Type I and II supersymmetric backgrounds

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Based on

Type I

U Gran, P Lohrmann, GP, hep-th/0602250
U Gran, D Roest, P Sloane, GP, hep-th/0703143
U Gran, D Roest, GP, to appear

Type IIB

U Gran, J Gutowski, GP, hep-th/0501177, hep-th/0505074, arXiv:0705.2208
U Gran, J Gutowski, D Roest, GP, hep-th/0606049, and to appear

Outline

Killing spinor equations

A parallel transport equation for the supercovariant connection $\mathcal{D}$

$$\delta\psi_A| = \mathcal{D}_A\epsilon = \nabla_A\epsilon + \Sigma_A(e, F)\epsilon = 0$$

and possibly algebraic equations

$$\delta\lambda| = \mathcal{A}(e, F)\epsilon = 0$$

where ∇ is the Levi-Civita connection, $\Sigma(e, F)$ a Clifford algebra element

$$\Sigma(e, F) = \sum_k \Sigma_{[k]}(e, F)\Gamma^{[k]}$$

e frame and F fluxes, ϵ spinor, Γ gamma matrices.

Can the KSE be solved without any assumptions on the metric and fluxes?

Holonomy

For generic D=11 [Hull, Duff, Liu] and IIB [Tsimpis, GP] backgrounds

$$\mathrm{hol}(\mathcal{D}) \subseteq SL(32, \mathbb{R})$$

because $\mathcal{R} = [\mathcal{D}, \mathcal{D}]$ takes values in $\mathfrak{sl}(32, \mathbb{R})$.

For backgrounds with L parallel and N Killing spinors, $N \leq L$,

$$\begin{aligned} \mathrm{hol}(\mathcal{D}) &\subseteq SL(32-L, \mathbb{R}) \ltimes \oplus_L \mathbb{R}^{32-L} \\ &= \mathrm{Stab}(\epsilon_1, \dots, \epsilon_L) \subset SL(32, \mathbb{R}) \end{aligned}$$

i.e. the isotropy group $\mathrm{Stab}(\epsilon_1, \dots, \epsilon_L)$ of any number of spinors in $SL(32, \mathbb{R})$ is non-trivial.
A consequence is

- There may be backgrounds with number L of parallel and so Killing spinors, however see preons ($N = 31$)

Gauge symmetry

The gauge symmetry G of the KSE are the (local) transformations such that

$$g^{-1}\mathcal{D}(e,F)g = \mathcal{D}(e^g, F^g)\ , \quad g^{-1}\mathcal{A}(e,F)g = \mathcal{A}(e^g, F^g)$$

i.e. preserve the form of the Killing spinor equations.

D=11 SUGRA: $G = Spin(10,1)$

IIB SUGRA: $G = Spin(9,1) \times U(1)$

Type I: $G = Spin(9,1)$

- Backgrounds related by a gauge transformation are identified
- In type II theories $G \subset\subset SL(32,\mathbb{R})$. In particular, 2 generic spinors ϵ_1, ϵ_2 in D=11 and IIB have isotropy group $\mathrm{stab}(\epsilon_1,\epsilon_2)$ in G, $\mathrm{stab}(\epsilon_1,\epsilon_2) = \{1\}$
- In type I, the gauge and the holonomy groups coincide, $Spin(9,1)$

Holonomy vs gauge symmetry

Isotropy groups of one spinor in $SL(32,\mathbb{R})$ and in the gauge group G [($D=11$)Bryant, J Figueroa O'Farrill; (IIB) U Gran, J Gutowski, GP].

$N=1$	Stab	stab
$D=11$	$SL(31,\mathbb{R})\ltimes\mathbb{R}^{31}$	$SU(5)$, $Spin(7)\ltimes\mathbb{R}^9$
IIB	$SL(31,\mathbb{R})\ltimes\mathbb{R}^{31}$	$Spin(7)\ltimes\mathbb{R}^8$, $SU(4)\ltimes\mathbb{R}^8$, G_2
TypeI	$Spin(7)\ltimes\mathbb{R}^8$	$Spin(7)\ltimes\mathbb{R}^8$

Isotropy groups for 31 spinors in $SL(32,\mathbb{R})$ and in the gauge group G.

$N=31$	Stab	stab
$D=11$	$\mathbb{R}^{31}$	$\{1\}$
IIB	$\mathbb{R}^{31}$	$\{1\}$

- Since the gauge group is much smaller that the holonomy group there are several geometric types for the same number of Killing spinors
- Identifying backgrounds with larger gauge groups may lead to significant simplification

Spinorial geometry

The ingredients of the spinorial method to solve the supergravity KSE [J Gillard, U Gran, GP] are

- Gauge symmetry of KSE
  It is used to choose the Killing spinor directions or their normals. Very effective for backgrounds with small and large number of susies
- Spinors in terms of forms
  Convenient notation
- An oscillator basis in the space of Dirac spinors
  Allows to extract the geometric information using the linearity of KSE

Type I; Gravitino

The gravitino Killing spinor equation is

$$\mathcal{D}\epsilon = \hat{\nabla}\epsilon = \nabla\epsilon + \frac{1}{2}H\epsilon = 0$$

where $\hat{\nabla}$ is a metric connection with skew-symmetric torsion H, and so for generic backgrounds

$$\text{hol}(\hat{\nabla}) = G = Spin(9,1)$$

In addition

$$\hat{\nabla}\epsilon = 0 \Rightarrow \hat{R}\epsilon = 0$$

So either

$$\text{Stab}(\epsilon) = \{1\} \Longrightarrow \hat{R} = 0$$

all spinors are parallel and M is parallelizable (group manifold if $dH = 0$) [Figueroa O'Farrill, Kawano, Yamaguchi] or

$$\text{Stab}(\epsilon) \neq \{1\} \Longrightarrow \epsilon \text{ singlets}$$

$\text{Stab}(\epsilon) \subset Spin(9,1)$ and $\text{hol}(\hat{\nabla}) \subseteq \text{Stab}(\epsilon)$.

Parallel spinors

L	$Stab(\epsilon_1, \dots, \epsilon_L)$	parallel ϵ
1	$Spin(7) \ltimes \mathbb{R}^8$	$1 + e_{1234}$
2	$SU(4) \ltimes \mathbb{R}^8$	1
3	$Sp(2) \ltimes \mathbb{R}^8$	$1, \ i(e_{12} + e_{34})$
4	$(SU(2) \times SU(2)) \ltimes \mathbb{R}^8$	$1, \ e_{12}$
5	$SU(2) \ltimes \mathbb{R}^8$	$1, \ e_{12}, \ e_{13} + e_{24}$
6	$U(1) \ltimes \mathbb{R}^8$	$1, \ e_{12}, \ e_{13}$
8	$\mathbb{R}^8$	$1, \ e_{12}, \ e_{13}, \ e_{14}$
2	G_2	$1 + e_{1234}, \ e_{15} + e_{2345}$
4	$SU(3)$	$1, \ e_{15}$
8	$SU(2)$	$1, \ e_{12}, \ e_{15}, \ e_{25}$
16	$\{1\}$	$\Delta^+_{\mathbf{16}}$

- There are differences with the holonomy groups that appear in the Berger classification
- There are compact and non-compact isotropy groups which lead to geometries with different properties
- There is a restriction on the number of parallel spinors. This is a difference with the type II case
- The isotropy group of more than 8 spinors is $\{1\}$

- Identify the Majorana-Weyl representation $\Delta^+_{\mathbf{16}}$ of $Spin(9,1)$ with a real subspace of $\Lambda^{\mathrm{ev}}(\mathbb{C}^5)$ with respect to the reality condition $\mathcal{R} = \Gamma_{6789}*$. For example, the spinor 1 is complex Weyl and the Majorana components are

$$1 + e_{1234}\ , \quad i(1 - e_{1234})$$

where $e_{1234} = e_1 \wedge e_2 \wedge e_3 \wedge e_4$ and e_i hermitian basis in $\mathbb{C}^5$.

Dilatino

The dilatino KSE is

$$d\Phi\zeta - \frac{1}{2}H\zeta = 0$$

Some of the parallel spinors $\epsilon_1, \dots, \epsilon_L$ may not solve the dilatino KSE. To choose the solutions $\zeta = \sum_r f_r \epsilon_r$ of the dilatino KSE use as gauge symmetry transformations,

$$\Sigma(\mathcal{P}) = \mathrm{Stab}(\mathcal{P})/\mathrm{Stab}(\epsilon_1, \dots, \epsilon_L)$$

where $\mathcal{P}$ is the L-plane of parallel spinors and $\mathrm{Stab}(\mathcal{P})$ are those transformations of $Spin(9,1)$ that preserve $\mathcal{P}$.

- Using the $\Sigma(\mathcal{P})$ groups the Killing spinors can be chosen in all cases. Moreover the dilatino Killing spinor can be solved.
- If $N > L/2$, it is convenient to use $\Sigma(\mathcal{P})$ to choose the normals to the Killing spinors.

L	$Stab(\epsilon_1, \ldots, \epsilon_L)$	$\Sigma(\mathcal{P})$
1	$Spin(7) \ltimes \mathbb{R}^8$	$Spin(1,1)$
2	$SU(4) \ltimes \mathbb{R}^8$	$Spin(1,1) \times U(1)$
3	$Sp(2) \ltimes \mathbb{R}^8$	$Spin(1,1) \times SU(2)$
4	$(SU(2) \times SU(2)) \ltimes \mathbb{R}^8$	$Spin(1,1) \times Sp(1) \times Sp(1)$
5	$SU(2) \ltimes \mathbb{R}^8$	$Spin(1,1) \times Sp(2)$
6	$U(1) \ltimes \mathbb{R}^8$	$Spin(1,1) \times SU(4)$
8	$\mathbb{R}^8$	$Spin(1,1) \times Spin(8)$
2	G_2	$Spin(2,1)$
4	$SU(3)$	$Spin(3,1) \times U(1)$
8	$SU(2)$	$Spin(5,1) \times SU(2)$
16	$\{1\}$	$Spin(9,1)$

- The $\Sigma(\mathcal{P})$ groups are a product of a *Spin* group and a R-symmetry group, reminiscent of lower-dimensional supergravities.

L	$\Sigma(\mathcal{P})$	N
1	$Spin(1,1)$	$1(1)$
2	$Spin(1,1)\times U(1)$	$1(1),\ 2(1)$
3	$Spin(1,1)\times SU(2)$	$1(1),\ 2(1),\ 3(1)$
4	$Spin(1,1)\times Sp(1)\times Sp(1)$	$1(1),\ 2(1),\ 3(1),\ 4(1)$
5	$Spin(1,1)\times Sp(2)$	$1(1),\ 2(1),\ 3(1),\ 4(1),\ 5(1)$
6	$Spin(1,1)\times SU(4)$	$1(1),\ 2(1),\ 3(1),\ 4(1),\ 5(1),\ 6(1)$
8	$Spin(1,1)\times Spin(8)$	$1(1),\ 2(1),\ 3(1),\ 4(1),\ 5(1),\ 6(1),\ 7(1),\ 8(1)$
2	$Spin(2,1)$	$1(1),\ 2(1)$
4	$Spin(3,1)\times U(1)$	$1(1),\ 2(2),\ 3(2),\ 4(1)$
8	$Spin(5,1)\times SU(2)$	$1(1),\ 2(2),\ 3(3),\ 4(6),\ 5(3),\ 6(2),\ 7(1),\ 8(1)$
16	$Spin(9,1)$	$8(2),\ 10(1),\ 12(1),\ 14(1),\ 16(1)$

- The cases noted in red are those for which all parallel spinors are Killing $N=L$, and the case in blue does not occur
- The number in parenthesis denotes the different geometries for a given N

Geometry

The number of $\hat{\nabla}$-parallel 1-forms depends on the topology of Stab(ϵ).

Stab	$1-$ forms	algebras
$K \ltimes \mathbb{R}^8$	e^-	$\mathbb{R}$
G_2	e^0, e^1, e^5	$\mathbb{R} \oplus^2 \mathfrak{u}(1),\ \mathfrak{sl}(2,\mathbb{R})$
$SU(3)$	e^0, e^1, e^5, e^6	$\mathbb{R} \oplus^3 \mathfrak{u}(1),\ \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{u}(1), \mathbb{R} \oplus \mathfrak{su}(2),\ \mathfrak{cw}_4$
$SU(2)$	$e^0, e^1, e^2, e^5, e^6, e^7$	$\mathbb{R} \oplus^5 \mathfrak{u}(1),\ \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2),\ \mathfrak{cw}_6$

- If Stab $= K \ltimes \mathbb{R}^8$ is non-compact, then the spacetime admits CR-type of structures ($L > 1$) which depend on K. Most of H is determined in terms of the geometry.
- If Stab $= K$ is compact, then the spacetime is a principal bundle equipped with a connection λ. The fibre is spanned by the orbits of the vector fields and base space a manifold admits a K-type of structure compatible with a connection with skew-symmetric torsion. All of H is determined in terms of the geometry.

Holonomy Reduction

Consider $SU(4) \ltimes \mathbb{R}^8$. Since $\mathrm{hol}(\hat{\nabla}) \subseteq SU(4) \ltimes \mathbb{R}^8$, the expected $\hat{\nabla}$-parallel forms are

$$e^- \,, \quad e^- \wedge \omega_I \,, \quad e^- \wedge \mathrm{Re}\,\chi \,, \quad e^- \wedge \mathrm{Im}\,\chi$$

However, the field equations,

$$dH = 0 \,, \quad \mathrm{hol}(\hat{\nabla}) \subseteq SU(4) \ltimes \mathbb{R}^8$$

imply that

$$\tau_1 = H_{+ij}\omega_I^{ij} e^+ \,, \quad \tau_2 = \partial_+ \Phi e^+ \,,$$
$$\tau_3 = \mathcal{N} \,, \quad \tau_4 = 2d\Phi - \theta_{\omega_I} \,,$$

which do not vanish for $N < L$, are ALSO $\hat{\nabla}$-parallel. The consequences for $K \ltimes \mathbb{R}^8$ cases are

- The existence of $N < L$ supersymmetric backgrounds requires that $\mathrm{hol}(\hat{\nabla}) \subset \mathrm{Stab}(\epsilon)$.
- If $\mathrm{hol}(\hat{\nabla}) = \mathrm{Stab}(\epsilon)$, then the gravitino KSE implies the dilatino one and ALL parallel are Killing $L = N$, *i.e.* there are no $N < L$ backgrounds

Status in IIB

The Killing spinor equations of IIB supergravity [Schwarz, West, Howe] have been solved in the following cases

- $N = 1$: Four different types of geometry. The Killing spinors are represented by orbits with isotropy groups $Spin(7) \ltimes \mathbb{R}^8$, $SU(4) \ltimes \mathbb{R}^8$ and G_2 [U Gran, J Gutowski, GP]
- $N = 31$: Supersymmetry enhances to $N = 32$ [U Gran, J Gutowski, D Roest, GP]
- $N = 32$: The maximally supersymmetric backgrounds are $\mathbb{R}^{9,1}$, $AdS_5 \times S^5$ [Schwarz], and the plane wave [Blau, J Figueroa O'Farrill, Hull, GP]. This list is complete [J Figueroa O'Farrill, GP].
- $N > 24$ backgrounds are homogeneous [J Figueroa O'Farrill, E. Hackett-Jones, G Moutsopoulos]
- Early indications suggest that $N > 28$ backgrounds are maximally supersymmetric [U Gran, J Gutowski, D Roest, GP]

Much more progress has been made under additional assumptions like choosing certain invariant Killing spinors or assuming that the spacetime is invariant under certain isometries.

$N = 31$ is not IIB

The 31 Killing spinors span a hyperplane and have a unique normal ν.
The gauge symmetry can be used to choose the normal ν as

stab(ν)	ν
$Spin(7) \ltimes \mathbb{R}^8$	$(f + ig)(e_5 + e_{12345})$
$SU(4) \ltimes \mathbb{R}^8$	$(f + ig)e_5 + (h + ik)e_{12345}$
G_2	$f(e_5 + e_{12345}) + if(e_1 + e_{234})$

Choose the Killing spinors orthogonal to ν. Then

$$\mathcal{A}\epsilon_r = 0 \,, \quad r = 1, \ldots 31$$

implies that

$$P = G = 0$$

The remaining KSE are linear over the complex numbers and so the number of Killing spinors preserved is even. So there are no $N = 31$ backgrounds in IIB (preons).

- There are no IIA preons [Bandos, Azcarraga, Varela]
- There are no $D = 11$ preons [U Gran, J Gutowski, D Roest, GP] and [J Figueroa O'Farrill, S Gadhia]

$N = 2$ with $P = G = 0$

This is a special case of $N = 1$ because if $P = G = 0$, then IIB backgrounds preserve even number of supersymmetries. There are four different types of geometry that can arise [U Gran, J Gutowski, GP].

stab(ϵ)	ϵ	Vectors/Forms
$Spin(7) \ltimes \mathbb{R}^8$	$(f + ig)(1 + e_{1234})$	e^-, $\nabla_A e^- = 0$
$SU(4) \ltimes \mathbb{R}^8$ (generic)	$(f + ig)1 + (h + ik)e_{1234}$	e^- Killing $e^- \wedge de^- = 0$
$SU(4) \ltimes \mathbb{R}^8$ (pure)	$f1$	e^- Killing $e^- \wedge de^- \neq 0$
G_2	$f(1 + e_{1234}) + if(e_{51} + e_{5234})$	$f^2 e^0$ Killing, $d(f^2 e^1) = d(f^2 e^5) = 0$

Moreover the transverse geometries to the lightcone for the $K \ltimes \mathbb{R}^8$ cases and the three-directions in the G_2 case are

$stab(\epsilon)$	Transverse structure	Conditions
$Spin(7) \ltimes \mathbb{R}^8$	Holonomy $Spin(7)$	$\tilde{d}\phi = 0$
$SU(4) \ltimes \mathbb{R}^8$ (generic)	Almost hermitian $SU(4)$	$W_1,\ W_2,\ W_3,\ W_4,\ W_5 \neq 0$ determined by Killing spinor
$SU(4) \ltimes \mathbb{R}^8$ (pure)	Relatively balanced $SU(4)$ Hermitian	$\tilde{d}\omega^3 = W_4 \wedge \omega^3,\ W_1 = W_2 = 0$ $W_4 = W_5$
G_2	co − symplectic G_2	$\tilde{d} \star \varphi = 0$

- AdS_5 black holes [Gutowski, Reall] uplift to pure $SU(4) \ltimes \mathbb{R}^8$ geometries
- The LLM [Lin, Lunin, Maldacena] solution admits a co-cymplectic G_2 structure

Conclusions

- The Killing spinor equations of type I supergravities have been solved in ALL cases. The conditions that are imposed on the geometry of these backgrounds by supersymmetry have been found.
- Much progress has been done in IIB. The Killing spinor equations have been solved for $N = 1$ and $N = 2$ ($P = G = 0$), and the geometry of the backgrounds has been understood.
- IIB $N = 31$ backgrounds are maximally supersymmetric. The $N = 32$ backgrounds have been classified. The $N > 24$ backgrounds are homogeneous, and it is likely that the $N > 28$ backgrounds are maximally supersymmetric.

Type I susy backgrounds

L	$\mathrm{S}tab(\epsilon_1, \ldots, \epsilon_L)$	$\Sigma(\mathcal{P})$
1	$Spin(7) \ltimes \mathbb{R}^8$	$Spin(1,1)$
2	$SU(4) \ltimes \mathbb{R}^8$	$Spin(1,1) \times U(1)$
3	$Sp(2) \ltimes \mathbb{R}^8$	$Spin(1,1) \times SU(2)$
4	$(SU(2) \times SU(2)) \ltimes \mathbb{R}^8$	$Spin(1,1) \times Sp(1) \times Sp(1)$
5	$SU(2) \ltimes \mathbb{R}^8$	$Spin(1,1) \times Sp(2)$
6	$U(1) \ltimes \mathbb{R}^8$	$Spin(1,1) \times SU(4)$
8	$\mathbb{R}^8$	$Spin(1,1) \times Spin(8)$
2	G_2	$Spin(2,1)$
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8	$SU(2)$	$Spin(5,1) \times SU(2)$
16	$\{1\}$	$Spin(9,1)$