The Geometry of Rational Conformal Field Theories

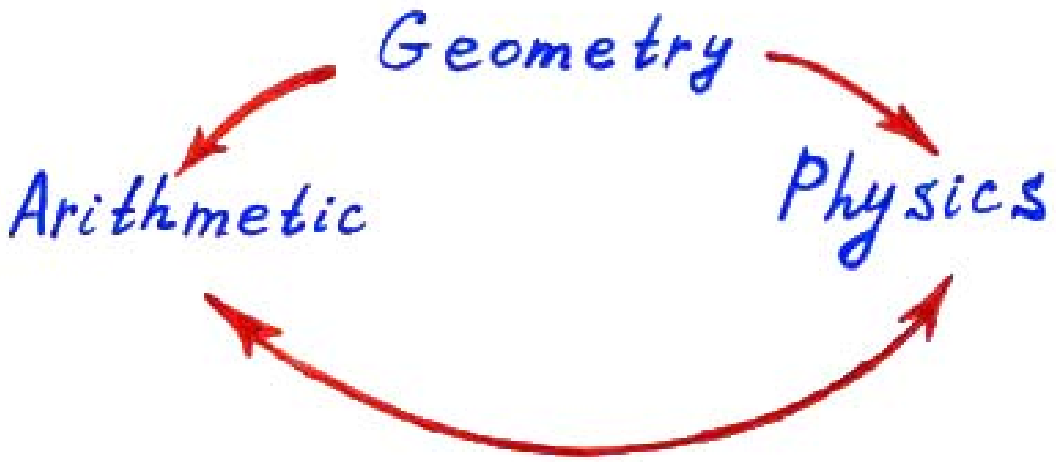
S.G. and C. Vafa

hep-th/0203213

Michael Atiyah (Bonn, 1984)

Commentary on the Article of Y. Manin:

"In recent years there has been a remarkable resurgence of the traditional links between mathematics and physics. A number of striking ideas and problems from theoretical physics have penetrated into various branches of mathematics, including areas such as algebraic geometry and number theory which are rarely disturbed by such outside influences.... The picture is best described by the following schematic diagram: "



Related Ideas:

C. Borcea '92

B.H. Lian, S.-T. Yau '96

P. Deligne '97

G. Moore '98

S.D. Miller and G. Moore '99

K. Wendland '00

P. Candelas, X. de la Ossa and F. Rodriguez-Villegas '00

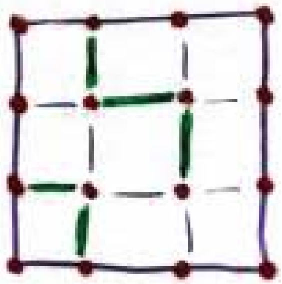
R. Schimmrigk '01

Y. Manin, M. Marcolli '02

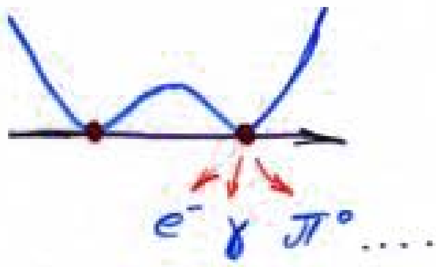
S. Kachru, M. Schulz, S. Trivedi '02

Conformal Field Theories

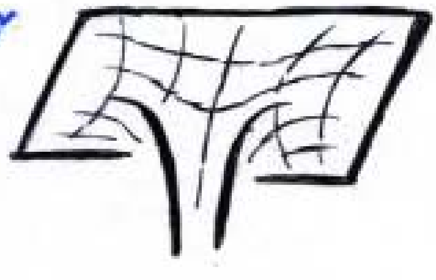
- Critical Phenomena in Statistical Mechanics



- Ground States in String Theory



- Dual to Gravity Backgrounds with $\Lambda < 0$ (AdS / CFT)



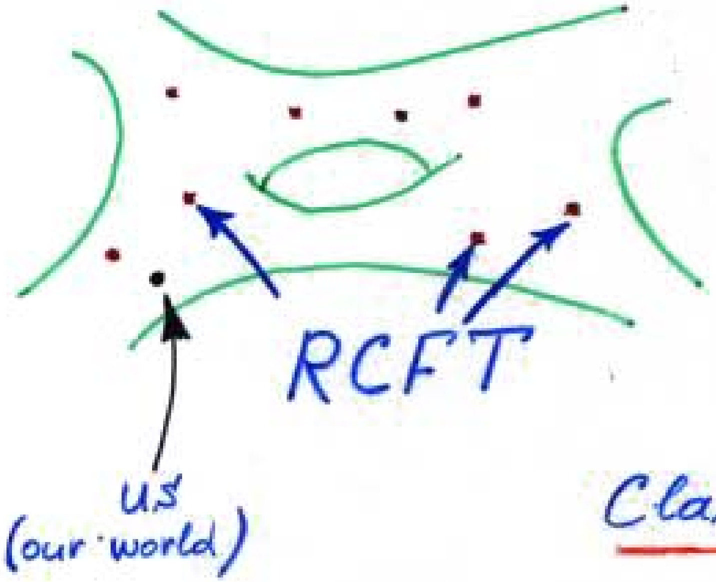
- Mathematical Interest (VOA's, modular tensor categories, etc.)

G. Segal
J. Fuchs, C. Schweigert

Rational Conformal Field Theories

Roughly speaking, .

Rational Theory ~ Exactly Solvable CFT



Moduli Space of Two-dimensional Conformal Field Theories

Classification of RCFT's?

Conjecture (D. Friedan & S. Shenker):

Rational Conformal Field Theories are Dense!

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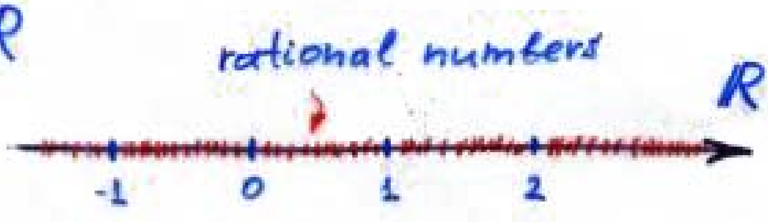
cf. $\mathbb{Q} \subset \mathbb{R}$



$\mathbb{R} = \left\{ \begin{matrix} \text{real} \\ \text{numbers} \end{matrix} \right\}$

RCFT (Algebraic Aspects)

* Chiral Algebras $\mathcal{A}$ and $\bar{\mathcal{A}}$ $(= \mathcal{A})$

$$\boxed{\text{Virasoro} \subseteq \mathcal{A}}$$

Virasoro algebra : $[L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12} n(n^2-1) \delta_{n+m}$

* Hilbert Space:

$$\boxed{\mathcal{H} = \oplus M_{j\bar{j}} V_j \otimes V_{\bar{j}}} \quad M_{j\bar{j}} \in \mathbb{Z}_{\geq 0}$$

Rationality $\Longrightarrow$ the spectrum of irreps V_j is finite dimensional, i.e. the exponent set $I := \{j\}$ is finite.

* One-loop Partition Function :

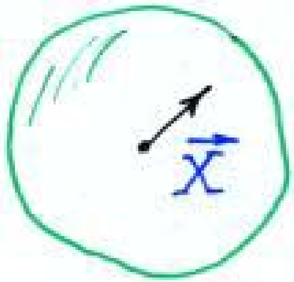
$$\boxed{Z(q, \bar{q}) = \sum M_{j\bar{j}} \chi_j(q) \bar{\chi}_{\bar{j}}(\bar{q})}$$

where

$$\chi_j(q) = tr_{V_j} q^{L_0 - c/24}$$

Conformal Sigma-Models

"target space" manifold M $\rightarrow$ two-dimensional field theory, σ-model:



$$S = \int d^2z \; G_{mn}(X)\, \partial X^m \bar{\partial} X^n$$

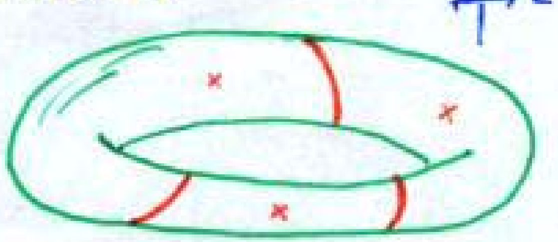
two-dimensional scaling invariance: $\Rightarrow$ Ricci-flat manifold M:

$\beta = 0$ $\qquad$ $R = 0$

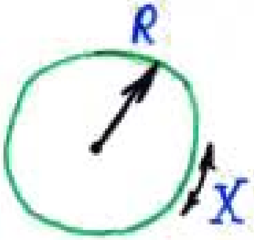
* Consider (super-)conformal field theory corresponding to sigma-model on a Ricci-flat (Calabi-Yau) space M

Q: When is such CFT rational?

Q: Geometric interpretation of Cardy states?



A Simple Example: $c=1$ CFT



Q: For what values of R the theory is rational?

$$\boxed{Z_R = \frac{1}{|\eta|^2} \sum q^{\frac{1}{2}p^2} \cdot \bar{q}^{\frac{1}{2}\bar{p}^2}}$$

$(p, \bar{p}) \in \Gamma^{1,1}$

$$\begin{cases} p = \frac{1}{\sqrt{2}}\left(\frac{n}{R} + mR\right) \\ \bar{p} = \frac{1}{\sqrt{2}}\left(\frac{n}{R} - mR\right) \end{cases}$$

A: CFT is rational when

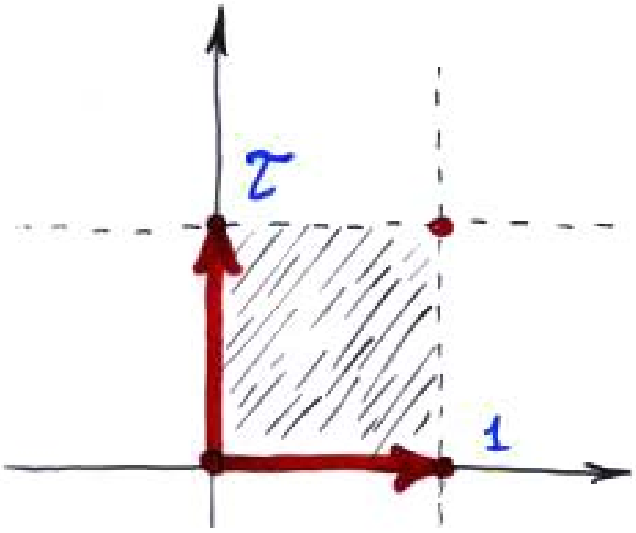
$$\boxed{R^2 = \frac{k}{l}} \qquad k, l \in \mathbb{Z}$$

Note, $R^2 \in \mathbb{Q}$ are dense in $\mathbb{R}^+$.

$I = \{1, 2, \ldots 2N\} \qquad N = k \cdot l$

$V_j \star V_k = V_{j+k} \rightsquigarrow \mathbb{Z}_{2N}$ fusion algebra

* $E = S^1_{R_1} \times S^1_{R_2}$



$\mathrm{CFT} = \mathrm{CFT}_1 \times \mathrm{CFT}_2$

This CFT is rational if and only if :

$$R_1 = \sqrt{\frac{k_1}{l_1}} \quad \text{and} \quad R_2 = \sqrt{\frac{k_2}{l_2}}$$

$$\tau = i\sqrt{\frac{k_1\, l_2}{k_2\, l_1}} \quad , \quad \rho = i\sqrt{\frac{k_1 k_2}{l_1\, l_2}}$$

$$\boxed{(k_2 l_1)\cdot\tau^2 + (k_1 l_2) = 0 \quad , \quad (l_1 l_2)\cdot\rho^2 + (k_1 k_2) = 0}$$

$D = -4 k_1 k_2 l_1 l_2$

In general,

$$\boxed{a\tau^2 + b\tau + c = 0} \qquad \begin{array}{l} a, b, c \in \mathbb{Z} \\ \gcd(a,b,c) = 1 \end{array}$$

with the discriminant:

$$\underline{D = b^2 - 4ac}\ , \qquad D < 0$$

Criterion for Rationality:

$$\boxed{RCFT \Longleftrightarrow \tau, \rho \in \mathbb{Q}(\sqrt{D})}$$

Imaginary Quadratic Number Fields

* Let's start with real numbers:

$$\alpha, \beta \in \mathbb{R}$$

$\Rightarrow$ then, we can construct complex numbers by introducing $i = \sqrt{-1}$:

$$\alpha + \beta \cdot i \in \mathbb{C}$$

* Remark: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$ are number fields.

* Similarly, we can construct imaginary quadratic number field $K = \mathbb{Q}(\sqrt{D})$ by supplementing $\mathbb{Q}$ with $\sqrt{D}$, $D < 0$

$$\alpha, \beta \in \mathbb{Q}$$

$$\alpha + \beta \cdot \sqrt{D} \in K = \mathbb{Q}(\sqrt{D})$$

$D < 0$ is the discriminant of K

Complex Multiplication (CM)

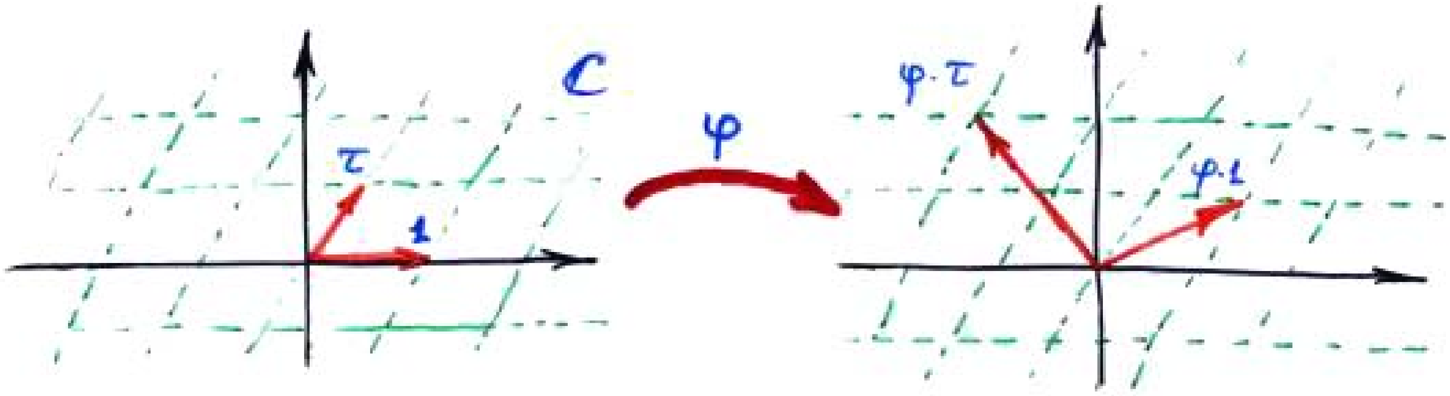
Consider endomorphisms of $E = \mathbb{C}/_{\mathbb{Z}\oplus\tau\mathbb{Z}}$

$$\boxed{\varphi : E \to E}$$

φ a finite degree map:

$$z \in \mathbb{C} \qquad\qquad \Rightarrow \qquad \begin{cases} \varphi\cdot 1 = m_1 + n_1\tau \\ \varphi\cdot\tau = m_2 + n_2\tau \end{cases}$$

$$\varphi : z \mapsto \varphi\cdot z$$



Notice, $\varphi \in \mathbb{Z}$ is always a (trivial) endomorphism.

In general, $\{\varphi\} = \boxed{End(E) = \begin{cases} \mathbb{Z} & \text{, no CM} \\ \mathbb{Z} + a\tau\mathbb{Z} & \text{CM} \end{cases}}$

$$CM \iff a\tau^2 + b\tau + c = 0$$

Classification of $c=2$ Rational Conformal Sigma Models

Rational Theory $\Longleftrightarrow$ E has complex multiplication, $\tau, \rho \in \mathbb{Q}(\sqrt{D})$

RCFT and Higher Dimensional Calabi-Yau

Rational Theory $\Longleftrightarrow$ Complex Multiplication

Conjecture: Sigma-model on Calabi-Yau M is rational if and only if M and its mirror W admit complex multiplication over the same # field.

* This criterion agrees with all Known examples:

  ✓ Complex Tori

  ✓ Toroidal Orbifolds

  ✓ Gepner Models $\longleftrightarrow$ Fermat Polynomials, e.g.

$$M: \quad z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 = 0 \quad \text{in } \mathbb{CP}^4$$

$(k=3)^5$ Gepner model

C. Borcea '92

T. Shioda '82

P. Deligne '82 '97

K. Wendland '00

Are CM points dense?

Generalized André-Oort Conjecture:

In order for a (sub)family of algebraic varieties to contain a dense set of CM-points, the corresponding moduli space has to be Shimura (sub-)variety, e.g.

$$SO(20,4;\mathbb{Z})\backslash SO(20,4)/SO(20)\times SO(4)$$

Manifold	dense CM-points
Complex Tori	+
K3	+
Calabi-Yau	−


PLEASE TAKE A NUMBER
3π
SEVEN!
$\sqrt{67}$
BLAIR

Open Problems

* Why Complex Multiplication?
  - Target Space Interpretation
  - Compactification with Fluxes
  - More Examples

* Families of Calabi-Yau Manifolds with Dense Sets of Rational / CM Points
  - K3 fibrations

* Geometric / Arithmetic Interpretation of Cardy States
  - K-theory

* Three-Dimensional Analogues / TQFT