BSSP-2025

[redacted]

LECTURE- 1

①

Dynamical Systems

[Classification of fixed points].

Newtonian System.

$q^i \quad 1 \leq i \leq N$ generalized coordinates

Equations of motion $$\frac{d^2 q_i}{dt^2} = f^i\left(\frac{d\bar{q}}{dt}, \bar{q}, t\right)$$

$i = 1, 2 \dots N$

Can be converted to set of 1st order equations.

Let $v_i = \frac{dq_i}{dt}$

$$\therefore \frac{dq_i}{dt} = v_i$$

$$\frac{dv_i}{dt} = f^i(\bar{v}, \bar{q}, t)$$

Define $x^i \quad i = 1, 2 \dots 2N$

Such that $x^1 = q^1, x^2 = q^2 \dots x^N = q^N, x^{N+1} = v^1, x^{N+2} = v^2 \dots x^{2N} = v^N$

Then $$\frac{dx^i}{dt} = F^i(\bar{x}, t)$$

$$F^i = v^i \quad , i = 1, 2 \dots N$$
$$= f^{i-N} \quad i = N+1, \dots 2N.$$

~~i.e dx~~

This is the most general equation of a dynamical system (with finite discrete degrees of freedom and continuous time).

Let $2N = n$

$$\frac{dx^i}{dt} = F^i(x, t) \quad i = 1, 2, .. n$$

$$F^i(\bar{x}, t) = F^i(\bar{x})$$ (no explicit time-dependence).

Defn. of an autonomous system.

Examples of dynamical systems.

① Newtonian system.

② Radioactive decay of a set of nuclei

③ Growth of a simple biological population

④ Chemical reactions $A + B \rightleftharpoons C + D$

$B + C \rightleftharpoons A + E$.

$$\frac{dN_A}{dt} = F_A(N_A \cdots N_E) \;\cdots\cdots\; \frac{dN_E}{dt} = f_E(N_A, \cdots N_E)$$

⑤ Renormalization group theory.

$\overline{E}_t$ $\underline{n=1}$ ③

$$\frac{dx}{dt} = F(x) \qquad \int_0^t dt = \int_{x_0}^{x} \frac{dx}{F(x)} \qquad x = x_0 \text{ at } t = 0$$

$$\Rightarrow t = t(x)$$

$$\Rightarrow x = x(t)$$

<u>General features</u>

Fixed points – Zeros of $F(x)$

$$x_F \qquad F(x_F) = 0$$

E_t Small perturbation from fixed point:

Let $x(t=0) = x_F + \epsilon$

$$\frac{dx}{dt} = F(x) = \underset{0}{\underset{\shortparallel}{F(x_F)}} + F'(x_F)(x - x_F) + \ldots$$

$$\frac{dx}{x - x_F} = F'(x_F)\,dt \Rightarrow x - x_F = c\,e^{F'(x_F)t}$$

At $t=0$ $x = x_F + \epsilon$

$$\therefore c = \epsilon$$

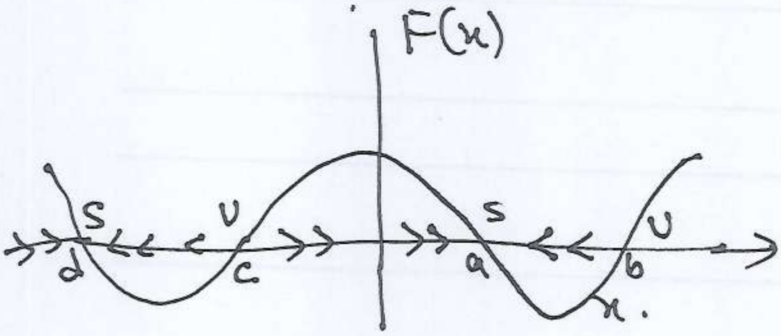
$$\therefore x(t) = x_F + \epsilon\, e^{F'(x_F)t}$$

IF $F'(x_F) < 0$ $x \to x_F$ as $t \to \infty$

Stable fixed point.

$F'(x_F) > 0$ x goes away from x_F as $t \to \infty$

– Unstable fixed point.



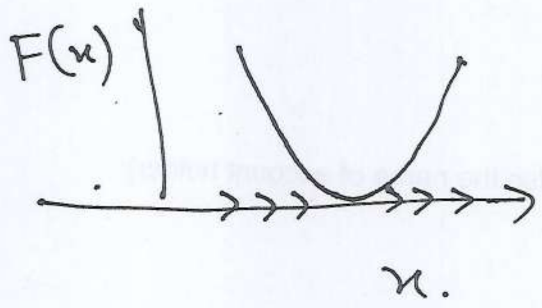
$a < x < b$ – Invariant set of points – if we start from anywhere inside the set, we will remain within the set at all times.

Domain of attraction of a stable fixed point – Set of points such that any motion starting in that set approaches the fixed pt. as $t \to \infty$.

eg Dom. of attraction of a $\to$ $c < x < b$.

⑤

Special case $F(x_f)=0 \quad F'(x_f)=0$



$$F''(x_f) > 0$$

For $x \simeq x_f \quad \frac{dx}{dt} = \frac{1}{2} F''(x_f)(x-x_f)^2$

$$x_{t=0} = x_f + \epsilon$$

Soln. $x = x_f + \frac{\epsilon}{1-\frac{1}{2}F''(x_f)\epsilon t}$

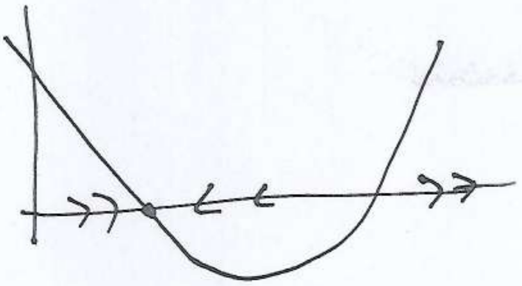
$\epsilon < 0 \quad x \to x_f$ as $t \to \infty$

$\epsilon > 0 \quad x$ goes away from x_f as $t \to \infty$

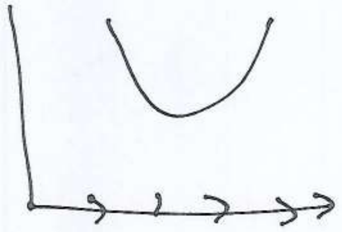
Small perturbation of F changes the nature of the fixed point

$$F \to \underbrace{F + \delta\,\phi(x)}_{\hookrightarrow \text{ arbitrary function}} = \tilde{F}$$

$\tilde{F}$

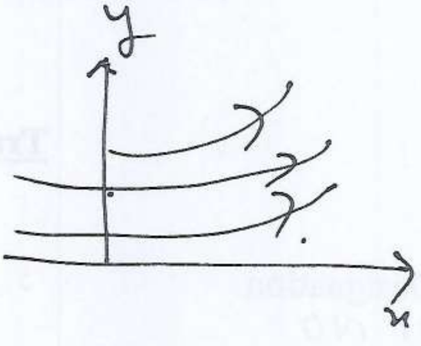


OR



⑥

$n=2$



$$\frac{dx}{dt} = F_1(x,y)$$

$$\frac{dy}{dt} = F_2(x,y)$$

$$\frac{dy}{dx} = \frac{F_2(x,y)}{F_1(x,y)} \rightarrow \text{Phase Curves.}$$

Fixed points: (x_F, y_F) is a fixed point if

$$F_1(x_F, y_F) = 0 \quad , \quad F_2(x_F, y_F) = 0$$

Motion near a fixed point.

$$\frac{dx}{dt} = \left.\frac{\partial F_1}{\partial x}\right|_F (x - x_F) + \left.\frac{\partial F_1}{\partial y}\right|_F (y - y_F)$$

$$\frac{dy}{dt} = \left.\frac{\partial F_2}{\partial x}\right|_F (x - x_F) + \left.\frac{\partial F_2}{\partial y}\right|_F (y - y_F)$$

Let $x - x_F = \tilde{x}$, $y - y_F = \tilde{y}$

$$\frac{d}{dt}\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$$

$$a = \left.\frac{\partial F_1}{\partial x}\right|_F \quad b = \left.\frac{\partial F_1}{\partial y}\right|_F \quad c = \left.\frac{\partial F_2}{\partial x}\right|_F \quad d = \left.\frac{\partial F_2}{\partial y}\right|_F$$

Let $\tilde{X} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$ $\quad R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\therefore \frac{d\tilde{X}}{dt} = R\tilde{X}$$

Diagonalize R :

$$\cancel{\Rightarrow} \quad |R - \lambda I| = 0$$

$$\Rightarrow (\lambda - a)(\lambda - d) - bc = 0$$

$$\Rightarrow \lambda^2 - (a+d)\lambda + ad - bc = 0$$

$$\Rightarrow \lambda^2 - \tau\lambda + \delta = 0.$$

$$\tau = \mathrm{Tr}[R] = a + d \qquad \delta = \mathrm{Det}[R] = ad - bc$$

$$\lambda = \frac{\tau}{2} \pm \sqrt{\frac{\tau^2}{4} - \delta} = \lambda_1, \lambda_2.$$

Let $R\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \lambda_1 \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}$ $\quad R\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \lambda_2 \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$

Then $R\begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix}^{-1} R \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$A^{-1} R A = \lambda$$

Let $\frac{\tau^2}{4} > \delta$ $\therefore$ λ_1, λ_2 real and distinct.

$\therefore$ A can be chosen to be real

$$\frac{d\tilde{x}}{dt} = R\tilde{x} \Rightarrow \frac{d}{dt}\underline{A^{-1}\tilde{x}} = A^{-1}RA\,\underline{A^{-1}\tilde{x}}$$

Let $\hat{x} = \begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix}$ $\therefore$ $\frac{d}{dt}\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix} = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix}$

$\frac{d\hat{x}}{dt} = \lambda_1 \hat{x}$, $\frac{d\hat{y}}{dt} = \lambda_2 \hat{y}$ $(\hat{x}(0), \hat{y}(0)) \simeq (0, 0)$

$\Rightarrow \hat{x}(t) = e^{\lambda_1 t}\hat{x}_0$ $\hat{y}(t) = e^{\lambda_2 t}\hat{y}_0$.

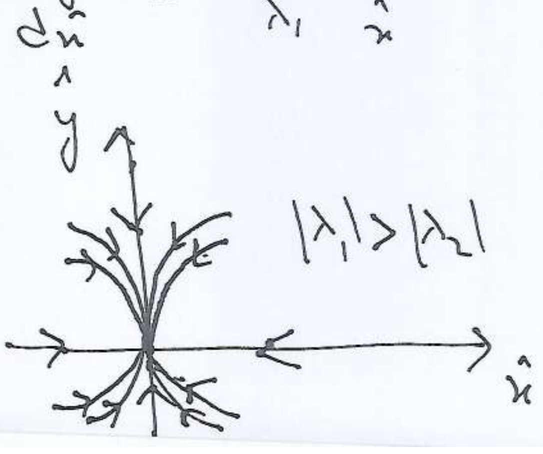
Note $\lambda_1 + \lambda_2 = \tau$ $\lambda_1\lambda_2 = \delta$

Case (I) $\lambda_1, \lambda_2 < 0 \Rightarrow \tau < 0,\ \delta > 0, + (\tau^2 > 4\delta)$

$\hat{x}, \hat{y} \to 0$ as $t \to \infty$

$\frac{d\hat{y}}{d\hat{x}} = \frac{\lambda_2}{\lambda_1}\frac{\hat{y}}{\hat{x}} \Rightarrow \ln \hat{y}/\hat{y}_0 = \frac{\lambda_2}{\lambda_1} \ln \hat{x}/\hat{x}_0$

$\Rightarrow \hat{y} = c\,(\hat{x})^{\lambda_2/\lambda_1}$



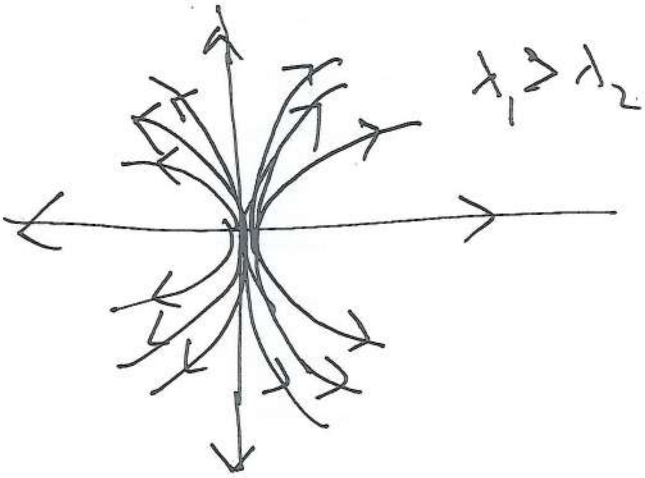
<u>Stable Node</u>

$\left(\frac{d\hat{y}}{d\hat{x}}\Big|_0 = \infty \text{ for } |\lambda_1| > |\lambda_2|\right)$

Case II $\lambda_1 > 0, \lambda_2 > 0 \Rightarrow \tau > 0, \delta > 0$
$+ (\tau^2 > 4\delta)$.

$\hat{x}, \hat{y} \to \infty$ as $t \to \infty$.

$$\frac{d\hat{y}}{d\hat{x}} = \frac{\lambda_2}{\lambda_1} \frac{\hat{y}}{\hat{x}} \Rightarrow \hat{y} = c(\hat{x})^{\lambda_2/\lambda_1}$$



$\lambda_1 > \lambda_2$ Unstable node

Case III $\lambda_1 > 0, \lambda_2 < 0 \Rightarrow \delta < 0 \quad (\tau^2 > 4\delta)$

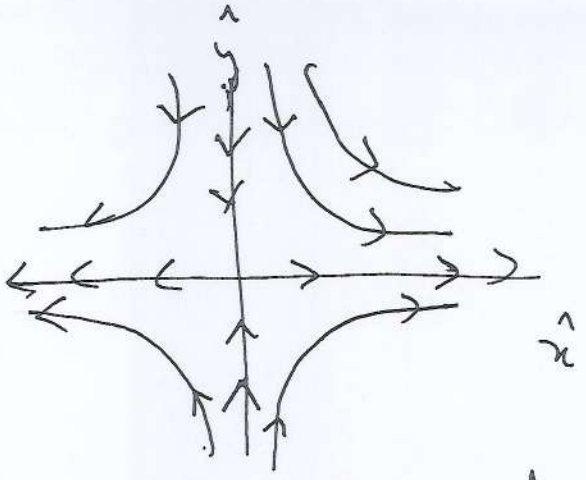
$\hat{x} = \hat{x}_0 e^{\lambda_1 t} \quad \hat{y} = \hat{y}_0 e^{\lambda_2 t}$

$\Rightarrow$ Unstable except along the line $\hat{x} = 0$.

Again $\hat{y} = c\,(\hat{x})^{\lambda_2/\lambda_1} \rightsquigarrow < 0$

$$= c \frac{1}{(\hat{x})^{|\lambda_2/\lambda_1|}}$$

If $|\lambda_1| = |\lambda_2| \quad \hat{x}\hat{y} = c$



Hyperbolic fixed point.

$\hat{x}=0$, $\hat{y}=0$] Separatrix : Phase curves that pass through a hyperbolic fixed point.

Case IV Let $\frac{\tau^2}{4} < \delta$

$$\lambda = \alpha \pm i\omega \quad \alpha = \frac{\tau}{2}, \quad \omega = \sqrt{\delta - \tau^2/4} .$$

$$\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix} = A^{-1}\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix} \rightarrow \text{Complex}$$

Let us try to ~~bring~~ transform instead to the following form

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} = B\begin{pmatrix}\alpha & -\omega\\ \omega & \alpha\end{pmatrix} B^{-1}$$

or $B^{-1}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}\alpha & -\omega\\ \omega & \alpha\end{pmatrix} B^{-1}$

Let $R\cancel{\xi}\hat{e} = (\alpha + i\omega)\hat{e}$

$$\Rightarrow R\left(\hat{e}^R + i\,\hat{e}^I\right) = (\alpha + i\omega)\left(\hat{e}^R + i\,\hat{e}^I\right)$$

$$= \cancel{\alpha}\ \alpha\hat{e}^R - \omega\hat{e}^I + i\left(\omega\hat{e}^R + \alpha\hat{e}^I\right)$$

$$\Rightarrow R\,\hat{e}^R = \alpha\,\hat{e}^R - \omega\,\hat{e}^I$$

$$R\,\hat{e}^I = \omega\hat{e}^R + \alpha\,\hat{e}^I$$

$$\Rightarrow R\begin{pmatrix}\hat{e}^R & \hat{e}^I\end{pmatrix} = \begin{pmatrix}\hat{e}^R & \hat{e}^I\end{pmatrix}\begin{pmatrix}\alpha & +\omega \\ -\omega & \alpha\end{pmatrix}$$

$$\therefore B = \begin{pmatrix}\hat{e}^R_1 & \hat{e}^I_1 \\ \hat{e}^R_2 & \hat{e}^I_2\end{pmatrix}. \qquad RB = B\begin{pmatrix}\alpha & \omega \\ -\omega & \alpha\end{pmatrix}$$

Let $B^{-1}\tilde{x} = \hat{x}$

$$\therefore \frac{d}{dt}\begin{pmatrix}\hat{x} \\ \hat{y}\end{pmatrix} = \begin{pmatrix}\alpha & +\omega \\ -\omega & \alpha\end{pmatrix}\begin{pmatrix}\hat{x} \\ \hat{y}\end{pmatrix}$$

Let $\hat{x} = r\cos\theta \quad \hat{y} = r\sin\theta$

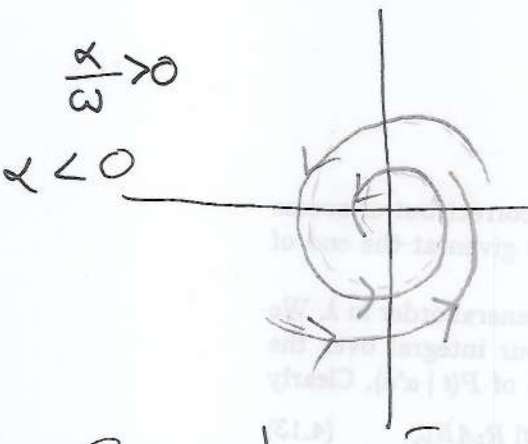
Then $\dfrac{dr}{dt} = \alpha r \qquad \dfrac{d\theta}{dt} = -\omega$

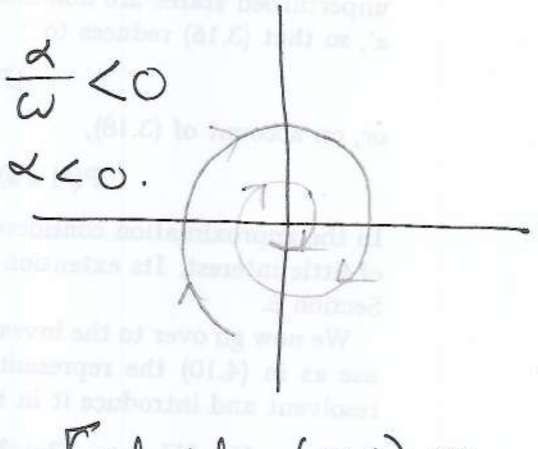
$$\Rightarrow \cancel{r}\ r = r_0 e^{\alpha t} \qquad \theta = \theta_0 - \omega t \qquad \left|\ \begin{matrix} r = r_0 \\ \theta = \theta_0 \end{matrix}\right. \text{ at } t = 0.$$

Phase Curves

⑫

$$\frac{dr}{d\theta} = \frac{-\alpha}{\omega} r \Rightarrow r = c e^{-\frac{\alpha}{\omega}\theta}$$





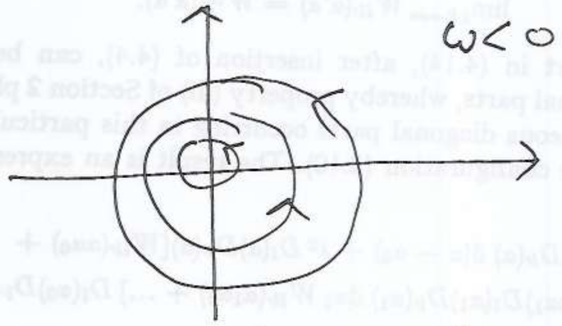
Spiral Fixed points [stable ($\alpha<0$) or unstable ($\alpha>0$)].

Special Cases : $\frac{\tau^2}{4} < \delta$, $\tau = 0$ $\omega \neq 0$

$\Rightarrow \alpha = 0.$

$$\therefore \frac{dr}{dt} = 0 \quad \frac{d\theta}{dt} = -\omega, \; r = r_0 \; \theta = \theta_0 - \omega t$$

Phase curve.



Elliptic fixed points (e.g S.H.O).

$\frac{T^2}{4} = \delta \rightarrow$ Two eigenvalues are equal and real. $\lambda = \frac{T}{2}$.

[It may ~~then~~ not ~~be~~ then be possible to diagonalize $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by a similarity transformation.]

However one can show that*

$$B^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} B = \begin{pmatrix} \lambda & 0 \\ c & \lambda \end{pmatrix}$$

For $B^{-1} = \begin{pmatrix} a-d & 2b \\ 2c & 0 \end{pmatrix}$ Check!

$$\hat{X} = B^{-1} X$$

$$\therefore \frac{d}{dt}\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ c & \lambda \end{pmatrix}\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}$$

$$\frac{d\hat{x}}{dt} = \lambda \hat{x} \qquad \frac{d\hat{y}}{dt} = c\hat{x} + \lambda \hat{y}$$

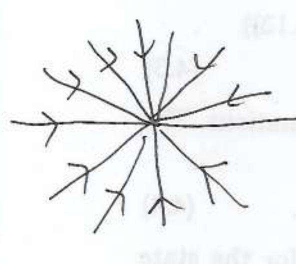
$$\hat{x} = \hat{x}_0 e^{\lambda t} \qquad \hat{y} = \hat{y}_0 e^{\lambda t} + c\hat{x}_0 t e^{\lambda t}.$$

⑭

$\underline{c = 0}$

$x = \hat{x}_0 e^{\lambda t} \quad y = \hat{y}_0 e^{\lambda t}$

$\frac{d\hat{y}}{d\hat{x}} = \frac{\hat{y}}{\hat{x}} \quad \hat{y} = \text{Const} \cdot \hat{x}.$



$\lambda < 0$ Stable Star

Unstable star $(\lambda > 0)$

$c \neq 0$

$x = \hat{x}_0 e^{\lambda t} \; ; \; y = (\hat{y}_0 + c\hat{x}_0 t) e^{\lambda t}$

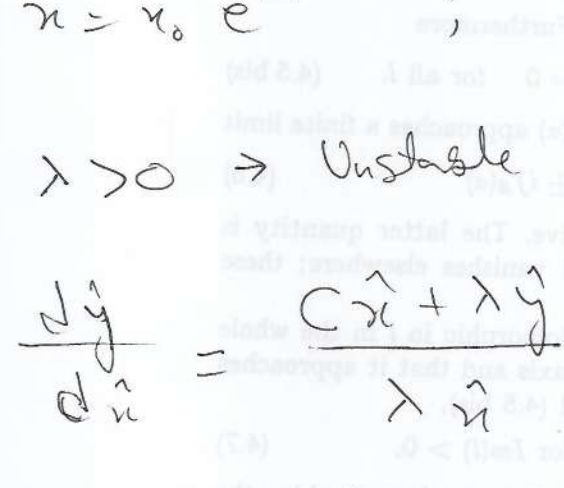
$\lambda > 0 \rightarrow$ Unstable $\quad \lambda < 0$ Stable.

$\frac{d\hat{y}}{d\hat{x}} = \frac{c\hat{x} + \lambda\hat{y}}{\lambda\hat{x}} \; ; \; \hat{y} = \frac{c}{\lambda}\hat{x}(\ln|\hat{x}| + k)$

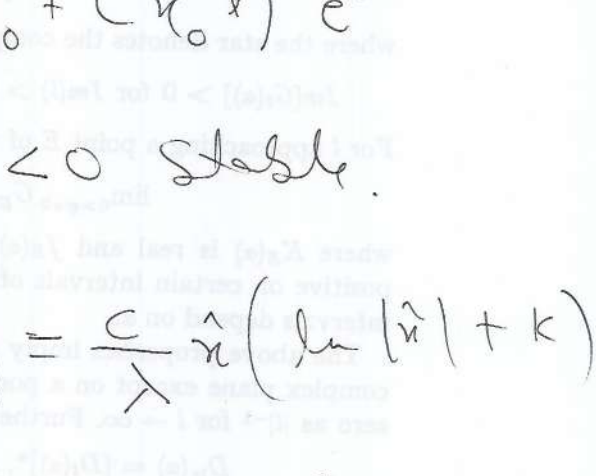
$\hat{x} > 0 \quad \frac{dy}{dx} = \frac{c}{\lambda}(\ln|\hat{x}| + k) + \frac{c}{\lambda}$

$\frac{c}{\lambda} < 0$

$\lambda < 0$



$\lambda < 0.$



IMPROPER NODES.

⑮

Stable Unstable Nodes
Hyperbolic Fixed points $\Big\}$ $\tau^2 > 4\delta$. λ_1, λ_2 real.

Spiral fixed point $\}$ $\tau^2 < 4\delta$. $\lambda = \alpha \pm i\omega$

Elliptic fixed point $\tau = 0$. $\lambda = \pm i\omega$

Stable Unstable Star
Improper node $\Big\}$ $\tau^2 = 4\delta$
Degenerate eigenvalues.

$$\tilde{x} = \begin{pmatrix} x - x_F \\ y - y_F \end{pmatrix} = \cancel{B} M \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \qquad M = A \text{ or } B$$

$$= \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

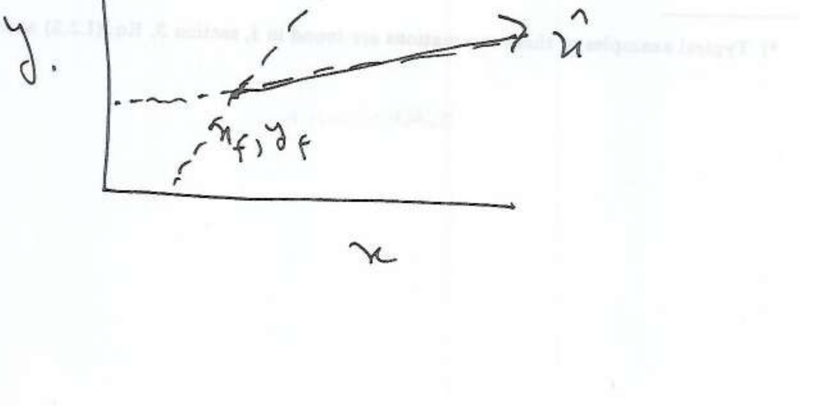
$$\therefore \; x - x_f = p\hat{x} + q\hat{y}$$

$$y - y_f = r\hat{x} + s\hat{y}$$

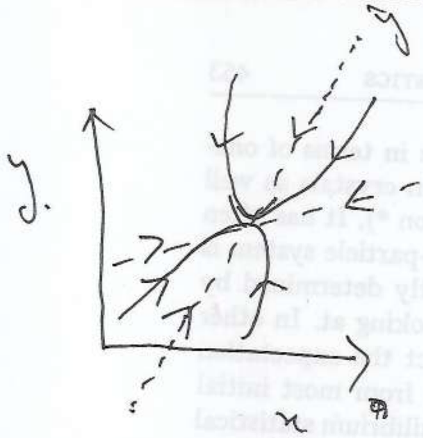
$\hat{x}$ axis $\Rightarrow \hat{y} = 0 \Rightarrow y - y_f = \frac{r}{p}(x - x_f)$

$\hat{y}$ axis $\Rightarrow \hat{x} = 0 \Rightarrow y - y_f = \frac{s}{q}(x - x_f)$

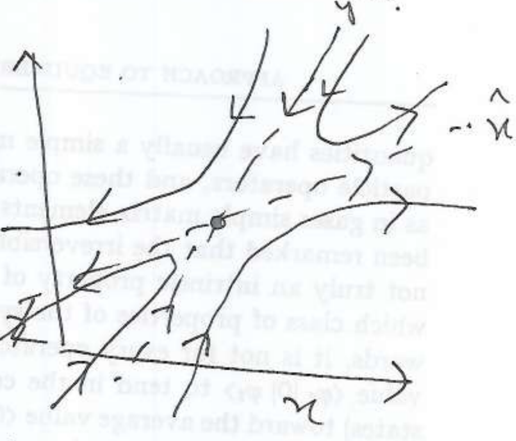
x-y plane



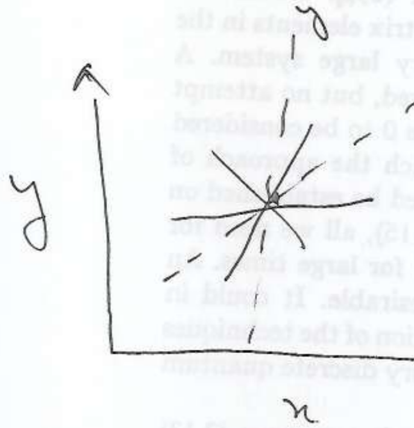
Phase Curves in original axes.



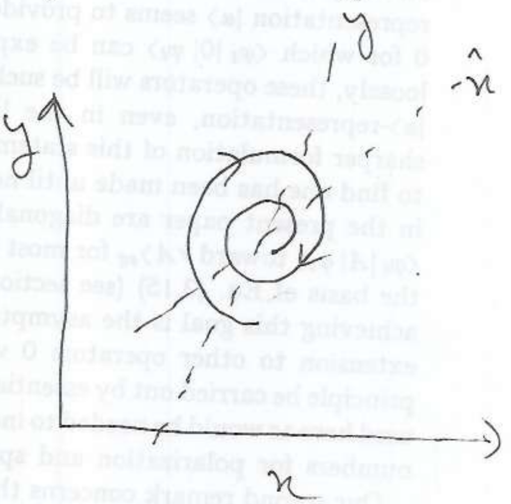
a. Stable node



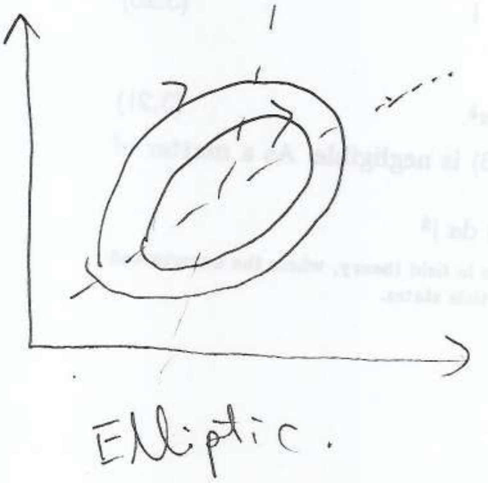
Hyperbolic fixed pt.



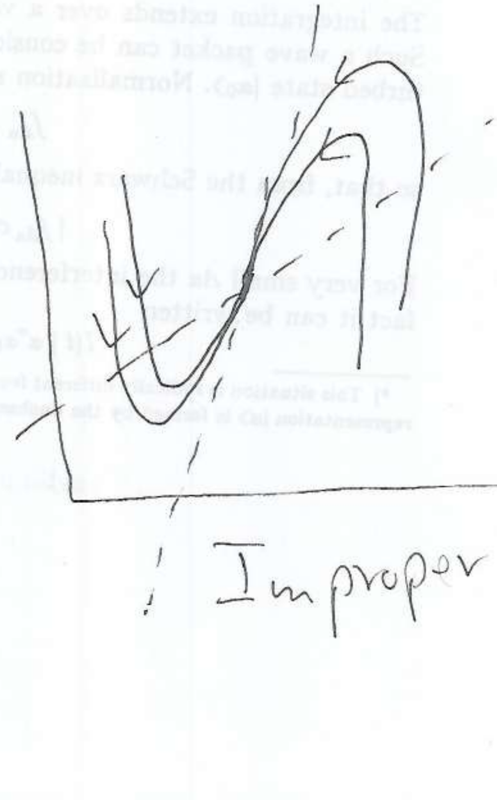
Star.



Spiral



Elliptic.



Improper

[redacted]

LECTURE-2

n^{th} order autonomous system.

$$\frac{dx^i}{dt} = F^i(\bar{x})$$

$\bar{x}_F$: Fixed point ⇒ $F^i(\bar{x}_F) = 0 \quad i = 1, 2 \ldots n.$

For small distances from the fixed point

$$\frac{dx^i}{dt} = \frac{\partial F^i}{\partial x^j}\Big|_F (x^j - x^j_F) + \cdots$$

$$\tilde{x}^i = x^i - x^i_F \qquad M^i_j = \frac{\partial F^i}{\partial x^j}\Big|_{x_F}$$

$$\frac{d\tilde{x}^i}{dt} = M^i_j \, \tilde{x}^j$$

In general we can bring M to the standard form

$A^{-1} M A \Rightarrow$ Diagonal (if possible)

$\det(M - \lambda I) = 0 \Rightarrow n$ eigenvalues

$\lambda_1, \lambda_2 \, \cdots \, \lambda_n$

Generic Case : (1) No two eigenvalues are equal

(2) No eigenvalue is 0.

Eigenvectors of M: $M\hat{a} = \lambda \hat{a}$

$$\Rightarrow M^i_j a^j = \lambda a^i$$

$$\Rightarrow M^i_j a^{j*} = \lambda^* a^{i*}$$

$\Rightarrow$ Complex eigenvalues come in pairs.

$\therefore$ General Case

$$\text{eval.} \begin{cases} 2p \text{ complex eigenvalues} : \alpha_{(1)} \pm i\omega_{(1)} \cdots \alpha_{(p)} \pm i\omega_{(p)} \\ n-2p \text{ real eigenvalues} : \lambda_{(1)}, \lambda_{(2)} \cdots \lambda_{(n-2p)} \end{cases}$$

$$\text{evec.} \begin{cases} b^i_{(1)} \pm i C^i_{(1)} \quad b^i_{(2)} \pm i C^i_{(2)} \cdots\cdots b^i_{(p)} \pm i C^i_{(p)} \\ a^i_{(1)}, a^i_{(2)}, \cdots a^i_{(n-2p)} \end{cases}$$

$$M^i_j a^j_{(k)} = \lambda_{(k)} a^i_{(k)} \quad 1 \le k \le n-2p$$

$$M^i_j \left(b^j_{(l)} \pm i C^j_{(l)} \right) = \left(\alpha_{(l)} \pm i\omega_{(l)} \right) \left(b^i_{(l)} \pm i C^i_{(l)} \right)$$

$$1 \le l \le p$$

Equating real and imaginary parts :

$$M^i_j b^j_{(l)} = \alpha_{(l)} b^i_{(l)} - \omega_{(l)} c^i_{(l)} \qquad 1 \le l \le p$$

$$M^i_j c^j_{(l)} = \omega_{(l)} b^i_{(l)} + \alpha_{(l)} c^i_{(l)}$$

$$\therefore M \begin{pmatrix} a^1_{(1)} & \cdots & a^1_{(n-2p)} & b^1_{(1)} & c^1_{(1)} & \cdots & b^1_{(p)} & c^1_{(p)} \\ a^2_{(1)} & & a^2_{(n-2p)} & b^2_{(1)} & c^2_{(1)} & & b^2_{(p)} & c^2_{(p)} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ a^n_{(1)} & & a^n_{(n-2p)} & b^n_{(1)} & c^n_{(1)} & & b^n_{(p)} & c^n_{(p)} \end{pmatrix}$$

$$= \underbrace{\left(\qquad \right)}_{A} \underbrace{\begin{pmatrix} \lambda_{(1)} & & & & & & & \\ & \ddots & & & & 0 & & \\ & & \lambda_{(n-2p)} & & & & & \\ & & & \alpha_{(1)} & \omega_{(1)} & & & \\ & & & -\omega_{(1)} & \alpha_{(1)} & & & \\ & 0 & & & & \ddots & & \\ & & & & & & \alpha_{(p)} & \omega_{(p)} \\ & & & & & & -\omega_{(p)} & \alpha_{(p)} \end{pmatrix}}_{D}$$

$$MA = AD \Rightarrow A^{-1}MA = D$$

Original equations

④

$$\frac{d\tilde{x}^i}{dt} = M^i_j \tilde{x}^j$$

or $$\frac{d\tilde{x}}{dt} = M\tilde{x} \qquad \tilde{x} = \begin{pmatrix} \tilde{x}^1 \\ \tilde{x}^2 \\ \vdots \\ \tilde{x}^n \end{pmatrix}$$

$$\therefore \frac{d}{dt} A^{-1}\tilde{x} = (A^{-1}MA)A^{-1}\tilde{x} = D\,A^{-1}\tilde{x}$$

Let $$A^{-1}\tilde{x} = \hat{x} = \begin{pmatrix} \hat{x}_{(1)} \\ \hat{x}_{(2)} \\ \vdots \\ \hat{x}_{(n-2p)} \\ \hat{y}_{(1)} \\ \hat{z}_{(1)} \\ \vdots \\ \hat{y}_{(p)} \\ \hat{z}_{(p)} \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} \hat{x}_{(1)} \\ \vdots \\ \vdots \\ \hat{z}_{(p)} \end{pmatrix} = D \begin{pmatrix} \hat{x}_{(1)} \\ \vdots \\ \vdots \\ \hat{z}_{(p)} \end{pmatrix}$$

$$\Rightarrow \quad \frac{d\hat{x}_{(1)}}{dt} = \lambda_{(1)}\hat{x}_{(1)}$$

$$\vdots$$

$$\frac{d\hat{x}_{(n-2p)}}{dt} = \lambda_{(n-2p)}\hat{x}_{(n-2p)}$$

$$\frac{d\hat{y}_{(1)}}{dt} = \alpha_{(1)}\hat{y}_{(1)} + \omega_{(1)}\hat{z}_{(1)}$$

$$\frac{d\hat{z}_{(1)}}{dt} = -\omega_{(1)}\hat{y}_{(1)} + \alpha_{(1)}\hat{z}_{(1)}$$

$$\vdots$$

$$\frac{d\hat{y}_{(p)}}{dt} = \alpha_{(p)}\hat{y}_{(p)} + \omega_{(p)}\hat{z}_{(p)}$$

$$\frac{d\hat{z}_{(p)}}{dt} = -\omega_{(p)}\hat{y}_{(p)} + \alpha_{(p)}\hat{z}_{(p)}$$

$\therefore \quad \hat{x}_{(k)} = \hat{x}_{(k)_0} e^{\lambda_{(k)} t} \qquad k = 1, 2 \ldots n-2p$

Let $\hat{y}_{(l)} = r_{(l)} \cos\theta_{(l)} \quad \hat{z}_{(l)} = r_{(l)} \sin\theta_{(l)}$

then $r_{(l)} = r_{(l)_0} e^{\alpha_{(l)} t}, \quad \theta_{(l)} = \theta_{(l)0} - \omega_{(l)} t$

$l = 1, 2 \ldots p$

The Fixed point is: ⑥

- ① Stable (strongly) iff $\lambda_{(k)} < 0 \quad \alpha_{(l)} < 0$ for all k, l

- ② Unstable if any $\lambda_{(k)} > 0$ or any $\alpha_{(l)} > 0$.

- ③ Stable if $\lambda_{(k)} < 0$ and $\alpha_{(l)} < 0$.

- ④ Some $\lambda_{(k)}, \alpha_{(l)} > 0$, Some other $\lambda_{(k)}, \alpha_{(l)} < 0$

- Generalization of Hyperbolic fixed points.

Note: ~~Under~~ Consider co-ordinate transformations,

$$y^i = \phi^i(\vec{x})$$

Then
$$\frac{dy^i}{dt} = \frac{\partial y^i}{\partial x^j}\frac{dx^j}{dt} = \frac{\partial \phi^i(\vec{x})}{\partial x^j} F^j(\vec{x}) = \tilde{F}^i(\vec{y})$$

At $\bar{y}_F = \phi(\bar{x}_F)$, clearly $\tilde{F}^i(\bar{y}_F) = 0$.

and
$$\tilde{M}^i_l = \frac{\partial \tilde{F}^i}{\partial y^l}\bigg|_{\bar{y}} = \frac{\partial \phi^i}{\partial x^j}\frac{\partial F^j}{\partial x^k}\frac{\partial x^k}{\partial y^l} + \left(\frac{\partial}{\partial y^l}\left(\frac{\partial \phi^i}{\partial x^j}\right)\right)\underset{\substack{\parallel \\ 0.}}{F^j\bigg|_{\bar{y}}}$$

$$= \frac{\partial y^i}{\partial x^j} M^j_k \frac{\partial x^k}{\partial y^l}.$$

⑦

Since $\frac{\partial y^i}{\partial x^j}\frac{\partial x^j}{\partial y^l} = \delta^i_l$.

$\therefore \tilde{M} = T M T^{-1}$

$\left(\therefore \tilde{M}^i_l = T^i_j M^j_k (T^{-1})^k_l.\right).$

$\therefore \tilde{M}$ and M have the same eigenvalues. and hence same nature of fixed points

(As expected)

Hamiltonian Systems.

$$\frac{dx^i}{dt} = F^i(\vec{x}) = \Omega^{ij}\frac{\partial H}{\partial x^j}, \quad 1 \leq i,j \leq n = 2N$$

with: (1) $H(\vec{x})$ is some function of $\vec{x}$

(2) $n = \text{even} = 2N$.

$$(3)\ \Omega = \begin{pmatrix} 0 & 1 & 0 & \\ -1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 \\ & & -1 & 0 \\ & & & & \ddots \end{pmatrix}$$

Let $$\begin{pmatrix} x^1 \\ x^2 \\ \cdot \\ \cdot \\ \vdots \\ x^{2N} \end{pmatrix} = \begin{pmatrix} q^1 \\ p^1 \\ q^2 \\ p^2 \\ \cdot \\ \vdots \\ q^N \\ p^N \end{pmatrix}$$

Then $$\frac{d}{dt}\begin{pmatrix} q^1 \\ \cdot \\ \cdot \\ \cdot \\ \vdots \\ p^N \end{pmatrix} = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & \ddots & \\ & & & \end{pmatrix} \begin{pmatrix} \partial H/\partial q_1 \\ \partial H/\partial p_1 \\ \cdot \\ \vdots \\ \partial H/\partial p_N \end{pmatrix}$$

$$\Rightarrow \quad \frac{dq^i}{dt} = +\frac{\partial H}{\partial p^i} \qquad \frac{dp^i}{dt} = -\frac{\partial H}{\partial q^i} \quad \Bigg\}$$ Hamilton's equations of motion.

Fixed point structure

$$\frac{dx^i}{dt} = \Omega^{ik}\frac{\partial H}{\partial x^k} = F^i$$

$$F^i(\bar{x}_F) = 0 \quad \Rightarrow \quad \frac{\partial H}{\partial x^j} = 0 \quad \text{at } \bar{x}_F .$$

$$M^i_K = \left.\frac{\partial F^i}{\partial x^k}\right|_{\bar{x}_F} = \Omega^{ij}\left.\frac{\partial^2 H}{\partial x^k \partial x^j}\right|_{\bar{x}_F}$$

$$\mathrm{Tr}\, M = M^i_i = \underset{\text{Asymmetric}}{\underset{\downarrow}{\Omega^{ij}}}\underset{\text{Symmetric}}{\underset{\downarrow}{\left.\frac{\partial^2 H}{\partial x^j \partial x^i}\right|_{\bar{x}_F}}} = 0.$$

⇒ Sum of eigenvalues of $M = 0$

Absolutely stable fixed point ⇒ Re(eigenvalue) < 0

∀ eigenvalue

e.Values $\lambda_{(1)} \cdots \lambda_{(m)}, \alpha_{(1)} \pm i\beta_{(1)} \cdots\cdots \alpha_{(p)} \pm i\beta_{(p)}$

$\lambda_{(k)} < 0, \quad \alpha_{(l)} < 0$ for all k, l

$\Rightarrow \sum \lambda_{(k)} + 2 \sum \alpha_{(l)} = 0$ for a Hamiltonian System.

Thus Hamiltonian Systems cannot have absolutely stable fixed point

Can get stable fixed point if all eigenvalues are purely imaginary

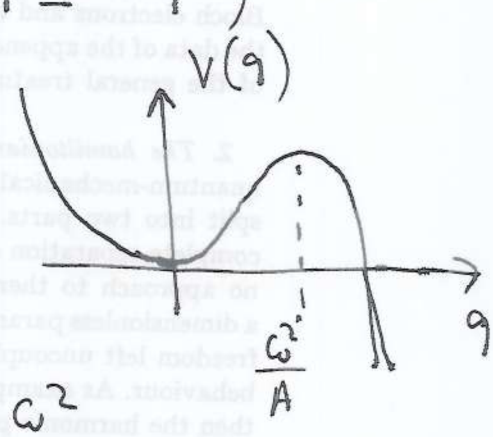
i.e $P = N$ $\alpha_{(l)} = 0$ for all l

e.Values $\Rightarrow \pm i\beta_{(1)} \cdots \pm i\beta_{(N)}$.

Two-dimensional Example.

$$H = \frac{p^2}{2m} + V(q) \qquad V(q) = \frac{1}{2}\omega^2 q^2 - \frac{1}{3}Aq^3$$

$$\frac{dq}{dt} = \frac{p}{m} \qquad \frac{dp}{dt} = -(\omega^2 q - Aq^2)$$



Fixed points:

$p = 0$

$\omega^2 q - Aq^2 = 0 \Rightarrow q = 0 \quad q = \frac{\omega^2}{A}$

$$M = \begin{pmatrix} \frac{\partial^2 H}{\partial q \partial p} & \frac{\partial^2 H}{\partial p^2} \\ -\frac{\partial^2 H}{\partial q^2} & -\frac{\partial^2 H}{\partial q \partial p} \end{pmatrix}_{x_F} = \begin{pmatrix} 0 & \frac{1}{m} \\ -\omega^2 + 2Aq_F & 0 \end{pmatrix}.$$

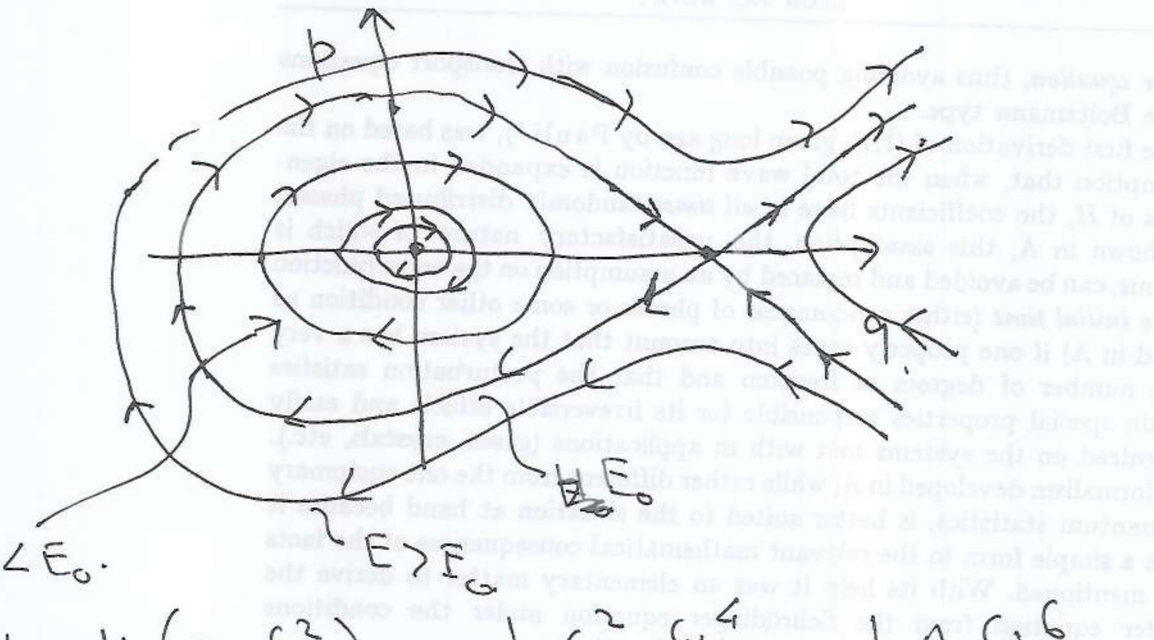
At $\bar{x}_F = (0, 0)$ $M = \begin{pmatrix} 0 & \frac{1}{m} \\ -\omega^2 & 0 \end{pmatrix}$ $\lambda = \pm i \frac{\omega}{\sqrt{m}}$

$\Rightarrow$ $\boxed{\text{Elliptic Fixed Point}}$

At $\bar{x}_F = (0, \frac{\omega^2}{A})$, $M = \begin{pmatrix} 0 & \frac{1}{m} \\ \omega^2 & 0 \end{pmatrix}$ $\lambda = \pm \frac{\omega}{\sqrt{m}}$

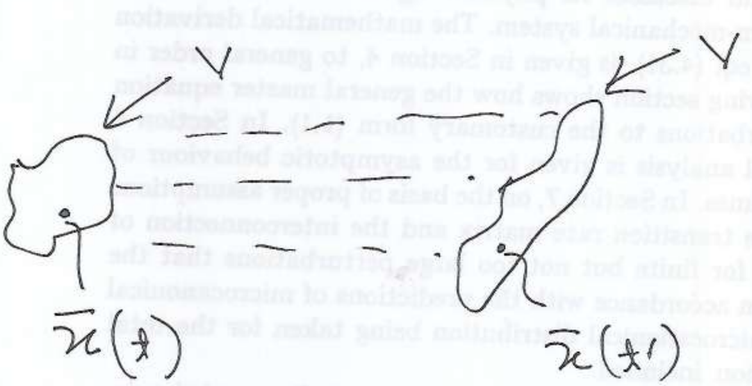
$\Rightarrow$ $\boxed{\text{Hyperbolic Fixed Point.}}$

Phase Curves.



$$E_0 = H\left(0, \frac{\omega^2}{A}\right) = \frac{1}{2}\omega^2 \cdot \frac{\omega^4}{A^2} - \frac{1}{3} A \frac{\omega^6}{A^3} = \frac{1}{6}\frac{\omega^6}{A^2}.$$

Liouville's Theorem.



$$V = \int_V \prod_i d\bar{x}^i(t) \qquad V' = \int_{V'} \prod_i dx^i(t')$$

$x^i(t)$ evolves to $x^i(t')$ under Hamiltonian dynamics.

②

$\therefore\ x^i(t')$: Function of $x^i(t)$ for fixed t, t'.

$$\therefore\ V' = \int_{V'} \prod_i dx^i(t') = \int_{V} J \prod_i dx^i(t)$$

Where $J(\bar{x}(t), t, t') = \det S$

$$S^i_j = \frac{\partial x^i(t')}{\partial x^j(t)}$$

For an infinitesimal volume δV

$$\delta V' = \int_{\delta V'} \prod_i dx^i(t') = J(\bar{x}(t), t, t') \int_{\delta V} \prod_i dx^i \quad ~~= J(\bar{x}, t, t') d~~$$

$$= J(\bar{x}(t), t, t')\, \delta V$$

↳ Ratio of final to initial small volume.

Let $t' = t + \delta t$

$$\frac{dx^i}{dt} = \Omega^{ij} \frac{\partial H}{\partial x^j} \Rightarrow x^i(t+\delta t) = x^i(t) + \Omega^{ij} \frac{\partial H}{\partial x^j} \delta t$$

$$\therefore\ \mathcal{S}^i_k\left(\bar{x}(t), t, t+\delta t\right) = \frac{\partial x^i(t+\delta t)}{\partial x^k(t)}$$

$$= \delta^i_k + \delta t\, \Omega^{ij} \underbrace{\frac{\partial^2 H}{\partial x^j \partial x^k}}_{A_k}$$

$$\therefore\ S = \begin{pmatrix} 1+\delta t A^1_1 & \delta t A^1_2 & \cdots & \\ \delta t\ A^2_1 & 1+\delta t A^2_2 & \delta t\ A^2_3 & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

$$J\left(\bar{x}(t), t, t+\delta t\right) = \mathrm{Det}\, S = 1 + \delta t\, \underset{\substack{\shortparallel \\ 0}}{\mathrm{Tr}\left[A\right]} + O(\delta t^2)$$

$$= 1 + O(\delta t^2)$$

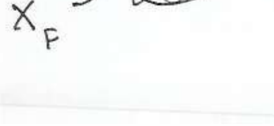
$$\therefore\ \frac{\delta V' - \delta V}{\delta V} = O(\delta t^2)$$

$$\frac{d}{dt}\left(\frac{\delta V'}{\delta V}\right) = 0. \quad \text{as} \quad \delta t \to 0.$$

$$\therefore\ \delta V = \text{Const.} \Big\}\ \text{Liouville's Theorem.}$$

Applications of Liouville's theorem.

(1) Absence of absolutely stable fixed point.

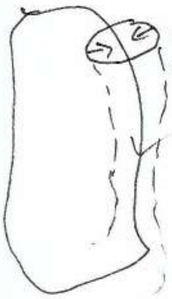


If $\bar{x}_F$ is absolutely stable

then δV evolves to $\mathrm{Vol}=0$ as $t\to\infty$. ⑭

This is not possible by Liouville's theorem.

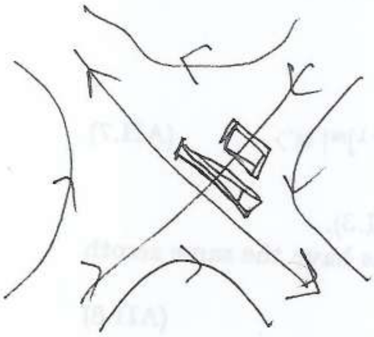
(2) Absence of absolutely stable limit cycle.



As $t\to\infty$ radial width $\to 0$

This contradicts Liouville's theorem

(3) Motion near a hyperbolic point.



(4) Equation for phase-space density.

$$\frac{d}{dt}(\rho\,\delta V)=0$$ conservation of total number of pts

$$\Rightarrow \frac{d\rho}{dt}=0 \quad \text{since } \frac{d}{dt}(\delta V)=0$$

$$\Rightarrow \frac{\partial\rho}{\partial t}=-\frac{\partial\rho}{\partial x^i}\dot{x}^i=-\frac{\partial\rho}{\partial x^i}\Omega^i_j\frac{\partial H}{\partial x^j}=-\{\rho,H\}$$

In general:

$$\frac{dx^i}{dt} = F^i(\vec{x})$$

$$\Rightarrow \quad x^i(t+\delta t) = x^i(t) + F^i(\vec{x})\,\delta t$$

$$\Rightarrow \quad \frac{\partial x^i(t+\delta t)}{\partial x^j(t)} = \delta^i_j + \frac{\partial F^i(\vec{x})}{\partial x^j}\,\delta t$$

$$\Rightarrow \quad J = 1 + \frac{\partial F^i}{\partial x^i}\,\delta t$$

$$\therefore \quad \cancel{d}\ \delta V(t+\delta t) = \delta V + \frac{\partial F^i}{\partial x^i}\,\delta t\,\delta V$$

$$\frac{d}{dt}\,\delta V(t) = \frac{\partial F^i}{\partial x^i}\,\delta V$$

$$\frac{d}{dt}(\rho\,\delta V) = 0$$

$$\Rightarrow \quad \frac{d\rho}{dt}\,\delta v + \rho\,\frac{d\,\delta v}{dt} = 0$$

$$\Rightarrow \quad \cancel{\frac{d\rho}{dt}}\ \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x^i}\,\dot{x}^i + \rho\,\frac{\partial F^i}{\partial x^i} = 0$$

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^i}(\rho F^i) = 0.$$

Continuity equation.