$\mathcal{N} = 1$ FLUX VACUA:
GEOMETRY AND NON GEOMETRY

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INTRODUCTION

- Compact Minkowski type II vacua with RR/NS fluxes are relevant for

$\longrightarrow$ moduli stabilisation, supersymmetry breaking

$\longrightarrow$ landscape of vacua

- $4d$ low energy effective sugra

$\longrightarrow$ consistency of the embedding in string theory (compactness, large volume limit)

- $10d$ examples are all related to Calabi-Yau's

$\longrightarrow$ conformal CY, compact versions of deformed conifold, T-duals ...

- Go beyond conformal CY's
  - use Generalised Complex Geometry

    $\longrightarrow$ exhaustive search of $\mathcal{N}=1$ Minkowski vacua on $6d$ nil and solvmanifolds (twisted tori)

    $\longrightarrow$ non geometric backgrounds

SUPERSYMMETRIC SOLUTIONS

10d equations of motion

$$ds^2_{(10)} = e^{2A(y)} ds^2_{(4)} + ds^2_{(6)}$$

$$F_{(10)} = F_{(6)} + \mathrm{vol}_4 \wedge \lambda(*F_{(6)})$$

$$\Updownarrow$$

10d SUSY variations

$$\delta\psi_i = 0$$

$$\delta\lambda_i = 0$$

$$\Downarrow$$

Generalised Calabi Yau

flux e.o.m

$$(d + H)(e^{4A} * F = 0)$$

$$d(e^{4A-2\phi} * H) = \mp e^{4A} F_n \wedge *F_{n+2}$$

$$\Downarrow$$

implied by SUSY

Bianchi identities

$$(d - H)F = \delta(source)$$

$$dH = 0$$

$$\Downarrow$$

no-go theorem

GENERALISED COMPLEX GEOMETRY

[hitchin 02; gualtieri 04]

- It treats tangent and cotangent bundle on the same footing

$$X = (v + \xi) \in T(M) \oplus T^*(M)$$

	T	$T \oplus T^*$
almost complex structure	$J : J^2 = -1_d$	$\mathcal{J} : \mathcal{J}^2 = -1_{2d}$
integrability	$\pi_+[\pi_-(v), \pi_-(w)] = 0$	$\Pi_+[\Pi_-(X), \Pi_-(Y)]_C = 0$
Clifford algebra	Cliff(6)	Cliff(6,6)
spinors	$(0, q)$ forms	(p, q) forms
pure spinor	vacuum of Cliff(6): η_0	vacuum of Cliff(6,6): Φ
	$\nabla_m \eta_0 = 0$	$d\Phi = 0$
	Calabi Yau	Generalised Calabi Yau

- $[,]_C \longrightarrow$ Courant bracket

$$[v + \xi,\, w + \eta] = [v, w] + \left\{ \mathcal{L}_v \eta - \mathcal{L}_w \xi - \frac{1}{2} d(i_v \eta - i_w \xi) \right\}$$

SUSY VARIATIONS AND GENERALISED COMPLEX GEOMETRY

- Define the bispinors

$$\Phi_+ = \eta^1_+ \otimes \eta^{2\,\dagger}_+ \quad \text{even forms} \qquad \epsilon_1 = \zeta_+ \otimes \eta^1_+ + \zeta_- \otimes \eta^1_-$$

$$\Phi_- = \eta^1_+ \otimes \eta^{2\,\dagger}_- \quad \text{odd forms} \qquad \epsilon_2 = \zeta_+ \otimes \eta^2_\mp + \zeta_- \otimes \eta^2_\pm$$

  - pure spinors
  - compatible spinors $\longrightarrow$ three common annihilators
  - define a SU(3) $\times$ SU(3) structure on $T \oplus T^*$

SU(3)	$\eta^1_+ = a\eta_+$	$\eta^2_+ = b\eta_+$	$\Phi_+ = \frac{a\bar{b}}{8} e^{-iJ}$ $\Phi_- = -i\frac{ab}{8}\Omega$
SU(2)	$\eta^1_+ = a\eta_+$	$\eta^2_+ = bz \cdot \eta_-$	$\Phi_+ = \frac{a\bar{b}}{8}\omega \wedge e^{z\wedge\bar{z}/2}$ $\Phi_- = -i\frac{ab}{8}e^{-ij} \wedge z$
degenerate SU(2)	$\eta^1_+ = a\eta_+$	$\eta^2_+ = c_1\eta_+ + c_2 z \cdot \eta_-$	$\Phi_+ = \frac{a}{8}(\bar{c}_1 e^{-ij} - i\bar{c}_2\omega) \wedge e^{z\wedge\bar{z}/2}$ $\Phi_- = -i\frac{a}{8}(\bar{c}_2 e^{-ij} + i\bar{c}_1\omega) \wedge z$

- Rewrite the SUSY variations as differential equations for the pure spinors on the internal manifold

  - IIA

$$(d - H\wedge)(e^{2A-\phi}\Phi_+) = 0$$

$$(d - H\wedge)(e^{2A-\phi}\Phi_-) = e^{2A-\phi} dA \wedge \bar{\Phi}_- + \frac{i}{8} e^{3A} * \lambda(F_A)$$

with $F_A = F_0 + F_2 + F_4 + F_6$ and $\lambda(F_n) = (-1)^{\mathrm{int}[n/2]} F_n$

  - IIB

$$(d - H\wedge)(e^{2A-\phi}\Phi_-) = 0$$

$$(d - H\wedge)(e^{2A-\phi}\Phi_+) = e^{2A-\phi} dA \wedge \bar{\Phi}_+ + \frac{i}{8} e^{3A} * \lambda(F_B)$$

with $F_B = F_1 + F_3 + F_5$

- General conditions for $\mathcal{N}=1$ flux vacua
  - topological condition $\longrightarrow$ SU(3) $\times$ SU(3) structure on $T \oplus T^*$

$$\Phi_+ = \eta^1_+ \otimes \eta^{2\,\dagger}_+ \qquad \Phi_- = \eta^1_+ \otimes \eta^{2\,\dagger}_-$$

  - integrability condition (SUSY)
    - one spinor is twisted closed

IIA/IIB $\quad (d - H\wedge)(e^{2A-\phi}\Phi_\pm) = 0 \longrightarrow$ twisted generalised Calabi Yau

    - the RR fields act as torsion

IIA/IIB $\quad (d - H\wedge)(e^{2A-\phi}\Phi_\mp) = e^{2A-\phi} dA \wedge \bar{\Phi}_\mp + \frac{i}{8} e^{3A} * \lambda(F_{A/B})$

	zero fluxes	fluxes
	T	$T \oplus T^*$
pure spinor	η_0	Φ
integrability	$\nabla_m \eta_0 = 0$	$\mathrm{d}\Phi = 0$
	Calabi Yau	Generalised Calabi Yau

EQUATIONS OF MOTION

- The flux e.o.m follow from the pure spinor equations
  - RR fluxes
    - split into real and imaginary part

$$(d - H\wedge)(e^{2A-\phi}\Phi_2) = e^{2A-\phi} dA \wedge \bar{\Phi}_2 + \frac{i}{8} e^{3A} * \lambda(F) \quad \begin{matrix} \nearrow & (d - H\wedge)(e^{A-\phi}\mathrm{Re}\Phi_2) = 0 \\ \searrow & (d - H\wedge)(e^{3A-\phi}\mathrm{Im}\Phi_2) = \frac{1}{8} e^{4A} * \lambda(F) \end{matrix}$$

    - commute λ and $*$

$$\lambda[(d - H\wedge)(e^{3A-\phi}\mathrm{Im}\Phi_2)] = \mp(d + H\wedge)(e^{3A-\phi}\lambda[\mathrm{Im}\Phi_2)] = \mp\frac{1}{8} e^{4A} * F$$

    - RR part is determined by metric and B field

  - Recently proved also for the NS flux [koerber, tsimpis 07]

BIANCHI IDENTITIES AND NO-GO THEOREM

- Bianchi identities
  - the scalar component $\Leftrightarrow$ no-go theorem

$$\int \langle (d - H\wedge)F,\ e^{3A-\phi}\mathrm{Im}\Phi_2 \rangle = \frac{1}{8}\int e^{4A}\,\langle F,\ *\lambda(F)\rangle < 0 \longleftarrow \text{TADPOLE}$$

    - need for negative charge sources $\longrightarrow$ O-planes
    - individual terms may correspond both to O-planes and D-branes
  - other components $\rightarrow$ conditions on the possible cycles wrapped by branes and orientifolds
- Orientifold action

O3/O7 and O6	$\rightarrow$	$\Omega_{\mathrm{WS}}(-)^{F_L}\sigma$	(IIA	$\sigma I = -I$)
O5/O9 and O4/O8	$\rightarrow$	$\Omega_{\mathrm{WS}}\sigma$	(IIB	$\sigma I = I$)

O3/O7	O5	O6
$\sigma(\Phi_+) = -\lambda(\bar{\Phi}_+)$	$\sigma(\Phi_+) = \lambda(\bar{\Phi}_+)$	$\sigma(\Phi_+) = -\lambda(\Phi_+)$
$\sigma(\Phi_-) = \lambda(\Phi_-)$	$\sigma(\Phi_-) = -\lambda(\Phi_-)$	$\sigma(\Phi_-) = \lambda(\bar{\Phi}_-)$

TWISTED TORI - NIL(SOLV)MANIFOLDS

- d-dimensional parallelisable manifolds $\rightarrow \exists\, d$ globally defined 1-forms

$$de^a = f^a_{bc} e^b \wedge e^c \qquad \text{(or dual vectors } [E_b, E_c] = f^a{}_{bc}\, E_a)$$

- Homogeneous spaces $\rightarrow f^a_{bc}$ are constant
- Structure constants of a real Lie algebra $\mathcal{G}$

$$d^2 e^a = 0 \quad \Rightarrow \quad f^a_{[bc} f^e_{d]a} = 0 \quad \text{(Jacobi identities)}$$

- Twisted identifications

$$M = \frac{G}{\Gamma} = \frac{\text{Lie Group}}{\text{discrete maximal subgroup}} \qquad G \text{ nilpotent (solvable)} \Rightarrow \text{Nil (Solv) manifold}$$

Ex 3d Heisenberg algebra $(0, 0, k \times 12)$

$$\left.\begin{aligned} de^1 &= 0 \\ de^2 &= 0 \\ de^3 &= k e^1 \wedge e^2 \end{aligned}\right\} \Rightarrow \begin{cases} e^1 = dx^1 \\ e^2 = dx^2 \\ e^3 = dx^3 + k x^1 \wedge dx^2 \\ (x^1, x^2, x^3) \sim (x^1, x^2 + a, x^3) \sim (x^1, x^2, x^3 + b) \sim (x^1 + c, x^2, x^3 - k c x^1) \end{cases}$$

- Complete classification
  - nilmanifold $\longrightarrow$ 34 algebras in 6d
  - solvmanifold $\longrightarrow$ 182 algebras in 6d
- Compactness
  look for M compact with discrete isotropy group ($\dim G = \dim M = 6$)
  - nilmanifold
    $f_{ba}^{a} = 0$ and $f_{bc}^{a} \in \mathrm{Z} \longrightarrow$ necessary and sufficient
  - solvmanifold
    $f_{ba}^{a} = 0$ and $f_{bc}^{a} \in \mathrm{Z} \longrightarrow$ necessary
    $\exists$ sufficiency criteria $\longrightarrow$ algebraic groups
    47 = 13 + 34 (nil) compact
- Curvature $R = -1/2 f_{abc} f_{abc}$
  - nilmanifold $\longrightarrow$ negative curvature
  - solvmanifold $\longrightarrow$ negative or zero curvature
- All nilmanifolds are Generalised Calabi Yau [cavalcanti, gualtieri 04]

TWISTED TORI AND $\mathcal{N}=1$ VACUA

- Without NS and RR fluxes nilmanifolds are
  - generalised Calabi-Yau
  - not solutions of the SUGRA equations of motion ($R<0$)

- Turn on fluxes and look for (at least) $\mathcal{N}=1$ solutions
  - consider SU(3) and SU(2) structure pair of pure spinors
  - global solution $\longrightarrow$ large volume limit
    $\longrightarrow$ left invariant forms
    $\longrightarrow$ algebraic equations
  - local solutions $\longrightarrow$ rescale local ones

$$e^m \longrightarrow e^{\sum_i (-1)^{sign_i(m)} A_i} e^m \qquad sign_i(m) = \begin{cases} +1 & \text{i-th source is along } e^m \\ -1 & \text{i-th source is orthogonal to } e^m \end{cases}$$

- Procedure to find global solutions
  - Determine the orientifold involutions compatible with a given algebra
  - Pure spinors
    - find a pure spinor Φ_1 $\longrightarrow$ $(d - H\wedge)\Phi_1 = 0$
      (always $\exists$ on nilmanifolds)
      $\longrightarrow$ compatible with the orientifold projection
    - find a pure spinor Φ_2 $\longrightarrow$ compatible with Φ_1
      (trivial structure group)
      (no obstructions in finding a compatible spinor)
      $\longrightarrow$ compatible with the orientifold projection
      $\longrightarrow$ $(d - H\wedge)\text{Re}\Phi_2 = 0$

    the two spinors determine the metric and the B-field
  - RR fluxes (tadpole)
    - compute F_{RR} $\longrightarrow$ $(d - H\wedge)\text{Im}\Phi_2 = *F_{\text{RR}}$
    - check Bianchi $\longrightarrow$ smeared sources

$$(d - H\wedge)F_{\text{RR}} = \sum_i Q_i(source)\text{vol}_\text{i}$$

SUMMARY OF SOLUTIONS

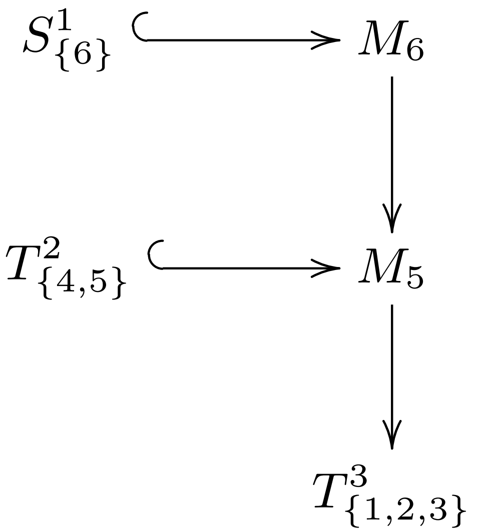
- Minkowski vacua
  - T-duals of conformal Calabi Yaus $\longrightarrow$ full solutions including the warp factor
  - Multi-source solutions $\longrightarrow$ delocalised solution (large volume only)

		IIA			IIB	
	algebras	O4	O6		O5	
		type 12	type 30	type 12	type 30	type 12
n	(0,0,0,12,13,23)		456			
n	(0,0,0,12,23,14-35)				$(45+26)^*$	
n	(0,0,0,0,12,14+23)				56	56
n	(0,0,0,0,12,34)				56	56
n	(0,0,0,0,12,13)				56	56
n	(0,0,0,0,13 + 42,14+23)				56	56
n	(0,0,0,0,0,12+34)	6			56	56
n	(0,0,0,0,0,0,12)	6				
s	(25,- 15,$\pm$45,$\mp$35,0,0)		$(136+246)^*$ $(146+236)^*$	$(136+246)^*$ $(146+236)^*$		$(13+24)^*$ $(14+23)^*$

- AdS vacua
  - IIA AdS_4 solutions $\longrightarrow$ generalised half flat manifolds

NON T-DUAL SOLUTION

- nilmanifold (0,0,0,12,23,14-35) with O5 in 45



- iterated torus fibration
- not T-dualisable to O3

  $\longrightarrow$ ∂_4, ∂_5 not isometries

- Pure spinors $\longrightarrow$ SU(3) structure

$$\Phi_- = -\frac{i}{8}\Omega \qquad \Omega = V^{1/2}(e^1 - ie^3) \wedge (e^2 + i\tau e^6) \wedge (e^4 + ie^5)$$

$$\Phi_+ = \frac{1}{8}e^{-iJ} \qquad J = (-t_1 e^1 \wedge e^3 + t_2 \tau_r e^2 \wedge e^6 + t_3 e^4 \wedge e^5)$$

- Fluxes $\longrightarrow$ only RR 3-form

$$F_3 = -(\tau_i e^2 - |\tau|^2 e^6) \wedge \left(t_2(e^1 \wedge e^4 - e^3 \wedge e^5) + \frac{t_3}{\tau_r}(e^1 \wedge e^5 + e^3 \wedge e^4) \right)$$

- Tadpole

$$dF_3 = -2\,|\tau|^2 \left(\frac{t_3}{\tau_r^2 t_1 t_2}\ \mathrm{vol1}^{1236} + \frac{t_2}{t_1 t_3}\ \mathrm{vol2}^{1345} \right)$$

$\rightarrow$ O5-planes along 45 and 26 (generated by $\{\Omega_{\mathrm{WS}}\,\sigma_1, \sigma_1\sigma_2\}$)

- Moduli (after the second O5 projection)

complex st. $\rightarrow \tau = \tau_r + i\tau_i$ Kähler $\rightarrow t_1, t_2, t_3$

  - positive definite metric $t_i > 0$
  - integrated Bianchi

$$t_1 = \frac{|\tau|^2}{16 g_s |\tau_r|}$$
$$t_2|\tau_r| = \frac{4\sqrt{g_s|\tau_r|}}{|\tau|} V^{1/2}$$
$$t_3 = \frac{4\sqrt{g_s|\tau_r|}}{|\tau|} V^{1/2}$$

can have $g_s << 1$, $V >> 1$ and all cycles large

- Complete solution with dilaton and warp factor is unknown $\rightarrow$ pb with intersecting sources

NON GEOMETRY AND COMPACTIFICATIONS

- Non geometrical backgrounds appear naturally in string compactifications $\to$ effective actions
  - T-duality invariant form of the IIA and IIB superpotentials [shelton, taylor, wecht 05]

$$H_{abc} \overset{T_a}{\longleftrightarrow} f^a_{bc} \overset{T_b}{\longleftrightarrow} Q_c^{ab} \overset{T_c}{\longleftrightarrow} R^{abc}$$

  - Gaugings in SU(3) $\times$ SU(3) effective actions $\to$ torsion and NS fluxes are not enough [grana, louis, waldram 06]
  - Lower dimensional supergravities with non abelian gauge groups (generalised Scherck-Schwarz) reduction [hull, dabholkar 02; hull 04;...]
- What about the 10 d interpretation?

THE 4D ALGEBRA

- 4d effective actions

$\longrightarrow$ the generalised fluxes appear in the gauge algebra of the vector fields

$$[v_a, v_b] = H_{abc}X^c + f^c{}_{ab}v_c$$
$$[v_a, X^b] = -f^b{}_{ac}X^c + Q^{bc}{}_a v_c$$
$$[X^a, X^b] = Q^{ab}{}_c X^c + R^{abc}v_c$$

$v_a \longrightarrow$ isometries

$X^a \longrightarrow$ B-field gauge transf.

- 10d perspective

$\longrightarrow$ Courant bracket algebra of the section of the generalised tangent bundle E

$\longrightarrow$ for parallelisable manifolds
  - T-duality is a symmetry of the sections of E
  - geometry and non geometry are in the relation to $T \oplus T^*$

A TOY MODEL: 3-TORUS WITH NS FLUX

- T^3 with a k units of NS flux

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 \qquad H = k dx^1 \wedge dx^2 \wedge dx^3 \qquad B = k x^1 \wedge dx^2 \wedge dx^3$$

- Generalised Tangent Bundle of M (extension of T by T^*)
  - the fibers are $T \oplus T^*$ with transition functions in $GL(R,d) \times \Omega^{2,cl}$

$$X = x + \hat{\xi} \quad \text{and on } U_{\alpha\beta} \begin{cases} B_\beta = B_\alpha + dA_{\alpha\beta} \\ x_\alpha + \hat{\xi}_\alpha = x_\beta + \hat{\xi}_\beta - i_X dA_{\alpha\beta} \end{cases}$$

  - relation to $T \oplus T^*$

$$(v+\xi)|_{U_\alpha} = v_\alpha + (\hat{\xi}_\alpha + i_v B_\alpha).$$

  - a basis for the sections

$$(\{E_i\};\{E^i\}) = (\partial_1, \partial_2 + kx^1 dx^3, \partial_3 - kx^1 dx^2; dx^1, dx^2, dx^3)\,.$$

- The Courant bracket yields the algebra $H_{ijk} = k\epsilon_{ijk}$

$$\begin{cases} [E_i, E_j] = H_{ijk} E^k \\ [E_i, E^j] = 0 \\ [E^i, E^j] = 0 \end{cases}$$

- Perform T-dualities $\rightarrow O(2,2) \subset O(3,3)$ symmetries of the sections of E
  - T-duality in x^3 ($\partial_3 \leftrightarrow dx^3$)
    - new basis for E

$$(\{\tilde{E}_i\};\{\tilde{E}^i\}) = (\partial_1, \partial_2 + kx^1\partial_3, \partial_3; dx^1, dx^2, dx^3 - kx^1 dx^2)$$

    - Courant bracket $\rightarrow$ Heisenberg algebra ($(0,0,k12)$)

$$\begin{aligned} &[E_i, E_j] = f_{ij}^k E_k \\ &[E_i, E^j] = -f_{ik}^j E^k \qquad \text{with } f_{12}^3 = k \\ &[E^i, E^j] = 0 \end{aligned}$$

    - all basis elements are well defined sections of $T \oplus T^*$

$\rightarrow$ the reduction to $T \oplus T^*$ requires $B = 0$

$\rightarrow$ nilmanifold (twisted torus)

$$\mathrm{d}s^2 = (dx^1)^2 + (dx^2)^2 + (dx^3 - kx^1 dx^2)^2 \qquad f_{yz}^x = k$$

T^2 fibration over S^1 $\rightarrow$ coordinates are patched with an $SL(2,Z)$ transformation

$(x^1, x^2, x^3) \sim (x^1, x^2 + 1, x^3) \sim (x^1, x^2, x^3 + 1) \sim (x^1 + c, x^2, x^3 - kx^2)$

- T-duality in x^2 ($\partial_2 \leftrightarrow dx^2$)
  - new basis

$$(\{\tilde{E}_i\};\{\tilde{E}^i\}) = (\partial_1, \partial_2, \partial_3; dx^1, dx^2 + kx^1\partial_3, dx^3 - kx^1\partial_2)$$

  - Courant bracket

$$\begin{aligned}
[E_i, E_j] &= 0 \\
[E_i, E^j] &= -Q_i{}^{jk} E_k \qquad \text{with } Q_3^{12} = k \\
[E^i, E^j] &= Q_k{}^{ij} E^k
\end{aligned}$$

  - the 1-forms are well defined but not the vectors

$\rightarrow$ no isomorphism with $T \oplus T^*$

$\rightarrow$ new isomorphism via bi-vector (compatible with Buscher)

$$(v + \xi)|_\alpha = (x_\alpha + \xi \lrcorner \beta_\alpha) + \hat{\xi}$$

but no gluing conditions

$\rightarrow$ not a manifold for $x^1 \sim x^1 + 1$ (non geometrical)

$$\mathrm{d}s^2 = \frac{1}{1+(kx^1)^2}((dx^3)^2 + (dx^2)^2) + (dx^1)^2 \qquad \beta^{23} = \frac{kx^1}{1+(kx^1)^2}$$

coordinates are patched with an $O(2,2)$ transformation (T-fold) [hull 04]

CONCLUSIONS

- Generalised Complex Geometry is good framework to describe flux backgrounds
  - geometric characterisation
  - interpretation of non geometrical backgrounds
  - explicit construction of supersymmetric backgrounds

- Search for Minkowski vacua on twisted tori
  - fewer (new) vacua found than naively expected $\rightarrow$ 1 nilmanifold and 4 models on the same solmanifold
  - all have intersecting sources and some moduli unfixed

- May also consider other options
  - non-algebraic groups
  - higher-dimensional algebras and continuous lattices
  - non-geometries