE S^3 = U(1) BUNDLE OVER S^2:

$$\boxed{ds_8^2 = dt^2 + \underbrace{\underbrace{a^2 (D\mu_i)^2}_{S^2} + \underbrace{b^2 (d\psi + A)^2}_{S^1}}_{S^3 \text{ bundle}} + \underbrace{c^2\, d\Omega_4^2}_{S^4}}$$

$\mu_i \mu_i = 1$

- SPIN(7) HOLONOMY ⇒

Cvetič, Gibbons, Lü, Pope

$$\boxed{\dot a = 1 - \frac{b}{2a} - \frac{a^2}{c^2}}, \quad \boxed{\dot b = \frac{b^2}{2a^2} - \frac{b^2}{c^2}}, \quad \boxed{\dot c = \frac{a}{c} + \frac{b}{2c}}$$

INTEGRABLE

↓

1. Previous AC solution as $a = b$ special case
2. A simple new solution $\mathbb{B}_8^\circ$ that is Asymptotically Locally Conical (ALC) – approaches $S^1 \times M_7$ locally at large distance – where M_7 is AC
3. A general 1-parameter family $\mathbb{B}_8$ of ALC metrics, with a parameter $0 \le \lambda \le \infty$ so that

$$\boxed{\lambda = \frac{\text{Radius of } S^1 \text{ at infinity}}{\text{Radius of } S^4 \text{ bolt}}}$$

$\mathbb{B}_8^\circ$:

$$\boxed{ds_8^2 = \frac{(r-\ell)\,dr^2}{(r-3\ell)(r+\ell)} + \tfrac{1}{4}(r-3\ell)(r+\ell)\, D\mu_i^2 + \frac{\ell^2 (r-3\ell)(r+\ell)}{(r-\ell)^2}(d\psi + A)^2 + \tfrac{1}{2}(r^2 - \ell^2)\, d\Omega_4^2}$$

$r \ge 3\ell \qquad \lambda = \tfrac{1}{2}$

FOR GENERAL λ :

$$\boxed{ds_8^2 = \frac{v f\, dz^2}{4z(1-z^2)(1-z)(v-2)} + \frac{(v-2)fz}{(1+z)v}\, D\mu_i^2 + \frac{4(v-2)zf}{(1+z)v^3}(d\psi+\mathcal{A})^2 + f\, d\Omega_4^2}$$

$$v = \frac{2k\sqrt{z}}{(1-z^2)^{1/4}} - 2z\; {}_2F_1\left[1, \tfrac{1}{2}, \tfrac{5}{4}, 1-z^2\right]$$

$\lambda = \lambda(k)$

$$f = \left(\frac{1+z}{1-z}\right)^{1/2} \exp \int^z \frac{dz'}{v(z')(1-z'^2)}$$

THIS 1-PARAMETER FAMILY IS RATHER TYPICAL OF WHAT ONE FINDS IN METRICS OF SPECIAL HOLONOMY.

$\lambda = 0$ ⟵ ALC METRICS ⟶ $\lambda = \infty$

$S^1 \times M_7$ GROMOV-HAUSDORFF LIMIT

ORIGINAL AC METRIC

- M_7 IS AN AC METRIC OF G_2 HOLONOMY, WITH $\mathbb{CP}^3 = S^2$ bundle over S^4 ORBITS.

  THIS IS ONE OF THREE AC G_2 METRICS FOUND BY Bryant & Salamon, Gibbons, Page, Pope

  $\boxed{\mathbb{CP}^3}$, $\boxed{SU(3)/T^2}$, $\boxed{S^3 \times S^3}$ ORBITS.

- $S^3 \times S^3$ ADMITS S^1 REDUCTION FROM M-THEORY TO IIA STRINGS

- CONE OVER $S^3 \times S^3$:

$$\boxed{ds_7^2 = dt^2 + \frac{1}{9}t^2(\Sigma_i \tfrac{1}{2}\sigma_i)^2 + \frac{1}{12}t^2\sigma_i^2}$$

(Cone over squashed Einstein $S^3 \times S^3$)

- ASYMPTOTICALLY CONICAL RESOLUTION:

$$\boxed{ds_7^2 = \frac{dr^2}{1-\ell^3/r^3} + \frac{1}{9}r^2\left(1-\frac{\ell^3}{r^3}\right)(\Sigma_i - \tfrac{1}{2}\sigma_i)^2 + \frac{1}{12}r^2\sigma_i^2}$$

$r \geq \ell$ $\quad$ R^4 bundle over S^3 $\quad$ Bryant & Salamon, Gibbons, Page, Pope

This has $SU(2)^3$ symmetry

- Various generalisations with reduced symmetry $SU(2)^2$ or $SU(2)^2 \times U(1)$ have been studied.

Brandhuber, Gomis, Gubser, Gukov, Cvetič, Gibbons, Lü, Pope, ...

- An $SU(2)^2 \times U(1)$ invariant ansatz of particular relevance for string theory is

$$\boxed{\begin{aligned} ds_7^2 = dt^2 &+ a^2\left[(\Sigma_1 + g\sigma_1)^2 + (\Sigma_2 + g\sigma_2)^2\right] + b^2\left[(\Sigma_1 - g\sigma_1)^2 + (\Sigma_2 - g\sigma_2)^2\right] \\ &+ c^2(\Sigma_3 - \sigma_3)^2 + f^2(\Sigma_3 + g_3\sigma_3)^2 \end{aligned}}$$

$\Phi_3 = \ldots$

Requiring G_2 holonomy, $d\Phi_3 = 0$, $d*\Phi_3 = 0$, gives a system of first-order equations

(Cvetič, Gibbons, Lü, Pope). (Related work by Brandhuber)

Non-Integrable (probably A. Dancer & M.Y. Wang)

- WE FIND THREE FAMILIES OF SMOOTH ALC SOLUTIONS, BY USING NUMERICAL INTEGRATION. EACH HAS A NON-TRIVIAL PARAMETER $0 \leq \lambda \leq \infty$.

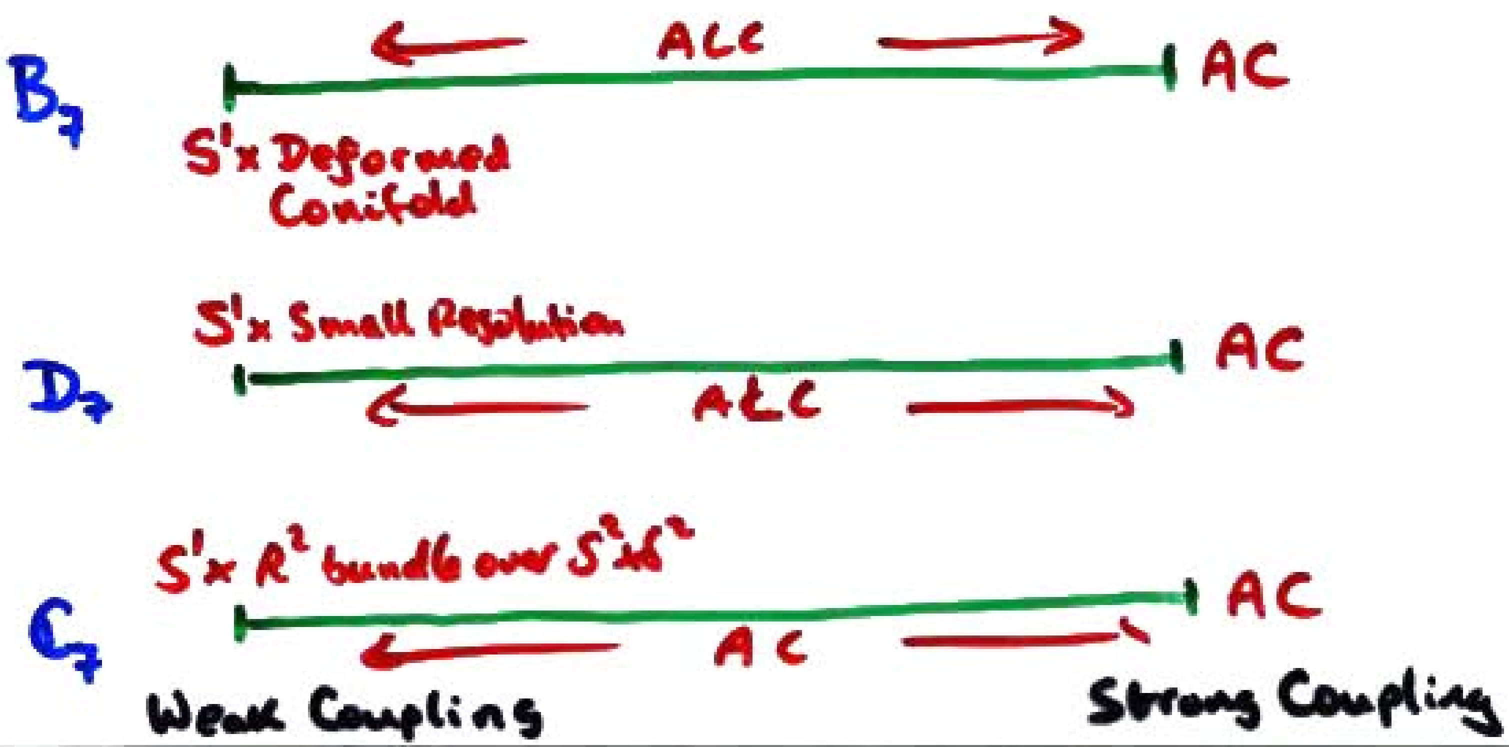
$$\boxed{\lambda \sim \frac{\text{Radius of } S^1 \text{ at infinity}}{\text{Radius of bolt at origin}}}$$

$\mathbb{B}_7$: Round S^3 bolt

$\mathbb{D}_7$: Squashed S^3 bolt $\sim \sigma_1^2 + \sigma_2^2 + \lambda^2 \sigma_3^2$

$\mathbb{C}_7$: $T^{p,q} = \dfrac{S^3 \times S^3}{U(1)_{p,q}}$ bolt

- The Gromov-Hausdorff limits give $S^1 \times$ Calabi-Yau:



Example $\mathbb{B}_7$ metrics

These have $g = 1$, $g_3 = 1$ $\Rightarrow$

$$\boxed{\begin{aligned}
\dot{a} &= \frac{a^2}{4bc} - \frac{b}{4c} - \frac{c}{4b} - \frac{f}{4a} \\
\dot{b} &= \frac{b^2}{4ac} - \frac{a}{4c} - \frac{c}{4a} + \frac{f}{4b} \\
\dot{c} &= \frac{c^2}{2ab} - \frac{a}{2b} - \frac{b}{2a} \\
\dot{f} &= \frac{f^2}{4a^2} - \frac{f^2}{4b^2}
\end{aligned}}$$

Brandhuber, Gomis, Gubser, Gukov

One exact solution in the 1-parameter ALC family is known (analogue of $\mathbb{B}_8^\circ$)

System apparently non-integrable ...

A General $SU(2)\times SU(2)$ Class

Z.W. Chong, M. Cvetič, G.W. Gibbons, H. Lü, C.N. Pope, P. Wagner

$$\rho = n\,\Sigma_1\wedge\Sigma_2\wedge\Sigma_3 + m\,\sigma_1\wedge\sigma_2\wedge\sigma_3 + x_1\, d(\sigma_1\Sigma_1) + \cdots$$

$$\omega = \sqrt{\frac{y_2 y_3}{y_1}}\;\sigma_1\wedge\Sigma_1 + \cdots$$

$$\boxed{\Phi_3 = dt\wedge\omega + \rho}$$

$x_i(t)$
$y_i(t)$

Gives G_2-invariant metric $g_{ab} = (\det\gamma)^{-\frac{1}{9}}\gamma_{ab}$
where

$$\gamma_{ab} = -\frac{1}{144}\,\Phi_{ac_1c_2}\,\Phi_{bc_3c_4}\,\Phi_{c_5c_6c_7}\,\varepsilon^{c_1\cdots c_7}$$

$\Rightarrow$

$$\boxed{\begin{aligned} ds_7^2 = dt^2 + \frac{1}{y_1}\Big[&(nx_1+x_2x_3)\Sigma_1^2 + (mn+x_1^2-x_2^2-x_3^2)\,\sigma_1\Sigma_1 \\ &+ (mx_1+x_2x_3)\,\sigma_1^2\Big] + \cdots \end{aligned}}$$

G_2 holonomy $\rightarrow$ $d\Phi_3 = 0,\; d{*}\Phi_3 = 0$ $\Rightarrow$

$$\boxed{\dot{x}_1 = \sqrt{\frac{y_2y_3}{y_1}}\,,\quad \dot{y}_1 = \frac{mn\,x_1 + (m+n)x_2x_3 + x_1(x_2^2+x_3^2-x_1^2)}{\sqrt{y_1y_2y_3}}}$$

and cyclic

$$\boxed{\begin{aligned} &4y_1y_2y_3 + m^2n^2 - 2mn(x_1^2+x_2^2+x_3^2) - 4(m+n)x_1x_2x_3 + x_1^4+x_2^4+x_3^4 \\ &-2x_1^2x_2^2 - 2x_2^2x_3^2 - 2x_3^2x_1^2 = 0 \end{aligned}}$$

Presumably non-integrable

- Existence of new solutions?

Summary

- Non-compact metrics of special holonomy can describe supersymmetric compactifications in String and M-theory near to singular points.

- The known regular examples with G_2 or Spin(7) holonomy are of cohomogeneity one.

- Simplest examples are Asymptotically Conical

- A more generic behaviour is where the metric is Asymptotically Locally Conical, looking like $S^1 \times M_{d-1}$ locally at large distance

$$\Updownarrow$$

String theory / M-theory

- First-order systems sometimes integrable. Other cases need more study.