Black Holes and Large Order Quantum Geometry

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Topological String Theory

- Coupling of 2d gravity to matter (ST-geometry) underlies string perturbation theory

- TST describes the coupling of 2d gravity to topological matter (subsector)

- Frequently integrable and a large N laboratory to study non-perturbative completions of string and gauge theory

- Calculates exact terms in sugra effective action $\longrightarrow$ string phenomenology and 5d/4d black hole physics

Central objects: Partition function

$$Z(\underline{t}) = e^{F(\underline{t})} \ .$$

Free energy

$$\begin{aligned} F(t) \ &= \sum_g \lambda^{2g-2} F_g(t) \ , \\ F_{open}(t,u) \ &= \sum_{g,h} \lambda^{2g-2+h} t^h F_{g,h}(u) \ . \end{aligned}$$

Question: Is Z, Z_{open} governed by an integrable structure?

In the *simplest model* target space $M =$pt: *2d gravity*.

Large N Matrix model description: *Kontsevich model*.

Manifestation of the integrable structure:

- $F(t)$ is the $\tau(t)$ function of the *KdV hierarchy*.
- $Z(t)$ fulfills the *Virasoro constraints*

$$L_m Z(t) = 0, \quad m > -1.$$

- *Boundary conditions:* $\langle \tau_0 \tau_0 \tau_0 \rangle = 1$, $\langle \tau_1 \rangle = \frac{1}{24}$ and integrability fix $Z(t)$ completely.

Coupling 2d gravity to topological matter. Twisted A-model on M (Kaehler):

$$F(t) = \sum_{g,\beta \in H_2(M,\mathbb{Z})} \lambda^{2g-2} e^{\beta t} r^g_\beta \ ,$$

where

$$r^g_\beta = \int_{\overline{\mathcal{M}}_{g}(M,\beta)} c^{vir}(g,\beta,M) \in \mathbb{Q}$$

are the Gromov-Witten invariants. Symplectic invariant closely related to integer invariants such as Donaldson-Thomas and Gopakumar-Vafa invariants.

Grothendieck-Hirzebruch-Rieman-Roch

$$\dim\overline{\mathcal{M}}_g(M,\beta) = c_1(M)\cdot\beta + (\dim(M) - 3)(1-g) \geq 0$$

Special in this GHRR dimension formula are

- Calabi-Yau manifolds as $c_1(M) = 0$.
- complex 3-folds.
- the genus one amplitude.

as then $\dim\overline{\mathcal{M}}_g(M,\beta) = 0 \;\longrightarrow r_g^\beta \neq 0$: a point counting problem sometimes solvable by localization.

$r_g^\beta \neq 0$ Calabi Yau 4-folds relevant for M/F-theory compactifications

- GHRR $\rightarrow r_g^\beta \neq 0$ only for $g = 0, 1$. This sector is solved in arXiv:math.ag/0702189 xwith R. Pandharipande and new integer meeting invariants defined.

Calabi-Yau 3-folds are the critical case.

- GHRR $\rightarrow r_g^\beta \neq 0$, $\forall g$

		non–compact CY(toric)		compact CY (AS toric)
A–model	localisation Pandharipande, Graber, Zaslow, Liu, Katz	✓		g=0 Kontsevich Givental,Yau,Lian g>0 ?
	large N duality Vertex Aganagic,Klemm Marino, Vafa	✓		?
	Relative G–W Pandharipande	✓		in principle g small Pandharipande Okounkov, Gathman
B–model	large N duality Matrix model Aganagic,Klemm Marino, Vafa	✓		?
	DT Okounkov, Maulik,Nekrasov, Pandharipande	✓		? Pandharipande, Thomas announced
	Holomorphic anomaly this talk	✓		g=0 Candelas della Ossa, Green,Parkes g small Bershadski,Cecotti, Ooguri,Vaf Katz, Klemm, Vafa g>0 this talk
	KS–H Action BCOV, Pestun, Witten		g =0,1	

Heterotic –II duality

K3–Fiber g=0 KLM, g=1, Harvey, Moore, all g: Gava, Narain, Taylor, Marino, Moore, Klemm, Kreuzer, Riegler, Scheidegger, Grimm Weiss 07
Maulik Pandharipande 07

Improvement on the B-model solution in the critical case:

After proper incorporation of

- space-time modularity
- $N = 2$ low energy effective action constraints as boundary conditions

integrability is as good as in the 2d gravity toy model.

T. W. Grimm, A. Klemm, M. Marino, and M. Weiss, arXiv:hep-th/0702187.

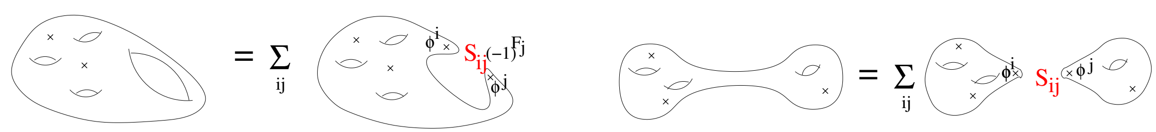
M. x. Huang, A. Klemm, and S. Quackenbush arXiv:hep-th/0612125.

M. Aganagic, V. Bouchard, and A. Klemm, hep-th/0607100.

M. x. Huang and A. Klemm, hep-th/0605195.

➪ *KdV hierarchy* $\longrightarrow$ *Holomorphic anomaly equations.*

$$
\begin{aligned}
\bar{\partial}_{\bar{k}} F_g &= \int_{\overline{\mathcal{M}}(g)} \partial\bar{\partial}\lambda \\
&= \tfrac{1}{2}\bar{C}^{ij}_{\bar{k}}\left(D_i D_j F_{g-1} + \textstyle\sum_{r=1}^{g-1} D_i F_r D_j F_{g-r}\right) \ .
\end{aligned}
$$



➪ *Virasoro constraints* $\longrightarrow$ *Wave function property of* Z

$$
\Big[\frac{\partial}{\partial \bar{T}^I} + \frac{i}{8}\bar{C}_I{}^{JK}\frac{\partial^2}{\partial X^J \partial X^K}\Big] Z(X,T,\bar{T}) = 0
$$

$$
\Big[\frac{\partial}{\partial T^I} + \frac{i}{2}X^J C_{IJ}{}^K \frac{\partial}{\partial X^K} + \frac{i}{2}C_{IJK}X^J X^K - \frac{i}{4}C_{IJ}{}^J\Big] Z = 0 \ .
$$

➪ *Boundary conditions* $\longrightarrow$ *Gap conditions* at bndr of moduli space.

The integrability equations come from factorization of *higher genus world-sheets*, but leaves

- an *holomorphic ambiguity* (functions)

- *s-t modularity* $\rightarrow$ *modular ambiguity* (discrete data)

- eventually fixed by *gap conditions*.

Implementation of interplay between w-s and s-t arguments requires

- an understanding of modular group Γ_M,

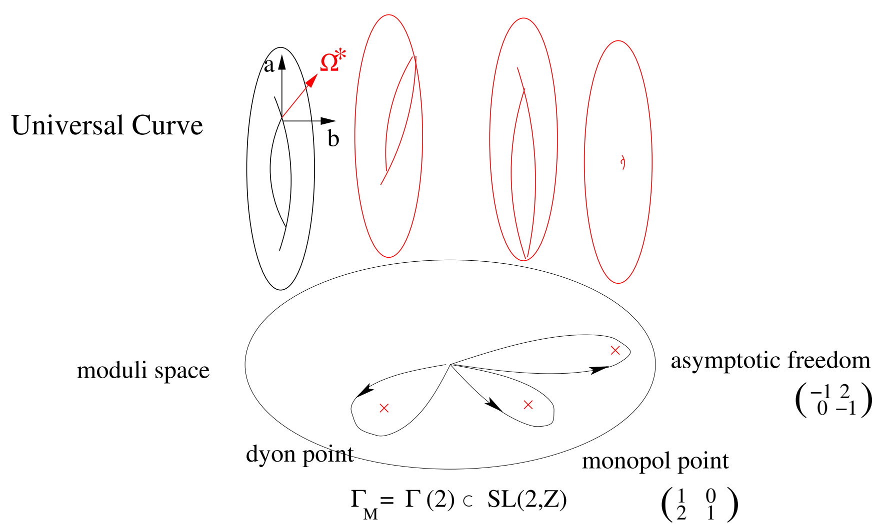
- control over the metaplectic transformation property of $Z(X,T,\bar{T})$ under Γ_M.

Easier the local case $\rightarrow$ discussed next as example.

Coupling Seiberg-Witten gauge theory to gravity HK

Geometric engineering realizes e.g. N=2 SU(2) as double scaling limit of TST on $\mathcal{O}(-2,-2) \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$.

The mirror is an elliptic curve with $\Gamma(2) \in \mathrm{SL}(2,\mathbb{Z})$ monodromy.

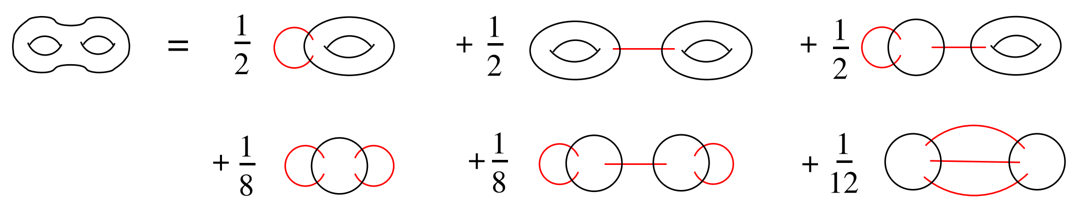


Modularity and WS degenerations:

➪ $F_g(\tau,\bar\tau)$ invariant under $\Gamma_M = \Gamma(2)$, e.g.

$$F_1 = -\log(\sqrt{\mathrm{Im}(\tau)}\eta\bar\eta)$$

➪ degenerations cap. by Feynmann rules:



➪ 'Propagator' transforms as form of weight 2 (derivative)

$$ = S = \frac{\partial}{\partial\tau}2F_1 = \frac{1}{12}\left(E_2 - \frac{3}{\pi\mathrm{Im}\tau}\right) =: \hat{E}_2$$

➪ $$F_g(\tau,\bar\tau)=\xi^{2g-2}\sum_{k=0}^{3(g-1)}\hat E_2^k(\tau,\bar\tau)c_k^{(g)}(\tau)=:\xi^{2g-2}f_g\ ,x$$

where $\xi=\frac{\theta_2^2}{1728\theta_3^4\theta_4^4}=\frac{1}{F_{aaa}^{(0)}}$ is of weight -3.

➪ Invariance means mathematically

$$\boxed{f_g\in\hat{\mathcal{M}}_{6(g-1)}(\hat E_2,\Delta,h)}$$

the *ring* of *almost holomorphic functions* of $\Gamma(2)$ of weight $6(g-1)$ *finitely generated* by

$$\boxed{(\hat E_2, h=\theta_2^4+2\theta_4^4, \Delta=\theta_3^4\theta_4^4)\ .}$$

Direct integration:

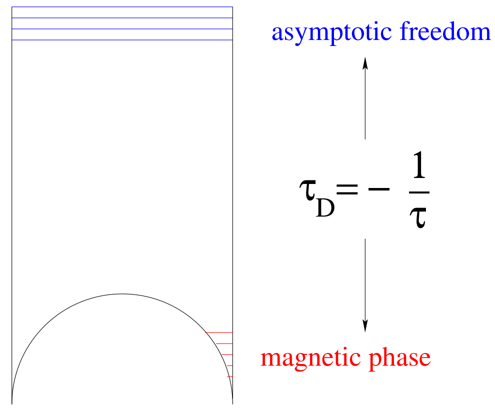
The only antiholomorphic dependence is in the $S \propto \hat{E}_2$:
$\frac{\partial}{\partial \bar{\tau}} \longrightarrow \frac{\partial}{\hat{E}_2}$:

$$\begin{aligned} \frac{1}{24^2}\frac{\mathrm{d}}{\mathrm{d}\hat{\mathrm{E}}_2} f_g &= d_\xi^2 f_{g-1} + \frac{1}{3}\frac{(\partial_\tau \xi)}{\xi} d_\xi f_{g-1} + \sum_{r=1}^{g-1} d_\xi f_r d_\xi f_{g-r}, \\ \text{with} \quad d_\xi f_k &= \partial_\tau f_k + \frac{k}{3}\frac{(\partial_\tau \xi)}{\xi} f_k \quad \text{Serre operator} \end{aligned}$$

- ➪ Only the degree 0 part in $\hat{E}_2$ remains undetermined. Ambiguity is a holomorphic modular form $c_0^{(g)}(\tau) \in \mathcal{M}_{6(g-1)}(\Delta, h)$.

- ➪ $\dim \left(\mathcal{M}_{6(g-1)}(h, \Delta)\right) = \left[\frac{3g}{2}\right]$ number of required boundary conditions

Global properties:



$\mathbb{F}(\Gamma(2))$

$$F_g^D(\tau_D, \bar{\tau}_D) = F_g(-\frac{1}{\tau_D}, -\frac{1}{\bar{\tau}_D})$$

- ST-instanton expansion

$$\mathcal{F}_g(\tau(a)) = lim_{\bar{\tau}\to\infty} F_g(\tau, \bar{\tau})$$

- Strong-coupling expansion

$$\mathcal{F}_g^D(\tau_D(a_D)) = lim_{\bar{\tau}_D\to\infty} F_g^D(\tau_D, \bar{\tau}_D)$$

Can be seen as metaplectic transformation on $\Psi = Z$

The strong coupling gap :

$$\mathcal{F}_g^D = \frac{B_{2g}}{2g(2g-2)a_D^{2g-2}} + \ldots + k_1^{(g)} a_D + \mathcal{O}(a_D^2)$$

$$\uparrow$$

$$2g-2 \text{ independent vanishing conditions}$$

$$\boxed{2g-2 > \left[\frac{3g}{2}\right]}$$

➪ theory completly solved

- Other methods: W-S Instanton localisation or vertex, S-T Inst. localisation Nekrasov, Flume, Poghossian, Nakajima, Okounkov, Yoshioka,. . .

- are perturbatively defined near $\frac{1}{a} = 0$. First coefficients check and confirm the gap.

Why the Gap ?

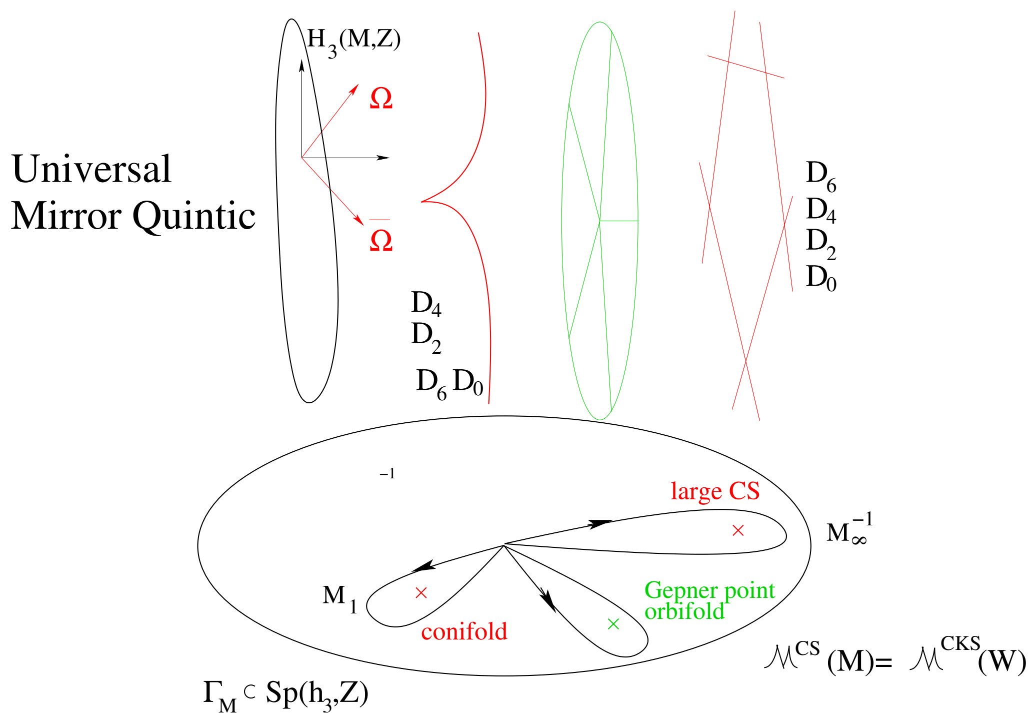
- Dijkgraaf & Vafa: SW is described by a matrix model: Typical in MM is a pole $\frac{1}{s^{2g-2}}$ from the measure followed by a regular perturbative expansion.

- String LEEA explanation: $F(\lambda, t)$ graviphoton couplings given by Schwinger-Loop calculation Antoniadis, Gava, Narain, Taylor, Gopakumar, Vafa. For one HM at conifold Strominger t_D mass of HM

$$F(\lambda, t_D) = \int_{\epsilon}^{\infty} \frac{\mathrm{d}s}{s} \frac{e^{-st_D}}{4\sin^2(s\lambda/2)} = \sum_{g=2}^{\infty} \left(\frac{\lambda}{t_D}\right)^{2g-2} \frac{(-1)^{g-1} B_{2g}}{2g(2g-2)} \; .$$

Compact Calabi-Yau HKQ

$$W = \sum_{i=1}^{5} x_i^5 - j_q^{\frac{1}{5}} \prod_{i=1}^{5} x_i = 0 \in \mathbb{P}^4,$$



$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 5 & -3 & 1 & -1 \\ -8 & -5 & 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_\infty^{-1} = \begin{pmatrix} -4 & 3 & -1 & 1 \\ 1 & 1 & 0 & 0 \\ 5 & -3 & 1 & -1 \\ 8 & -5 & 0 & 1 \end{pmatrix}.$$

generate a discrete subgroup of $\Gamma_M = \mathrm{Sp}(4, \mathbb{Z})$ acting on $H^3(W, \mathbb{Z})$ on periods $\Pi(z) = \int_\Gamma \Omega(z)$ fullfilling

$$[\theta^4 - 5 j_q^{-1} \prod_{i=1}^{4} (\theta + i)]\, \Pi(z) = 0, \qquad \theta := -j_q \frac{\mathrm{d}}{\mathrm{dj_q}}.$$

Properties of Γ_M, even if of finite index unknown, but we can build modular objects using the periods and special geometry.

E.g. from the mirror map an analog of j-function, $q = \exp(\int_C \omega) = \exp(\Pi_1(j_q)/\Pi_0(j_q))$

$$\begin{aligned} j_q &= \tfrac{1}{q} + 770 + 421375\, q + 274007500\, q^2 + 236982309375\, q^3 + \ldots \\ (j_e &= \tfrac{1}{q} + 744 + 196884\, q + 21493760\, q^2 + 864299970\, q^3 + \ldots) \end{aligned}$$

Regularity at the Gepner point is maintained if we introduce $P_g = \xi^{g-1} F_g$, where $\xi = \frac{j_q}{1-j_q} = j_q X$. From the *gap behaviour* of the F_g at the conifold $j_q = 1$ and from regularity at the large CS, we conclude that the *holomorphic* and *modular ambiguity* in P_g is given by

$$\boxed{c_0^{(g)} = \sum_{i=0}^{3g-3} a_i X^i}$$

Intermediate Jacobians

- We have *invariant coordinates* parametrizing $\mathcal{M}$

$$j_k, \quad k = 1, \ldots h_{21}$$

- Local *projective coordinates*, suitable for the Wilsonian action

$$t^k = \frac{\Pi_k}{\Pi_0}, \quad k = 1, \ldots h_{21}$$

There two maps ϕ_h and ϕ_{ah} from $\mathcal{M}$ to two *intermediate*

Jacobians $\mathcal{J}_{GH}$ and $\mathcal{J}_W$

$$\phi_h : \quad \mathcal{M} \to \tau_{IJ} = \frac{\partial^2 F_0}{\partial X^I \partial X^J}$$
$$\phi_{ah} : \quad \mathcal{M} \to \mathcal{N}_{IJ} = \bar{\tau}_{IJ} + 2i\frac{\mathrm{Im}\tau_{IK} X^K \mathrm{Im}\tau_{IL} X^L}{X^L \mathrm{Im}\tau_{KL} X^K},$$

X^I, $I = 1, \ldots h_{21} + 1$ a choice of A-periods.

➪ τ_{IJ} correponds to C.C. on $H_3(M, \mathbb{Z})$ and $\mathrm{Im}\tau_{IJ}$ has signature $(1, h_{21})$

➪ $\mathcal{N}_{IJ}$ correponds to $*$ on $H_3(M, \mathbb{Z})$ and $\mathrm{Im}\mathcal{N}_{IJ}$ has signature $(h_{21} + 1)$

$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(h^3, \mathbb{Z})$. acts naturally on both
$\tilde{\tau} = (A\tau + B)(C\tau + D)^{-1}$, $\tilde{\mathcal{N}} = (A\mathcal{N} + B)(C\mathcal{N} + D)^{-1}$.

On $\mathcal{J}_{GH}$ the formalism to treat an-holomorphic part generalizes immediatly from the local case ABK,GKMW

$$F_1 = -\frac{1}{2}\log\det \mathrm{Im}\tau_{IJ} - \log|\Phi(\tau)| + f_1 + \bar{f}_1$$

- $E_2(\tau)$ generalizes to $E^{IJ}(\tau) = -i\frac{\partial\Phi}{\partial\tau_{IJ}}$.
- $\hat{E}_2(\tau,\bar{\tau})$ generalizes to $\hat{E}_2^{IJ}(\tau,\bar{\tau}) = E^{IJ}(\tau) - \frac{1}{2}\mathrm{Im}\tau^{IJ}$ transforming as tensor-form under $\mathrm{SP}(h^3,\mathbb{Z})$.
- $F_g = \sum_{k=0}^{3g-3} \hat{E}^{I_1J_1}\ldots\hat{E}^{I_kJ_k} c^{(g)}_{I_1J_1\ldots I_kJ_K}$

- The an-holomophic part is related to the 'propagators' of BCOV S, S^i, S^{ij} as

$$\hat{E}_2^{IJ}(\tau,\bar{\tau}) = (X^I \chi_i^J) \begin{pmatrix} S & -S^i \\ -S^i & S^{ij} \end{pmatrix} \begin{pmatrix} X^J \\ \chi_j^J \end{pmatrix}$$

The generators of the ring of almost holomorphic modular tensor forms of Γ_M are not known, but Yau, Yamaguchi `hep-th/0406078`, showed following BCOV that the F_g can be written as polynomials in 3

an-holomophic and one holomorphic generator

$$A_p \ := \frac{(j\partial_j)^p G_{j,\bar{j}}}{G_{j\bar{j}}}, \quad B_p := \frac{(j\partial_j)^p e^{-K}}{e^{-K}}, \quad p = 1, \ldots$$
$$C : \ = C_{jjj} j^3, \quad X = \frac{1}{1-j}$$

➪ Special geometry & Picard-Fuchs eq. truncate to A_1, B_1, B_2, B_3, X.

➪ One combination does not appear in $P_g = C^{g-1} F_g$. $B_1 = u$, $A_1 = v_1 - 1 - 2u$, $B_2 = v_2 + uv_1$, $B_3 = v_3 - uv_2 + uv_1 X - c_1 uX$

➪ The P_g are degree $3g - 3$ weighted inhomogeneous polynomials in v_1, v_2, v_3, X,

➪ hol. anom. eq.

$$(\partial_{v_1} - u\partial_{v_2} - u(u+X)\partial_{v_3})\, P_g = -\frac{1}{2}\left(P^{(2)}_{g-1} + \sum_{r=1}^{g-1} P^{(1)}_r P^{(1)}_{g-r}\right)$$

Boundary conditions:

➪ Gap at the conifold $j = 1$

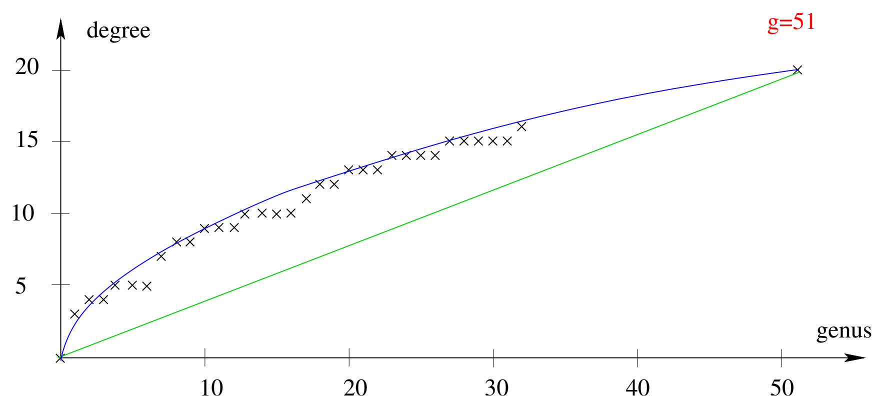
$$\mathcal{F}_g^D = \frac{B_{2g}}{2g(2g-2)t_D^{2g-2}} + k_g^1 + \mathcal{O}(t_D)$$

provides $2g-2$ conditions.

➪ Regularity at Gepner point $j = 0$ provides $\left[\frac{3(g-1)}{5}\right]$

conditions $\longrightarrow \left[\frac{2(g-1)}{5}\right]$ unkowns.

➪ Castelnouvo's bound for GV invariants at large radius. From aqjunction formula in $\mathbb{P}^4$ ones find there are no genus g curves for $d \leq \sqrt{g}$



genus	degree=18
0	144519433563613558831955702896560953425168536
1	491072999366775380563679351560645501635639768
2	826174252151264912119312534610591771196950790
3	866926806132431852753964702674971915498281822
4	615435297199681525899637421881792737142210818
5	306990865721034647278623907242165669760227036
6	109595627988957833331561270319881002336580306
7	28194037369451582477359532618813777554049181
8	5218039400008253051676616144507889426439522
9	688420182008315508949294448691625391986722
10	63643238054805218781380099115461663133366
11	4014173958414661941560901089814730394394
12	166042973567223836846220100958626775040
13	4251016225583560366557404369102516880
14	61866623134961248577174813332459314
15	451921104578426954609500841974284
16	1376282769657332936819380514604
17	1186440856873180536456549027
18	2671678502308714457564208
19	-59940727111744696730418
20	1071660810859451933436
21	-13279442359884883893
22	101088966935254518
23	-372702765685392
24	338860808028
25	23305068
26	-120186
27	-5220
28	-90
29	0

5d Black hole microstate counting

M. x. Huang, A. Klemm, M. Marino, and A. Tavanfar, arXiv:0704.2440 [hep-th].

➪ macroscopic description of 5d black holes Beckenrigde, Myers,..

$$S_0 = 2\pi\sqrt{\mathcal{Q}^3 - m^2},$$

$\mathcal{Q}$ graviphoton charge of the black hole.

➪ This charge is fixed by attractor mechanism Strominger, Ferrara,..

$$\mathcal{Q} = \left(\frac{2}{9\kappa}\right)^{\frac{1}{3}} d.$$

➪ R^2 term in Wald's formula one expects contribution

$$S_g \sim \chi \mathcal{Q}^{\frac{3}{2}-g}$$

microscopic entropy of 5d spinning BH conjectured by

Katz,Klemm, Vafa

$$S(d,m) = \log(\Omega(Q,m)). \tag{1}$$

with

$$\Omega(d,m) = \sum_r \binom{2r+2}{m+r+1} n_d^r. \tag{2}$$

This should agree with the macroscopic result in the large charge limit $Q = d \gg 1$ and $\mathcal{Q} = d \gg m$.

- $J = m = 0$

$$f(d) = \frac{\log(\Omega(d,0))}{d^{\frac{3}{2}}} \rightarrow \frac{4\pi}{3\sqrt{2\kappa}}$$

agree within 3%.

- For $J = m = 1$ the agreement is in 10% range. Microscopic perdiction lower.
- We find indications that the R^2 contributions is confirmed in micro counting
- Indictions that the Dennef, Moore scaling $k \sim 2$

bi-cubic in $\mathbb{P}^5$: