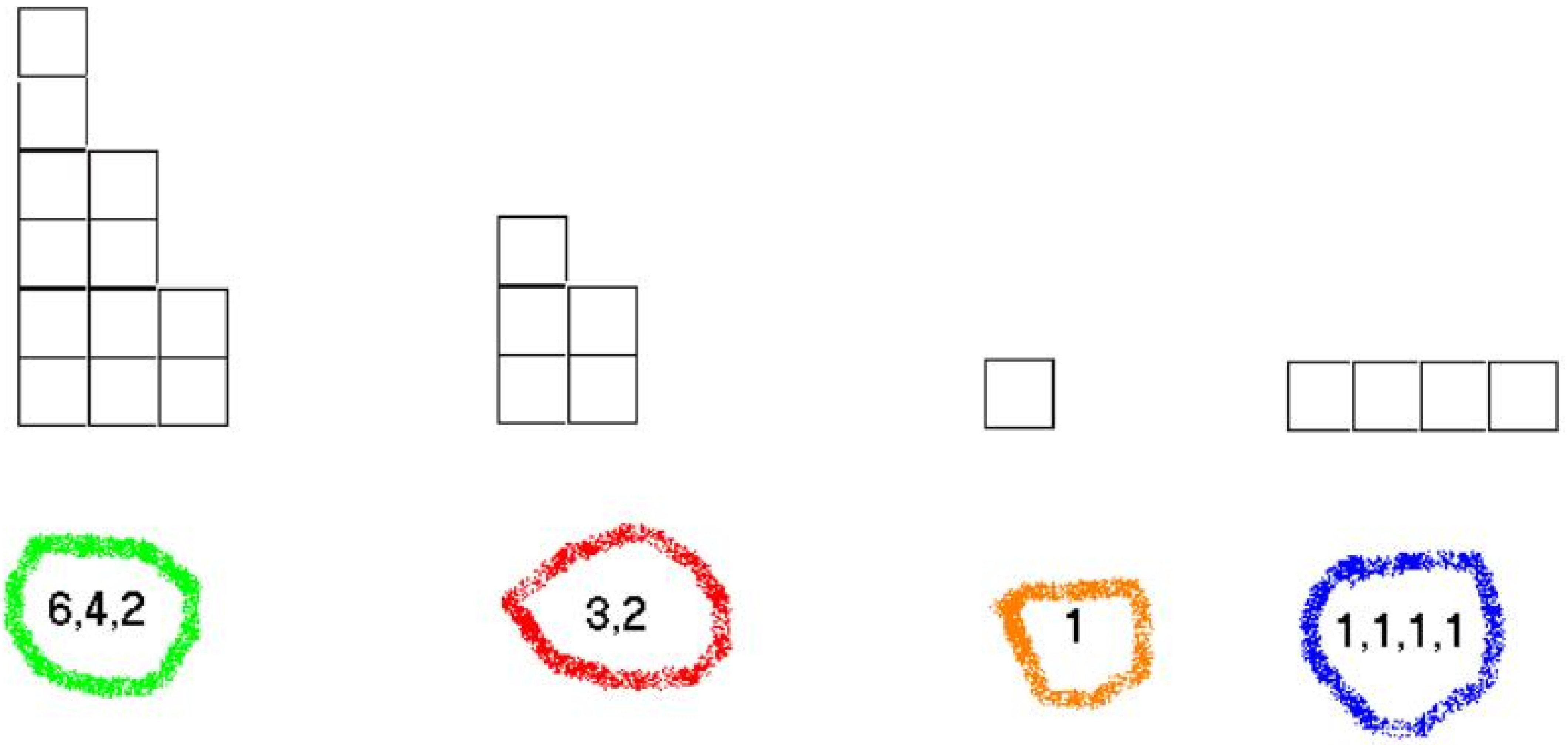
SEIBERG–WITTEN PREPOTENTIAL

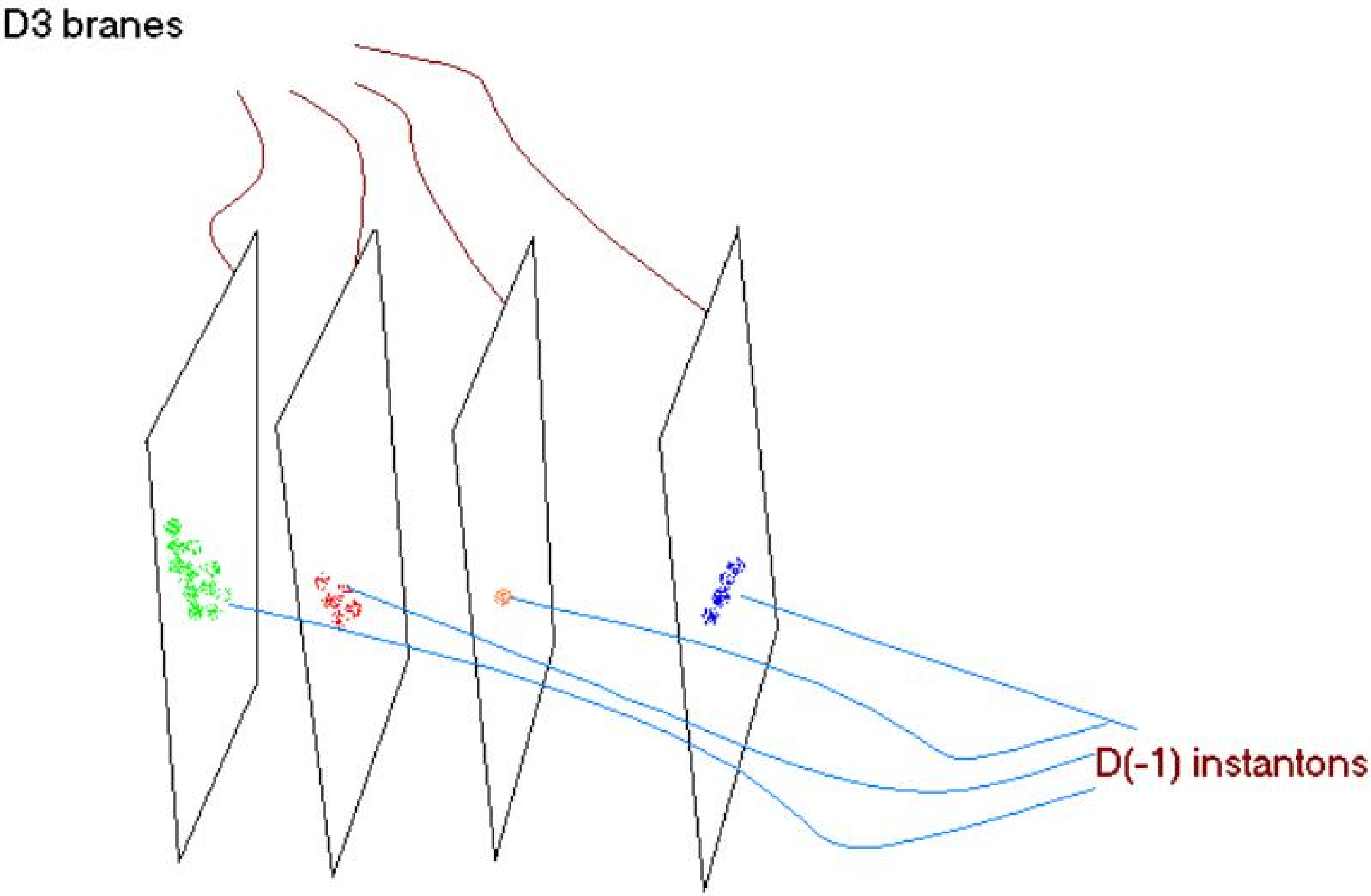
from instanton counting

STRINGS'02

NIKITA NEKRASOV



Partition of Young diagrams contributing to the prepotential
for gauge group SU(4), instanton charge 22



Fixed points on the instanton moduli space

D-brane picture

Four dimensional
$\mathcal{N} = 2$ super Yang-Mills theory
Gauge group $G = U(N)$

Field content:

Vector multiplet: $(A_\mu, \psi^i_\alpha, \psi^i_{\dot\alpha}, \phi, \bar\phi)$ – all in the adjoint of G.

Symmetries:

Lorentz $SU(2)_L \times SU(2)_R \times$R-symmetry $SU(2)_I$.

Supersymmetries:

$Q^{i\alpha}, Q^{i\dot\alpha}$

Twisted notations

Drop the full $SU(2)^3$ symmetry – keep only

$$SU(2)_L \times SU(2)_d,$$

$SU(2)_d$ –diagonal in $SU(2)_R \times SU(2)_I$.

Bosons unchanged,

Fermions: $\psi_\mu, \chi^+_{\mu\nu}, \eta$;

Superspace: $\theta^\mu, \bar{\theta}^+_{\mu\nu}, \bar{\theta}$

Superfield: $\Phi = \phi + \theta^\mu \psi_\mu + \frac{1}{2}\theta^\mu\theta^\nu F^-_{\mu\nu} + \dots$

Susys: $Q, Q^+_{\mu\nu}, G_\mu$:

$$\{Q, G_\mu\} = \partial_\mu$$

$$\{Q^+_{\mu\nu}, G_\lambda\} = \tfrac{1}{2}\left[\epsilon_{\mu\nu\lambda\kappa}\partial_\kappa + \delta_{\mu\lambda}\partial_\nu - \delta_{\nu\lambda}\partial_\mu\right]$$

Action:

$$S = \int_{\mathbb{R}^4} \frac{\tau_0}{2\pi i} \mathrm{Tr} F\wedge F + \{Q, \mathrm{Im}\tau_0 \chi F^+ + \psi \star d_A \bar\phi + \eta[\phi, \bar\phi]\}$$

$$= \int d^4x d^4\theta \frac{\tau_0}{2\pi i} \mathrm{Tr}\Phi^2 + \{Q, \dots\}$$

Bare coupling:

$$\tau_0 = \frac{\theta_0}{2\pi} + \frac{4\pi i}{e_0^2}$$

We want to determine The low-energy effective action

ϕ VEV breaks G to $T \Longrightarrow$ massless T vector multiplet

$$\mathcal{A}_l = a_l + \theta^\mu \psi_{\mu,l} + \theta^\mu \theta^\nu (dA_l)^-_{\mu\nu} + \ldots$$

$$l = 1, \ldots N, \qquad \sum_l \mathcal{A}_l = 0$$

Low-energy effective action is $\mathcal{N} = 2$ susy, and derives from a prepotential:

$$S_{low-energy} = \int d^4 x d^4 \theta \ \mathcal{F}(\mathcal{A}; \Lambda) \ + \ c.c =$$

$$\int_{\mathbb{R}^4} \tau_{lm} F_l^- \wedge F_m^- + \bar{\tau}_{lm} F_l^+ \wedge F_m^+ + \mathrm{Im}\tau_{lm} da_l \star d\bar{a}_m$$

$$+fermions$$

where $\tau_{lm} = \frac{\partial^2 \mathcal{F}}{\partial a_l \partial a_m}$

and $\Lambda \sim m e^{\frac{2\pi i \tau_0}{2N}}$ is dynamically generated scale.

The prepotential $\mathcal{F}$ has perturbative one-loop; and nonperturbative instanton corrections:

$$\mathcal{F} = \mathcal{F}^{pert} + \mathcal{F}^{inst}$$

$$\mathcal{F}^{pert} = -\frac{1}{8\pi i} \sum_{l \neq m} (a_l - a_m)^2 \log \left(\frac{a_l - a_m}{\Lambda} \right)^2$$

$$\mathcal{F}^{inst} = \sum_{k=1}^{\infty} \Lambda^{2Nk} \mathcal{F}_k$$

which is what we are after.

Idea of the calculation
a short-cut, really

Fix a translationally invariant symplectic form ω on $\mathbb{R}^4$:

$$\omega = dx^1 \wedge dx^2 + dx^3 \wedge dx^4$$

The choice of ω defines a complex structure on $\mathbb{R}^4$, thus identifying it with $\mathbb{C}^2$ with complex coordinates z_1, z_2 given by: $z_1 = x^1 + ix^2$, $z_2 = x^3 + ix^4$:

$$\omega = \frac{i}{2}\left(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2\right)$$

The form ω is invariant under the action of the group $U(2) \subset SO(4)$ of rotations.

The action of the maximal torus $\mathbf{T}^2 \in U(2)$ is generated by the Hamiltonian

$$H = \epsilon_1 |z_1|^2 + \epsilon_2 |z_2|^2$$

Correlation function of our interest:

$$Z(a,\epsilon) = \langle \exp \frac{1}{(2\pi i)^2} \int_{\mathbb{R}^4} \mathcal{O} \rangle_a$$

$$\mathcal{O} = \omega \wedge \mathrm{Tr}\left(\phi F + \tfrac{1}{2}\psi\psi\right) - H\,\mathrm{Tr}\left(F \wedge F\right)$$

$\langle \ldots \rangle_a$ is the vev in the a-vac:

$\langle \phi \rangle \sim a \in \mathbf{t}$. More precisely, a will be the central charge of $\mathcal{N} = 2$ algebra corresponding to the W-boson states.

our observable suppresses the widely separated instantons

$\Longrightarrow$ solves the problem of noncompactness of point-like instanton moduli space, anticipated

in [LosevNekrasovShatashvili'97'98].

One can expand $Z(a,\epsilon)$ as a sum over different instanton sectors:

$$Z(a,\epsilon) = \sum_{k=0}^{\infty} \Lambda^{2kN}\ Z_k(a,\epsilon)$$

Idea of the UV evaluation

Using the anomaly in the $U(1)$ R-symmetry, together with various susy arguments one can show that the expectation value of our observable can be expressed as follows:

$$Z_k(a,\epsilon) = \frac{1}{(2kN)!} \int_{\mathcal{M}_{k,N}} \Omega^{\wedge 2kN} \; e^{-\mathbf{H}}$$

i.e. as an integral over the moduli space $\mathcal{M}_{k,N}$ of $U(N)$ instantons on $\mathbb{R}^4$ of charge k;

the measure is defined with the help of the symplectic form on $\mathcal{M}_{k,N}$ which is inherited from ω on $\mathbb{R}^4$:

$$\Omega = \int_{\mathbb{R}^4} \omega \wedge \mathrm{Tr}\delta A \wedge \delta A \ ;$$

and $\mathbf{H}$ is the Hamiltonian

$$\mathbf{H} = \epsilon_1 H_1 + \epsilon_2 H_2 + \sum_l a_l \mathbf{h}_l$$

generating the $U(1)^N \times \mathbf{T}^2$ action on $\mathcal{M}_{k,N}$:

$$e^{i\tau\mathbf{H}} : \qquad A^{lm}_{\mathbf{i}}(z_1, z_2, \bar{z}_1, \bar{z}_2) \mapsto$$

$$e^{i\tau(a_l - a_m + \epsilon_{\mathbf{i}})} A^{lm}_{\mathbf{i}}(e^{i\tau\epsilon_1} z_1, e^{i\tau\epsilon_2} z_2, e^{-i\tau\epsilon_1} \bar{z}_1, e^{-i\tau\epsilon_2} \bar{z}_2)$$

Duistermaat-Heckman

formula helps to evaluate the integral like the one we have:

$$\int_{X^{2d}} e^{\Omega - \mathbf{H}} = \sum_{f : d\mathbf{H}|_f = 0} \frac{e^{-\mathbf{H}(f)}}{\prod_{n=1}^{d} w_n[f]}$$

which reduces the integral to the counting of the fixed points of the $U(1)^N \times \mathbf{T}^2$ action and the weights $w_i[f]$:

$$T_f X = \bigoplus_{n=1}^{d} \mathbb{R}^2_{w_i[f]}$$

$$w_n[f] = \epsilon_1 w_{n,\mathbf{1}}[f] + \epsilon_2 w_{n,\mathbf{2}}[f] + \sum_l a_l w_{n,l}[f]$$

with $w_{n,\mathbf{i}}[f] \in \mathbb{Z}$. The DH formula assumes that the symplectic manifold X^{2d} is smooth and the fixed points are isolated, and that no fixed points sit "at infinity".

This is not strictly speaking true for the instanton moduli space. It is, however, true for the moduli space of noncommutative instantons. Our luck is the independence of the prepotential on the noncommutativity parameter.

The fixed point counting can be nicely summarized by a contour integral.

This contour integral also can be obtained by transforming the integral over the ADHM moduli space of our observable evaluated on the instanton configuration, by adding $\tilde{Q}$-exact terms,

$$\tilde{Q} = Q + i\epsilon_{\mathbf{i}} \left(z_{\mathbf{i}} G_{\mathbf{i}} - \bar{z}_{\bar{\mathbf{i}}} G_{\bar{\mathbf{i}}} \right)$$

as in [MooreNekrasovShatashvili'97'98]:

$$Z_k(a,\epsilon) = \frac{1}{k!}\frac{(\epsilon_1+\epsilon_2)^k}{(2\pi i\ \epsilon_1\epsilon_2)^k}\times$$

$$\oint \prod_{I=1}^{k} \frac{\mathrm{d}\phi_I}{P(\phi_I)P(\phi_I+\epsilon_1+\epsilon_2)}\times$$

$$\prod_{1\le I<J\le k} \frac{\phi_{IJ}^2(\phi_{IJ}^2-(\epsilon_1+\epsilon_2)^2)}{(\phi_{IJ}^2-\epsilon_1^2)(\phi_{IJ}^2-\epsilon_2^2)}$$

where:

$$P(x) = \prod_{l=1}^{N}(x-a_l)$$

ϕ_{IJ} denotes $\phi_I - \phi_J$

contour integrals

the poles at $\phi_{IJ} = \epsilon_1, \epsilon_2$ are avoided by shifting $\epsilon_{1,2} \to \epsilon_{1,2} + i0$, those at $\phi_i = a_l$ similarly by $a_l \to a_l + i0 \Longrightarrow$ **Evaluate by residues**

The poles are labelled as follows.

Let $k = k_1+k_2+\ldots+k_N$ be a partition of the instanton charge in N summands which have to be non-negative (but may vanish), $k_l \geq 0$. In turn, for all l such that $k_l > 0$ let Y_l denote a partition of k_l:

$$k_l = k_{l,1} + \ldots k_{l,\nu^{l,1}}, \qquad k_{l,1} \geq k_{l,2} \geq \ldots \geq k_{l,\nu^{l,1}} > 0$$

Pictorially one represents these partitions by the Young diagram with n_l columns of the lengths $k_{l,1}, \ldots k_{l,n_l}$.

In total we have k boxes distributed among N Young tableaux (some of which could be empty, i.e. contain zero boxes). Let us label these boxes somehow (the ordering is not important as it is cancelled in the end by the factor $k!$). Let us denote the collection of N Young diagrams by $\vec{Y} = (Y_1, \ldots, Y_N)$. We denote by $|Y_l| = k_l$ the number of boxes in the l'th diagram, and by $|\vec{Y}| = \sum_l |Y_l| = k$.

The pole of the contour integral corresponding to $\vec{Y}$ is at ϕ_s with s labelling the box (α, β) in the l'th Young tableau (so that $0 \leq \alpha \leq \nu^{l,\beta},\ 0 \leq \beta \leq k_{l,\alpha}$) equal to:

$$\vec{Y} \longrightarrow \phi_s = a_l + \epsilon_1(\alpha - 1) + \epsilon_2(\beta - 1)$$

Picture "Young diagrams"

The poles correspond to the fixed points of the action of the groups $U(N) \times \mathbf{T}^2$ on the moduli space $\mathcal{M}_{k,N}$.

Physically they correspond to the $U(N)$ (noncommutative) instantons which split as a sum of $U(1)$ noncommutative instantons corresponding to N commuting $U(1)$ subgroups of $U(N)$.

The charge k_l is the instanton charge of the $U(1)$ instanton in the l'th subgroup.

Moreover, these abelian instantons are of special nature – they are fixed by the group of space rotations.

If they were commutative (and therefore point-like) they had to sit on top of each other, and the space of such point-like configurations would have been rather singular.

Fortunately, upon the noncommutative deformation the singularities are resolved.

The instantons cannot sit quite on top of each other.

Instead, they try to get as close to each other as the uncertainty principle lets them.

Now let us fix a configuration $\vec{Y}$ and consider the corresponding contribution to the integral over instanton moduli. Technically it is convenient to work with $\epsilon_1 = -\epsilon_2 = \hbar$ (after calculating the residues)

the contribution of $\vec{Y}$:

$$R_{\vec{Y}} = \prod_{(l,i)\neq(m,j)} \left(\frac{a_{lm} + \hbar(k_{l,i} - i - k_{m,j} + j)}{a_{lm} + \hbar(j - i)} \right) =$$

$$\mathrm{Det} \left(\frac{A_{\vec{Y}}}{A_{\vec{Y}_0}} \right)^2$$

where $A_{\vec{Y}}, A_{\vec{Y}_0}$ are $\nu \times \nu$ matrices, $\nu = \sum_l n_l$, given by $(1 \leq j \leq n_m)$:

$$A_{\vec{Y}}^{(l,i)|(m,j)} = (a_l + \hbar(k_{l,i} - i))^{m-1+N(j-1)}$$
$$A_{\vec{Y}_0}^{(l,i)|(m,j)} = (a_l - \hbar i)^{m-1+N(j-1)}$$

IR evaluation

Write our observable as the superspace integral:

$$\mathcal{O} = \omega \wedge \mathrm{Tr}\left(\phi F + \tfrac{1}{2}\psi\psi\right) - H\mathrm{Tr}\,(F \wedge F) =$$

$$-\frac{1}{8\pi^2}\int d^4x d^4\theta\ \mathcal{H}(x,\theta)\mathrm{Tr}\Phi^2$$

$$\mathcal{H}(x,\theta) =$$

$$\epsilon_1((x^1)^2 + (x^2)^2) + \epsilon_2((x^3)^2 + (x^4)^2) + \theta^1\theta^2 + \theta^3\theta^4$$

Crucial observation: Adding $\mathcal{O}$ to the action is nothing but change of coupling

$$\tau_0 \to \tau_0 + \frac{1}{2\pi i}\mathcal{H}(x,\theta)$$

Adiabatic approximation: If $\mathcal{H}$ was a constant, the IR dynamics had simply been that of SW action with the replacement

$$\Lambda \to \Lambda e^{-\mathcal{H}(x,\theta)/2N}$$

However, we should remember that $\mathcal{H}$ is not constant, and consider this renormalization as valid up to terms in the effective action containing derivatives of $\mathcal{H}$.

Keeping this in mind we arrive at the standard Seiberg-Witten effective action determined by the superspace-dependent prepotential

$$\mathcal{F}\left(a;\Lambda e^{-\mathcal{H}/2N}\right) = \mathcal{F}(a;\Lambda e^{-H/2N})+$$

$$\frac{\omega}{2N}\frac{d}{d\mathrm{log}\Lambda}\mathcal{F}(a;\Lambda e^{-H/2N})+$$

$$\frac{1}{8N^2}\omega\wedge\omega\frac{d^2}{(d\mathrm{log}\Lambda)^2}\mathcal{F}(a;\Lambda e^{-H/2N})$$

This prepotential is then integrated over the superspace (together with the conjugate terms) to produce the low-energy action.

Go to extreme infrared – scale the metric on $\mathbb{R}^4$ by a very large factor t (keeping ω intact). On flat $\mathbb{R}^4$ the only term which may contribute to the correlation function in question in the limit $t \to \infty$ is the last term;

the rest contains couplings to the gauge fields – contractions only from loop diagrams – suppressed by inverse powers of t. The last term, on the other hand, gives:

$$Z(a;\epsilon) = \exp - \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \omega \wedge \omega \frac{d^2}{(d\log\Lambda)^2} \mathcal{F}(a; \Lambda e^{-H/2N}) +$$

$$+ O(\epsilon_{1,2})$$

where we used the fact that the derivatives of H are proportional to $\epsilon_{1,2}$. Finally

$$d^4x \to \frac{1}{\epsilon_1 \epsilon_2} H dH,$$

and the integrals kill the log part of $\mathcal{F}$ leaving us with:

$$\boxed{Z(a;\epsilon) = \exp \frac{\mathcal{F}^{inst}(a;\Lambda) + O(\epsilon_{1,2})}{\epsilon_1 \epsilon_2}}$$

This is our main formula from IR

Synthesis

Now we can formulate our main result:

There exists a function $\mathcal{F}^{inst}(a|\epsilon_1,\epsilon_2)$, analytic in ϵ_1,ϵ_2 at $\epsilon_1,\epsilon_2=0$, such that:

$$\exp\left(\frac{1}{\epsilon_1\epsilon_2}\mathcal{F}^{inst}\left(a,\Lambda|\epsilon_1,\epsilon_2\right)\right)=\sum_{k=0}^{\infty}\Lambda^{2kN}\sum_{\vec{Y},\ |\vec{Y}|=k}R_{\vec{Y}}$$

The value $\mathcal{F}^{inst}(a,\Lambda)=\mathcal{F}^{inst}(a|0,0)$ at $\epsilon_1=\epsilon_2=0$ is the instanton part $\mathcal{F}^{inst}$ of the prepotential of $SU(N)$ $\mathcal{N}=2$ Yang-Mills theory, expanded in the quasiclassical region $a\to\infty$.

Experimental checks

Previously in the literature on YM integrals: [Mattis et al][Hollowood][Dorey et al] —- up to two instantons

The first three instantons...

We shall now give the formulae for the first three instanton contributions to the prepotential for the general $SU(N)$ case.

By straightforward application of our rules we arrive at the following expressions for the moduli integrals Z_1, Z_2, Z_3:

$$\epsilon_1\epsilon_2 Z_1 = \sum_l S_l$$

$$4(\epsilon_1\epsilon_2)^2 Z_2 = \sum_l S_l \left(S_l^{(+\hbar)} + S_l^{(-\hbar)} \right) + \sum_{l \neq m} \frac{2 S_l S_m}{\left(1 - \frac{\hbar^2}{a_{lm}^2}\right)^2}$$

and

$$36(\epsilon_1\epsilon_2)^3 Z_3 =$$

$$\sum_l S_l S_l^{(+\hbar)} S_l^{(+2\hbar)} + S_l S_l^{(-\hbar)} S_l^{(-2\hbar)} + 4 S_l S_l^{(+\hbar)} S_l^{(-\hbar)} +$$

$$\sum_{m\neq l} \frac{9 S_m S_l S_l^{(+\hbar)}}{\left(1 - \frac{2\hbar^2}{(a_{lm}(a_{lm}+\hbar))}\right)^2} + \frac{9 S_m S_l S_l^{(-\hbar)}}{\left(1 - \frac{2\hbar^2}{(a_{lm}(a_{lm}-\hbar))}\right)^2} +$$

$$\sum_{n\neq m,l} \frac{6 S_n S_m S_l}{\left(\left(1 - \frac{\hbar^2}{a_{lm}^2}\right)\left(1 - \frac{\hbar^2}{a_{ln}^2}\right)\left(1 - \frac{\hbar^2}{a_{mn}^2}\right)\right)^2}))$$

where

$$S_l^{(p\hbar)} = \frac{1}{\prod_{m\neq l}(p\hbar + a_{lm})^2}$$

$\Longrightarrow$ we derive (by taking the logarithm):

$$
\begin{aligned}
\mathcal{F}_1 &= \sum_l S_l \\
\mathcal{F}_2 &= \sum_l \frac{1}{4} S_l S_l^{(2)} + \sum_{l\neq m} \frac{S_l S_m}{a_{lm}^2} + O(\hbar^2) \\
\mathcal{F}_3 &= \sum_l \frac{S_l}{36} \left(S_l S_l^{(4)} + 2 S_l^{(1)} S_l^{(3)} + 3 S_l^{(2)} S_l^{(2)} \right) + \\
&\quad \sum_{l\neq m} \frac{S_l S_m}{a_{lm}^4} \left(5 S_l - 2 a_{lm} S_l^{(1)} + a_{lm}^2 S_l^{(2)} \right) + \\
&\quad \sum_{l\neq m\neq n} \frac{2 S_l \; S_m S_n}{3 (a_{lm} a_{ln} a_{mn})^2} \left(a_{ln}^2 + a_{lm}^2 + a_{mn}^2 \right) \\
&+ O(\hbar^2)
\end{aligned}
$$

where

$$
S_l^{(k)} = \hbar^{-k} \partial_p^k |_{p=0} S_l^{(p\hbar)}
$$

Four and five instantons

To collect more experimental data-points we have considered the case of the gauge groups $SU(2)$ and $SU(3)$ with fundamental matter. We have computed explicitly the prepotential for four and five instantons and found a perfect agreement with the results of [ChanD'Hoker'99; D'HokerKricheverPhong'96; EdelsteinGomez-ReinoMas'99; EdelsteinMariñoMas'98] .

Adjoint, fundamental, and other matters

So far we were discussing pure $\mathcal{N} = 2$ gauge theory.

Our formalism adapts easily to the theories with matter in various representations.

The ϵ-integrals reflect both the topology of the moduli space of instantons and also of the matter bundle.

The latter is the bundle of the Dirac zero modes in the representation of interest. For the adjoint representation, and on $\mathbb{R}^4$ this bundle can be identified with the tangent bundle to the moduli space of instantons. It has a $U(1)$ symmetry. The equivariant Euler class of the tangent bundle is the instanton measure in the case of massive matter.

`This reasoning leads to the following`

ϵ-`integral`:

for adjoint representation:

$$Z_k = \frac{1}{k!}\left(\frac{(\epsilon_1+\epsilon_2)(\epsilon_1+m)(\epsilon_2+m)}{2\pi i\ \epsilon_1\epsilon_2 m(\epsilon-m)}\right)^k \times$$

$$\oint \prod_{I=1}^{k} \frac{\mathrm{d}\phi_I\ P(\phi_I+m)P(\phi_I+\epsilon-m)}{P(\phi_I)P(\phi_I+\epsilon)} \times$$

$$\times \prod_{I<J} \frac{\phi_{IJ}^2(\phi_{IJ}^2-\epsilon^2)}{(\phi_{IJ}^2-\epsilon_1^2)(\phi_{IJ}^2-\epsilon_2^2)} \times$$

$$\frac{(\phi_{IJ}^2-(\epsilon_1-m)^2)(\phi_{IJ}^2-(\epsilon_2-m)^2)}{(\phi_{IJ}^2-m^2)(\phi_{IJ}^2-(\epsilon-m)^2)}$$

for N_f fundamental multiplets:

$$Z_k = \frac{1}{k!} \frac{\epsilon^k}{(2\pi i \epsilon_1 \epsilon_2)^k} \oint \prod_{I=1}^{k} \frac{\mathrm{d}\phi_I \; Q(\phi_I)}{P(\phi_I)P(\phi_I + \epsilon)} \times$$

$$\prod_{1 \le I < J \le k} \frac{\phi_{IJ}^2(\phi_{IJ}^2 - \epsilon^2)}{(\phi_{IJ}^2 - \epsilon_1^2)(\phi_{IJ}^2 - \epsilon_2^2)}$$

where

$$Q(x) = \prod_{f=1}^{N_f} (x + m_f)$$

We agree with the calculations in [D'HokerPhong, D'HokerChan] – for adjoint matter up to two instantons (all that is available in the literature), for fundamental up to five.

for the fundamental + symmetric tensor representation

$$
Z_k = \frac{1}{k!}\frac{\epsilon^k}{(2\pi i \epsilon_1\epsilon_2)^k}\times
$$

$$
\oint \prod_{I=1}^{k} \frac{\mathrm{d}\phi_I \; Q(\phi_I)\prod_{l,I}(a_l+\phi_I+m)}{P(\phi_I)P(\phi_I+\epsilon)}\times
$$

$$
\prod_{1\leq I<J\leq k} \frac{\phi_{IJ}^2(\phi_{IJ}^2-\epsilon^2)}{(\phi_{IJ}^2-\epsilon_1^2)(\phi_{IJ}^2-\epsilon_2^2)}\times
$$

$$
\prod_{I\leq J} \frac{(\phi_I+\phi_J+\epsilon_1+m)(\phi_I+\phi_J+\epsilon_2+m)}{(\phi_I+\phi_J+m)(\phi_I+\phi_J+\epsilon_1+\epsilon_2+m)}
$$

and with $I<J$ in the last line for the antisymmetric tensor.

These formulas were checked against the one-instanton results of [Schnitzer et al.]

Prospects for future

`Instead of going into detailed exposition`

`of various interesting topics which emerge`

`we simply list them:`

1. ϵ_1, ϵ_2 twist of gauge theory has interpretation in the M-engineering of gauge theories via

$$CY \times \mathbf{S}^1 \times \mathbb{R}^4$$

compactification where we twist $\mathbb{R}^4$ as we go around the circle, and also twist the fermions on CY by $\epsilon_1 + \epsilon_2$. In IIA language this means placing NS5 branes and suspended D4's on Melvin's 4d universe.

2. The perturbative part $\mathcal{F}^{pert}$ also has ϵ-deformed version, which can be derived from the previous remark, or, more formally, by studying equivariant K-theory of the instanton moduli space:

$$\mathcal{F}^{1-loop}(a;\epsilon_{1,2}) =$$
$$\sum_{l\neq m}\int_0^\infty \frac{ds}{s}\frac{\left(\exp(-sa_{lm})-1-\frac{1}{2}s^2a_{lm}^2\right)}{(e^{s\epsilon_1}-1)(e^{s\epsilon_2}-1)}$$

3. The full ϵ-deformed prepotential has a physical meaning as capturing the $\mathcal{F}_g R^2 F^{2g-2}$ terms in the effective action, describing the couplings to the graviphoton field strength $F \sim \hbar$ (for $\epsilon_1+\epsilon_2=0$).
4. In other terms, the coefficients $\mathcal{F}_g$ in the expansion:

$$Z(a,\Lambda,\hbar) = \exp - \sum_{g=0}^{\infty} \hbar^{2g-2}\mathcal{F}_g(a,\Lambda)$$

   should be viewed as a genus g topological string amplitude in the type A topological string on the local A_{N-1} singularity fibered over $\mathbb{P}^1$ in the geometrical engineering limit [Vafa].

5. The latter quantity obeys the holomorphic anomaly equation, which should be easier to understand then the genuine CY_3 one [BCOV, Witten], as this one should be derived from the identification of $Z(a;\epsilon)$ with the τ-function of KP/Toda hierarchy, given by the partition function of the free fermions living on the SW curve.

6. These chiral fermions are the fermionized chiral bosons which in turn descend from the chiral two-form of the M5-brane (or NS5-brane in IIA setup [KlemmLercheMayrVafa]) subject to the ϵ-twisted background

$$\boxed{Z(a,\Lambda;\hbar) = \int \mathcal{D}\phi \, \exp\left(-\frac{1}{4\pi}\int_{\Sigma} \partial\phi\bar{\partial}\phi + \frac{1}{\hbar}\Omega \wedge \bar{\partial}\phi \right)}$$

where Σ is the SW curve, and Ω is the SW differential, and ϕ is a compact boson.

In the limit $\hbar \to 0$ we arrive at the Krichever formula for

$$\mathcal{F}^{SW}(a) = \int_{\Sigma} \Omega \wedge \bar{\partial}\frac{1}{\partial}\Omega$$

On the coming pages we remind a few facts about SW solution and ADHM construction.

Reminder 1.
Seiberg-Witten solution

Originally $\mathcal{F}$ was calculated from the constraints imposed on the geometry of the moduli space of vacua by electro-magnetic duality [SeibergWitten'94, Klemm-LercheTheisenYankielowisz'94, ArgyresFaraggi'94].

It was found that $\mathcal{F}$ is encoded in the family of Toda-system spectral curves:

$$w + \frac{\Lambda^{2N}}{w} = \mathrm{Det}(\lambda - \phi)$$

associated to the moduli space of vacua of gauge theory [GorskyMarshakovMironovMorozovKrichever'95].

These curves have genus $N-1$. Almost like in mirror symmetry calculation, one computes the periods of a meromorphic differential

$$\Omega = \frac{1}{2\pi i}\lambda\frac{dw}{w}$$

over a base of A and B cycles:

$$a_s = \oint_{A_s}\Omega, \qquad \frac{\partial\mathcal{F}}{\partial a_s} = \oint_{B_s}\Omega$$

From these formulas one can extract $\mathcal{F}_k$ by a painful yet straightforward procedure.

Reminder 2.
Instanton measure and its localization

The moduli space $\mathcal{M}_{k,N}$ of instantons with fixed framing at infinity has dimension $4kN$.

It has the following convenient **description.**

Two complex vector spaces V and W of the complex dimensions k and N respectively. These spaces should be viewed as Chan-Paton spaces for $D(p-4)$ and Dp branes in the brane realization of the gauge theory with instantons.

Let us also denote by L the two dimensional complex vector space, which we shall identify with the Euclidean space $\mathbb{R}^4 \approx \mathbb{C}^2$ where our gauge theory lives.

Then the ADHM [AtiyahDrinfeldHitchinManin] data consists of the following maps between the vector spaces:

$$V \xrightarrow{\tau} V \otimes L \oplus W \xrightarrow{\sigma} V \otimes \Lambda^2 L$$

where

$$\tau = \begin{pmatrix} B_2 \\ -B_1 \\ J \end{pmatrix}, \qquad \sigma = \begin{pmatrix} B_1 & B_2 & I \end{pmatrix}$$

$$B_{1,2} \in \mathrm{End}(V), \;\; I \in \mathrm{Hom}(W, V), \;\; J \in \mathrm{Hom}(V, W)$$

The ADHM construction presents the moduli space of $U(N)$ instantons on $\mathbb{R}^4$ of charge k as a hyperkähler quotient of the space of operators B_1, B_2, I, J by the action of the group $U(k)$ for which V is a fundamental representation, and $B_{1,2}$ transform in the adjoint, I in the fundamental, and J in the anti-fundamental representations.

More precisely, the moduli space of proper instantons is obtained by taking the quadruples $(B_{1,2}, I, J)$ obeying the so-called ADHM equations:

$$\mu_c = 0, \qquad \mu_r = 0,$$

where:

$$\begin{aligned}\mu_c &= [B_1, B_2] + IJ \\ \mu_r &=[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J\end{aligned}$$

and with the additional requirement that the stabilizer of the quadruple in $U(k)$ is trivial. This produces a non-compact hyperkähler manifold $M_{k,N}$ of instantons with fixed framing at infinity.

The non-compactness of the moduli space of instantons is of both ultraviolet and of infrared nature. The UV non-compactness has to do with the instanton size, which can be made arbitrarily small. The IR non-compactness has to do with the non-compactness of $\mathbb{R}^4$ *which permits the instantons to run away to infinity.*

Curing non-compactness

The UV problem can be solved by relaxing the condition on the stabilizer, thus adding the so-called point-like instantons. A point of the hyperkähler space $\tilde{M}_{k,N}$ with orbifold singularities which one obtains in this way (Uhlenbeck compactification) is an instanton of charge $p \leq k$ and a set of $k-p$ points on $\mathbb{R}^4$:

$$\tilde{M}_{k,N} = M_{k,N} \cup M_{k-1,N} \times \mathbb{R}^4 \cup$$

$$\cup M_{k-2,N} \times Sym^2(\mathbb{R}^4) \cup \ldots \cup Sym^k(\mathbb{R}^4)$$

This space is still singular, and its singularities can be resolved by passing to the so-called **noncommutative instantons** [NekrasovSchwarz'98] or **torsion free sheaves** [Gieseker, Nakajima, Grojnowski, LosevMooreNekrasovShatashvili'95] which solve deformed ADHM equations $\mu_r = \zeta_r, \mu_c = 0$.

Symmetries of instanton moduli space

The ADHM construction gives rise to the instantons with fixed gauge orientation at infinity (fixed framing). $G = SU(N)$ acts on their moduli space $\mathcal{M}_{N,k}$ by rotating the gauge orientation. Also, the group of Euclidean rotations of $\mathbb{R}^4$ acts on $\mathcal{M}_{N,k}$.