Operator spectrum and spontaneous symmetry breaking in SYK-like models

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Talk based on:

J. Kim, I. R. Klebanov, GT, W. Zhao, PRX, 9, (2019)

I. R. Klebanov, A. Milekhin, GT, W. Zhao, JHEP, 162, (2020)

M. Tikhanovskaya, H. Guo, S. Sachdev, GT, PRB, 103, (2021)

Plan:

- Introduction to Random SYK Models

- Operators in SYK models and their effects

- Coupled Majorana and complex SYK models and symmetry breaking

Introduction to Random SYK Models

- A general non-relativistic Hamiltonian of interacting particles:

$$H = \sum_a H_a^{(1)}(\mathbf{r}_a) + \sum_{a,b} U^{(2)}(\mathbf{r}_a, \mathbf{r}_b) + \sum_{a,b,c} U^{(3)}(\mathbf{r}_a, \mathbf{r}_b, \mathbf{r}_c) + \dots$$

where $H_a^{(1)} = -\frac{\Delta_a}{2m} + U(\mathbf{r}_a)$

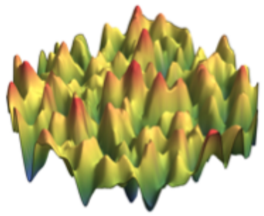
- In the second quantization representation it has the form (assume fermions)

$$H = \sum_i \varepsilon_i c_i^\dagger c_i + \sum_{i,j,k,l} J_{ij,kl} c_i^\dagger c_j^\dagger c_k c_l + \dots \qquad \{c_i, c_j^\dagger\} = \delta_{ij}$$

where $J_{ij,kl} = \int d\mathbf{r}_a d\mathbf{r}_b \varphi_i^*(\mathbf{r}_a)\varphi_j^*(\mathbf{r}_b) U^{(2)}(\mathbf{r}_a, \mathbf{r}_b)\varphi_k(\mathbf{r}_a)\varphi_l(\mathbf{r}_b)$

And ε_i , $\varphi_i(\mathbf{r})$ eigenvalues and eigenfunctions of $H^{(1)}$

$\varphi_i(\mathbf{r})$



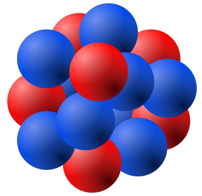
- If $\varphi_i(\mathbf{r})$ are irregular and "random" we can assume that $J_{ij,kl}$ are random

Complex SYK model in Nuclear physics

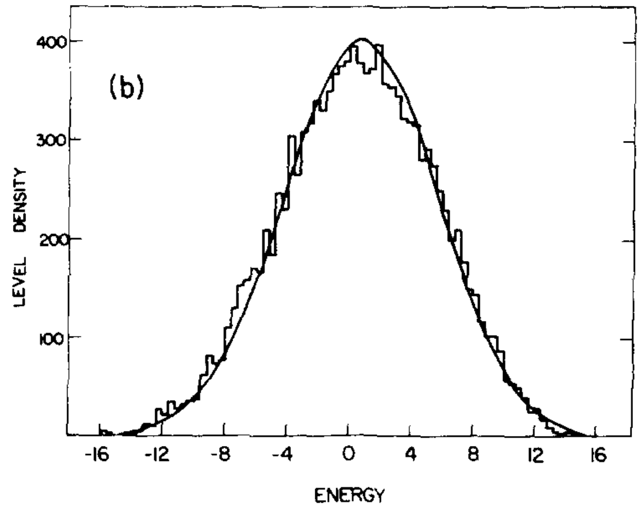
- Two-body random interaction model was introduced in Nuclear Physics in 70s

[J. French and S. Wong '70,
O. Bohigas and J. Flores '71]
[S. Sachdev '15]

$$H = \sum_{i,j,k,l=1}^{N} J_{ij,kl} c_i^\dagger c_j^\dagger c_k c_l \qquad \{c_i^\dagger, c_j\} = \delta_{ij}$$

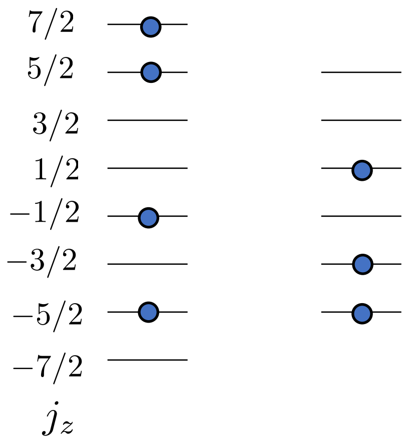


- Energy histogram



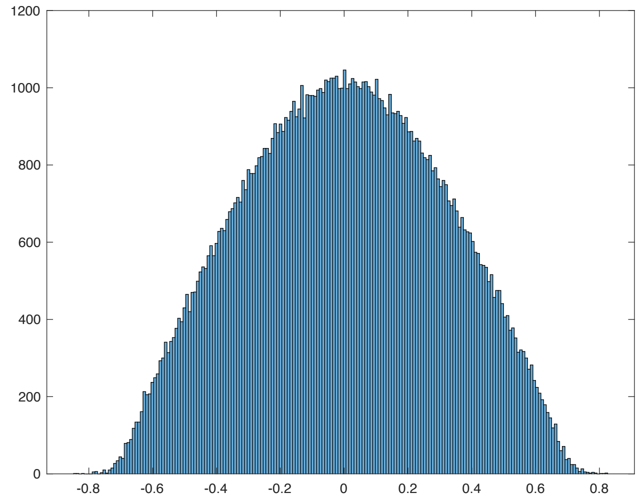
[J. French and S. Wong '70]

$(f_{7/2}, f_{5/2})$ orbitals



7 identical particles
fixed total momentum

- Reproduction of it



- The main conclusion at that time – it is not the Wigner semicircle!

Random models for quantum dots

- Hamiltonian of interacting particles in quantum dot:

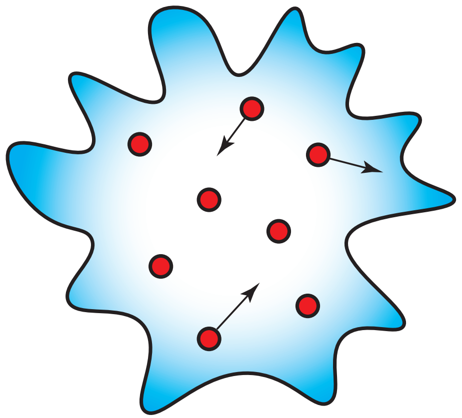
$$H_0 + H_1 = \sum_i \epsilon_i c_i^\dagger c_i + \sum_{ijkl} J_{ij,kl} c_i^\dagger c_j^\dagger c_k c_l$$

$$J_{ij,kl} = \int dx dx' V(x - x') \varphi_l^*(x) \varphi_k^*(x') \varphi_j(x) \varphi_i(x')$$

$J_{ij,kl}$ is a random quantity with zero average

How does this interaction affect free particles?

Quantum dot



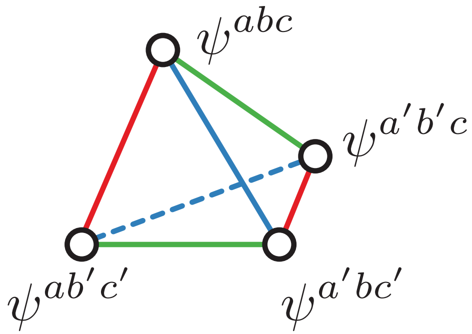
[B. Altshuler, Y. Gefen, A. Kamenev, L. Levitov, '96
Y. Alhassid, Ph. Jacquod, A. Wobst, '00
A. Lunkin, A. Kitaev, M. Feigel'man, '20, …]

Non-random large N models

- There are non-random models with the same large N limit

$$H = \frac{J}{4N^{3/2}} \sum_{a,a',\dots}^{N} \psi^{abc}\psi^{ab'c'}\psi^{a'bc'}\psi^{a'b'c}$$

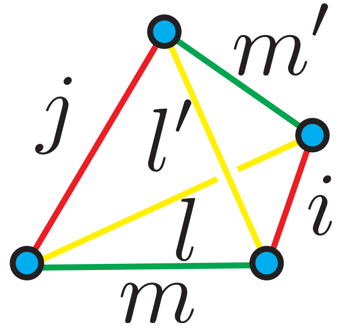
$$\{\psi^{abc}, \psi^{a'b'c'}\} = \delta^{aa'}\delta^{bb'}\delta^{cc'}$$



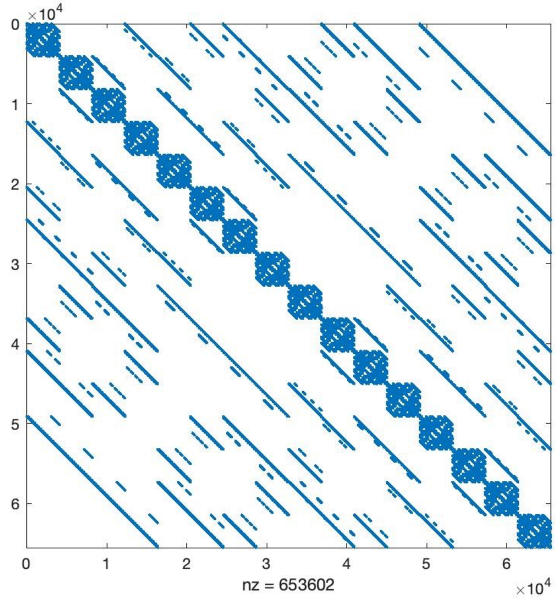
[R. Gurau'10,
S. Carrozza, A. Tanasa'15,
E. Witten '16,
I. R. Klebanov, GT'16]

$$H = \frac{J}{N^{3/2}} \sum_{i,l,m\dots} c^{\dagger}_{ilm} c^{\dagger}_{il'm'} c_{jlm'} c_{jl'm}$$

$$\{c_{ilm}, c^{\dagger}_{jl'm'}\} = \delta_{ij}\delta_{ll'}\delta_{mm'}$$



Tensor model Hamiltonian for 32 Majorana particles



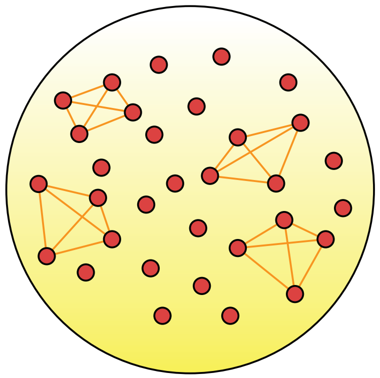
- Models of these types are called tensor models

- Randomness is replaced by addition of two "orbital" indices with a specific "tetrahedron" contraction:

The Sachdev-Ye-Kitaev model

- Hamiltonian: $H_{\mathrm{SYK}} = \frac{1}{4!}\sum_{ijkl}^{N} J_{ijkl}\psi^i\psi^j\psi^k\psi^l$

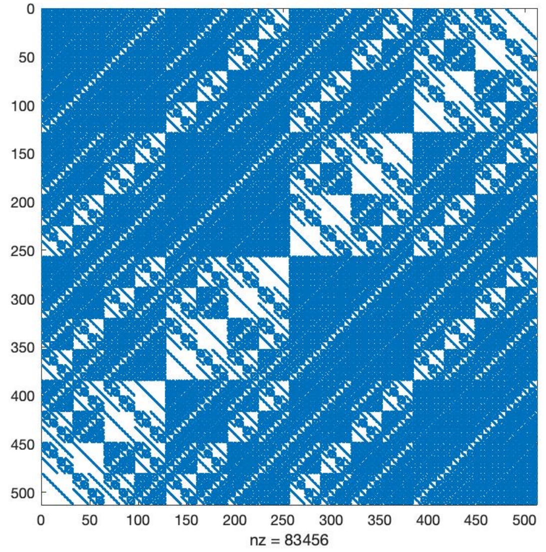
[S. Sachdev, J.Ye ‘93, A.Georges, O. Parcollet, S. Sachdev ’01, A.Kitaev ’15]



- N Majorana fermions: $\{\psi^i, \psi^j\} = \delta^{ij}$
- J_{ijkl} are antisymmetric Gaussian random $\overline{J_{ijkl}} = 0 \quad \overline{J_{ijkl}^2} = \frac{6J^2}{N^3}$
- The Hilbert space dimension is $2^{N/2}$
- One can compute numerically spectrum using Jordan-Wigner representation:

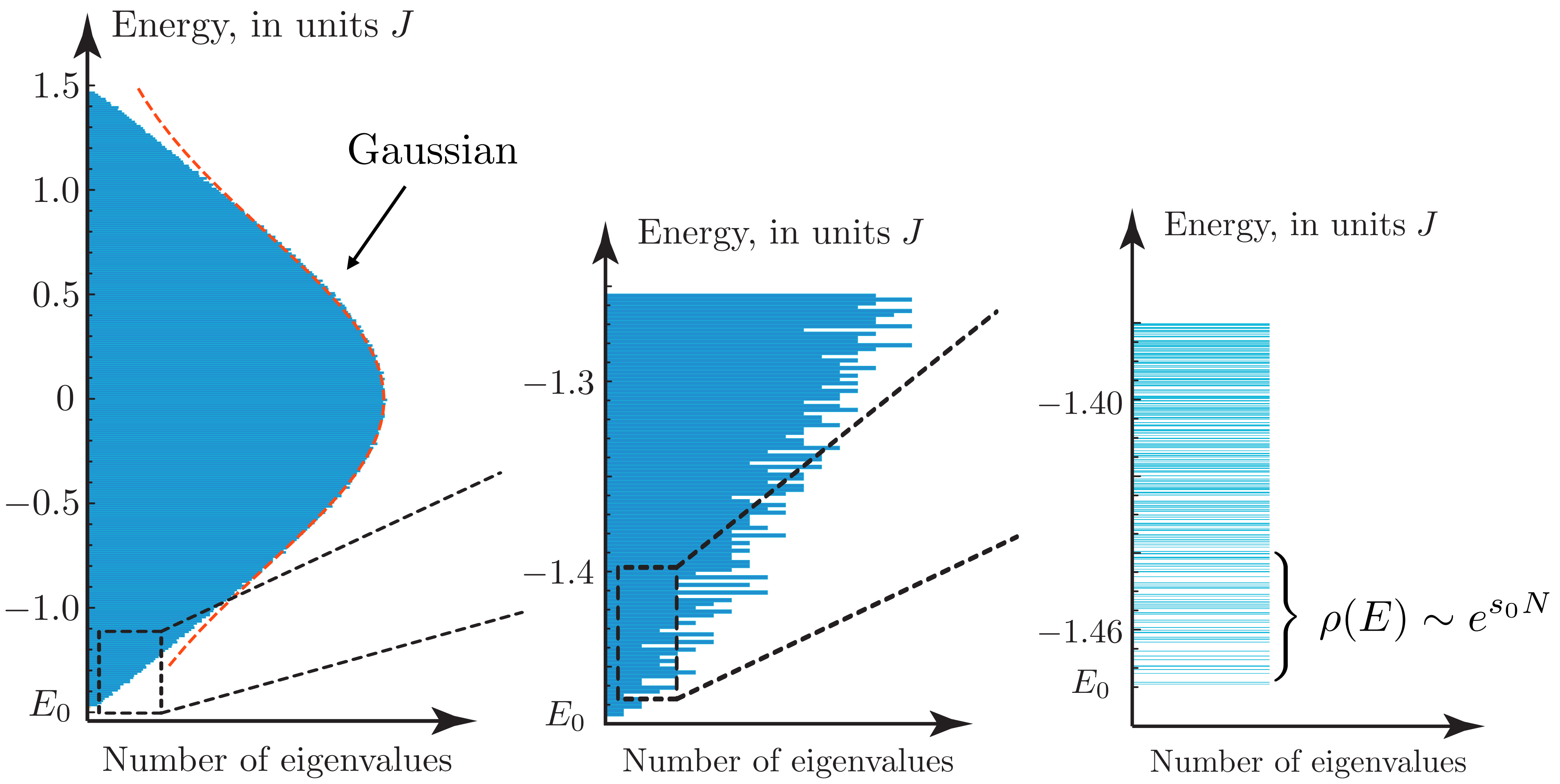
$$\psi^1 = \sigma_x \otimes \mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$$
$$\psi^2 = \sigma_y \otimes \mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$$
$$\psi^3 = \sigma_z \otimes \sigma_x \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$$
$$\psi^4 = \sigma_z \otimes \sigma_y \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$$
$$\cdots$$

SYK Hamiltonian for $N = 18$



Spectrum of the SYK model

- Energy levels for N=32 Majorana q=4 SYK model: 65536 energy levels



- s_0 is zero temperature entropy $\quad s_0 \approx 0.23$

- Most of the random SYK-like models admit scale invariant solution in IR for the two-point functions at large N limit

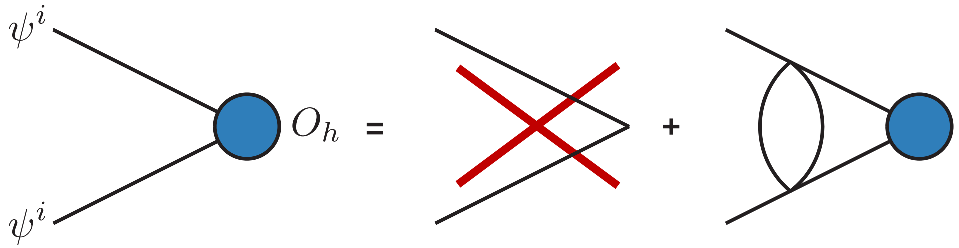
$$G(\tau) = \langle \mathrm{T}\psi(\tau)\psi(0)\rangle \to b\frac{\mathrm{sgn}(\tau)}{|J\tau|^{2\Delta_\psi}} \qquad \Delta_\psi = 1/4$$

- This scale invariance is related to high density of energy levels near the ground state

- To check theoretically if such a scaling invariant solution is stable and find corrections to it one has to analyze spectrum of two-particle operators

Operators in the SYK model

- For the two-particle operators $O_{h_n} = \psi^i \partial_\tau^{2n+1} \psi^i$ we consider the 3pt function $v(\tau_1, \tau_2, \tau_0) = \langle \psi^i(\tau_1) \psi^i(\tau_2) O_h(\tau_0) \rangle$ and can derive the Bethe-Salpeter equation using large N:



[D. Gross, V. Rosenhaus '16]

- This Bethe-Salpeter equation determines h and reads

$$v(\tau_1, \tau_2, \tau_0) = \int d\tau_3 d\tau_4 K(\tau_1, \tau_2; \tau_3, \tau_4) v(\tau_3, \tau_4, \tau_0)$$

$$K(\tau_1, \tau_2; \tau_3, \tau_4) = -3J^2 G(\tau_{13}) G(\tau_{24}) G(\tau_{34})^2 \qquad G(\tau) \to G_c(\tau) = b \frac{\mathrm{sgn}(\tau)}{|J\tau|^{2\Delta_\psi}} \qquad \Delta_\psi = 1/4$$

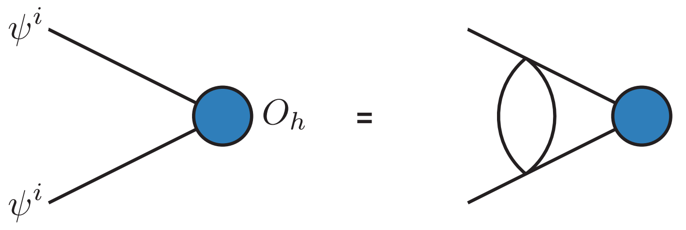
- Assume that in IR 3pt function has conformal form

$$v(\tau_1, \tau_2, \tau_0) = \langle \psi^i(\tau_1) \psi^i(\tau_2) O_h(\tau_0) \rangle = \frac{c_h \mathrm{sgn}(\tau_1 - \tau_2)}{|\tau_0 - \tau_1|^h |\tau_0 - \tau_2|^h |\tau_1 - \tau_2|^{2\Delta_\psi - h}}$$

the Bethe-Salpeter equation reduces to $1 = g(h) \quad g(h) = -\frac{3}{2} \frac{\tan(\frac{\pi}{2}(h - \frac{1}{2}))}{h - 1/2}$

Operators in the SYK model

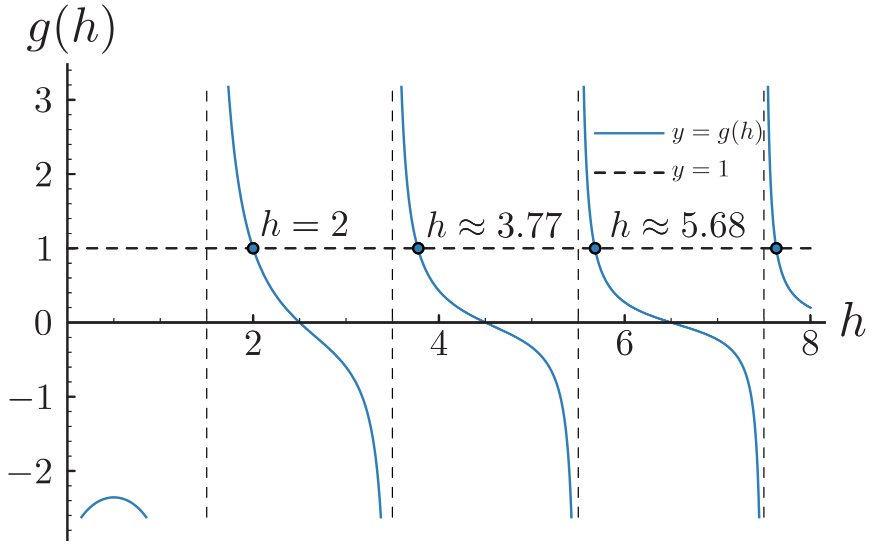
- The Bethe-Salpeter equation:



$$O_{h_n} = \psi^i \partial_\tau^{2n+1} \psi^i$$

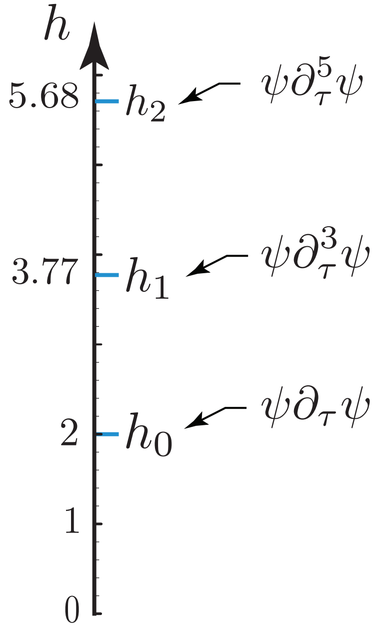
$$1 = g(h)$$

- Graphical solution of this equation gives operator spectrum



$$g(h) = -\frac{3}{2} \frac{\tan\left(\frac{\pi}{2}\left(h - \frac{1}{2}\right)\right)}{h - 1/2}$$

Scaling dimensions



- The first solution is $h = 2$; breaks conformal invariance.

- The higher scaling dimensions are

$h \approx 3.77,\ 5.68,\ 7.63,\ 9.60, \ldots$ approaching $h_n \to n + 1/2$

[J. Maldacena, D. Stanford '16
D. Gross, V. Rosenhaus '16]

SYK as Nearly CFT

- We can think about SYK as CFT perturbed by irrelevant operators $O_{h_n}(\tau) = \psi^i \partial_\tau^{2n+1} \psi^i$

$$S_{\text{SYK}} = S_{\text{SYK CFT}} + \sum_h \frac{g_h}{J^{h-1}} \int_0^\beta d\tau O_h(\tau)$$

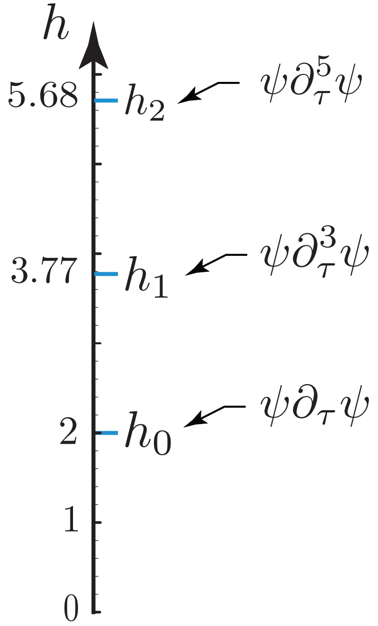
[A. Kitaev '15,
J. Maldacena, D. Stanford '16]
[D. Gross, V. Rosenhaus'17]

where g_h are couplings

$$\{O_{h_0}, O_{h_1}, O_{h_2}, \ldots\} \qquad O_{h_1}(\tau) O_{h_2}(0) = \sum_h c^h_{h_1 h_2} |\tau|^{h-h_1-h_2} O_h(0)$$

$$\langle \mathrm{T}\psi(\tau)\psi(0)\rangle_{\text{CFT}} = \frac{1}{Z}\int D\psi\, \psi(\tau)\psi(0) e^{S_{\text{SYK CFT}}} = b\frac{\text{sgn}(\tau)}{|J\tau|^{1/2}}$$

Scaling dimensions



- Next order J corrections to the Green's function are given by conformal perturbation theory:

$$\langle \mathrm{T}\psi(\tau)\psi(0)\rangle_{\text{SYK}} = \langle \mathrm{T}\psi(\tau)\psi(0)\rangle_{\text{CFT}} + \sum_h \frac{g_h}{J^{h-1}} \int d\tau_3 \langle \psi(\tau)\psi(0) O_h(\tau_3)\rangle$$

$$+ \sum_{h,h'} \frac{g_h g_{h'}}{2J^{h+h'-1}} \int d\tau_3 d\tau_4 \langle \psi(\tau)\psi(0) O_{h_3}(\tau_3) O_{h'}(\tau_4)\rangle + \ldots$$

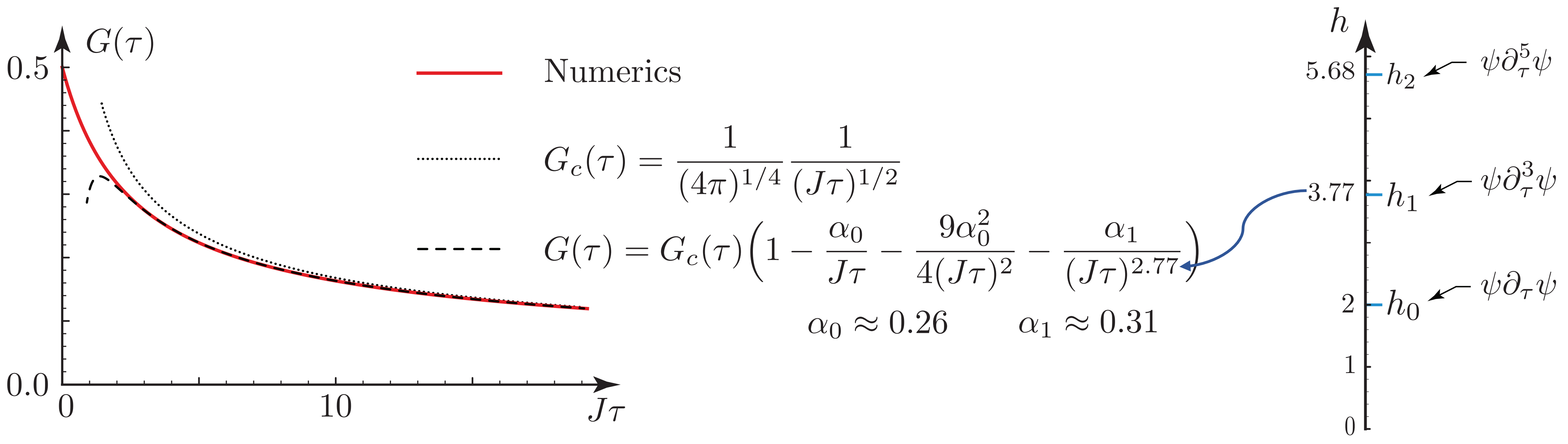
SYK two-point function

- General formula for the full SYK two-point function

$$G(\tau) = G_{\rm conf}(\tau)\left(1 - \sum_h \frac{v_h \alpha_h}{|J\tau|^{h-1}} - \sum_{h,h'} \frac{a_{hh'}\alpha_h\alpha_{h'}}{|J\tau|^{h+h'-2}} - \sum_{h,h',h''} \frac{a_{hh'h''}\alpha_h\alpha_{h'}\alpha_{h''}}{|J\tau|^{h+h'+h''-3}}\right)$$

- Comparison with numerics at zero temperature:

Scaling dimensions



- We can detect irrational scaling exponent numerically

Thermodynamics of the SYK model

- We can think about SYK as CFT perturbed by irrelevant operators $O_{h_n}(\tau) = \psi^i \partial_\tau^{2n+1} \psi^i$

$$S_{\text{SYK}} = S_{\text{SYK CFT}} + \sum_h \frac{g_h}{J^{h-1}} \int_0^\beta d\tau O_h(\tau)$$

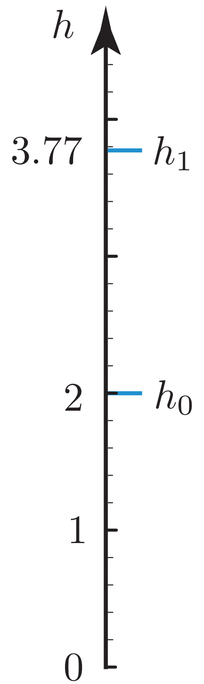
[A. Kitaev '15,
J. Maldacena, D. Stanford '16]
[D. Gross, V. Rosenhaus'17]

where g_h are couplings

- Free energy for the SYK model using conformal perturbation theory:

$$\beta F_{\text{SYK}} = \beta F_{\text{CFT}} + \sum_h g_h \int_0^\beta d\tau \langle O_h \rangle_\beta - \frac{1}{2} \sum_{h,h'} g_h g_{h'} \int_0^\beta d\tau_1 d\tau_2 \langle O_h(\tau_1) O_{h'}(\tau_2) \rangle_\beta + \dots$$

Scaling dimensions



- Computing the integrals we find

$$\beta F_{\text{SYK}} = -N\left((\beta J)\epsilon_0 + s_0 + \frac{\alpha_0}{\beta J} + \frac{\alpha_0^2}{(\beta J)^2} + \dots + \frac{\alpha_1^2}{(\beta J)^{5.54}} + \dots\right)$$

Schwarzian coupling

ground energy

zero temperature entropy

specific heat

Thermodynamics of the complex SYK model

- In the complex SYK we have two sets of operators $O_{h_n}(\tau) = c^\dagger \partial_\tau^{2n+1} c$ and $O_{\tilde{h}_n}(\tau) = c^\dagger \partial_\tau^{2n} c$

$$S_{\text{cSYK}} = S_{\text{cSYK CFT}} + \mu \int_0^\beta d\tau c^\dagger c + \sum_h \frac{g_h}{J^{h-1}} \int_0^\beta d\tau O_h(\tau) + \sum_{\tilde{h}} \frac{g_{\tilde{h}}}{J^{\tilde{h}-1}} \int_0^\beta d\tau O_{\tilde{h}}(\tau)$$

- Perturbation by conserved U(1) charge $Q = c^\dagger c$ is a marginal deformation of the theory

- The free energy reads

$$\beta F_{\text{cSYK}} = -N\left((\beta J)\epsilon_0 + s_0 + \frac{\alpha_0}{\beta J} + \frac{\alpha_0^2}{(\beta J)^2} + \frac{\tilde{\alpha}_1^2}{(\beta J)^{3.3}} + \dots\right)$$

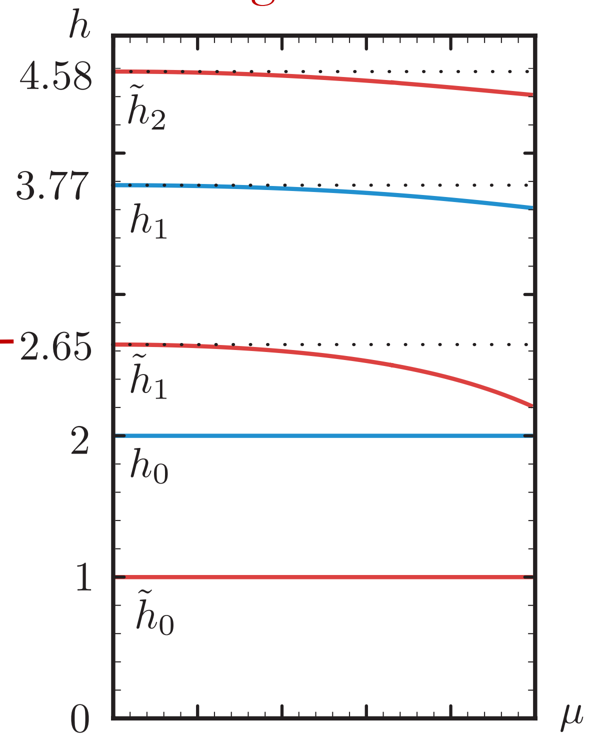
Schwarzian coupling

ground energy

zero temperature entropy

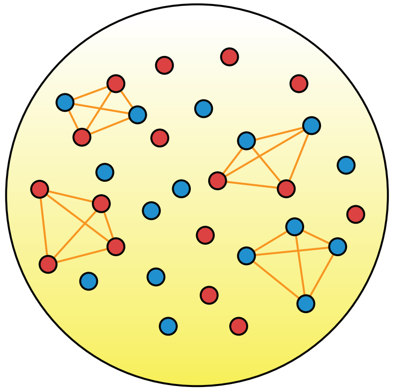
specific heat

Scaling dimensions



- In both SYK and complex SYK the small temperature thermodynamics is governed by $h = 2$ operator

Coupled Majorana SYK models



- Consider two types of Majorana fermions ψ_1^i and ψ_2^i with Hamiltonian:

$$H = \frac{1}{4!}\sum_{i,j,k,l}^{N} J_{ijkl}(\psi_1^i\psi_1^j\psi_1^k\psi_1^l + \psi_2^i\psi_2^j\psi_2^k\psi_2^l + 6\alpha\psi_1^i\psi_1^j\psi_2^k\psi_2^l)$$

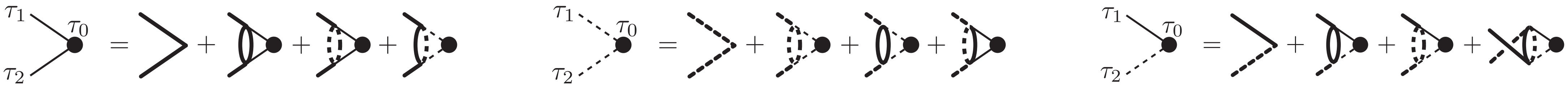
$$\{\psi_1^i, \psi_1^j\} = \{\psi_2^i, \psi_2^j\} = \delta^{ij} \qquad \{\psi_1^i, \psi_2^j\} = 0$$

- J_{ijkl} are Gaussian random, exactly as in the SYK models $\overline{J_{ijkl}} = 0 \qquad \overline{J_{ijkl}^2} = \frac{6J^2}{N^3}$

- α is the interaction strength. Using symmetry one can show $-1 \leq \alpha \leq 1/3$

- We assume that $G_{\psi^1\psi^1}(\tau) = G_{\psi^2\psi^2}(\tau) = G(\tau) \qquad G_{\psi^1\psi^2}(\tau) = 0$

- Under this assumption we find that $G(\tau)$ coincides with SYK two-point function and we can find anomalous dimensions of various bilinear operators

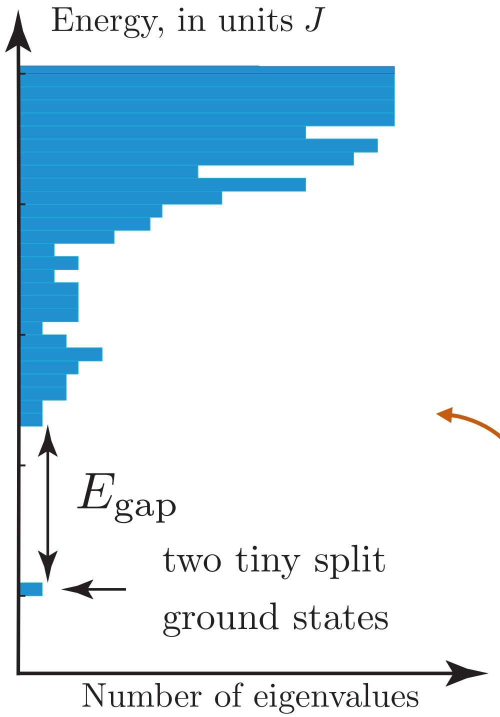


Operator spectrum

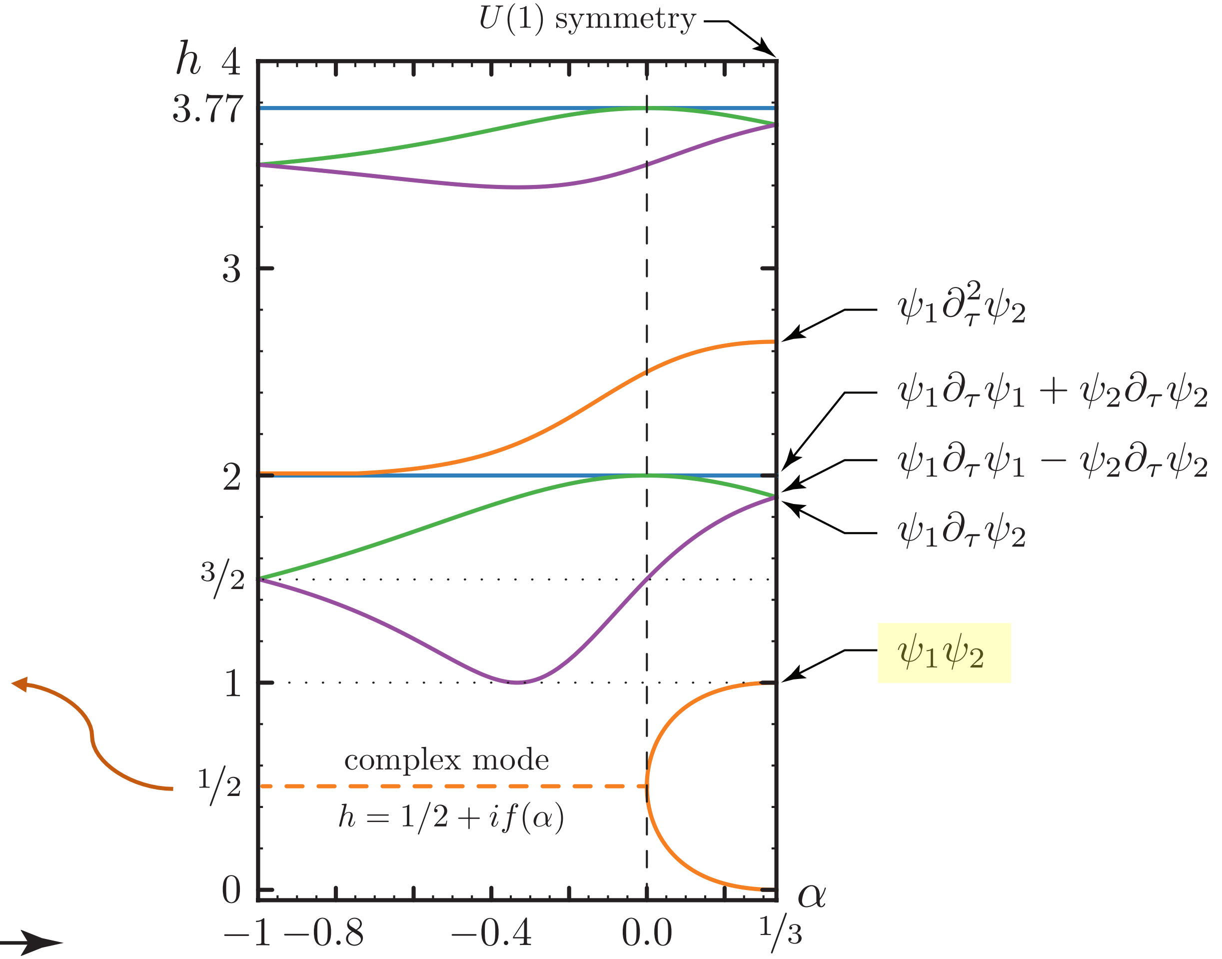
- Hamiltonian: $H = \frac{1}{4!}\sum_{i,j,k,l}^{N} J_{ijkl}(\psi_1^i\psi_1^j\psi_1^k\psi_1^l + \psi_2^i\psi_2^j\psi_2^k\psi_2^l + 6\alpha\psi_1^i\psi_1^j\psi_2^k\psi_2^l)$
- For $0 \leq \alpha \leq 1/3$ the model is similar to the SYK

  The operator $O = i\psi_1^i\psi_2^i$ is relevant.
  If we perturbed by it $H \to H + i\mu\psi_1^i\psi_2^i$
  the spectrum will be gapped [J. Maldacena, X. Qi'18]

- For $-1 \leq \alpha \leq 0$
  the operator $O = i\psi_1^i\psi_2^i$
  has complex anomalous
  dimension and acquires a vev
  The Z_2 symmetry
  is spontaneously broken



Scaling dimensions



Coupled complex SYK models

- Hamiltonian of the coupled complex SYK models

$$H = \sum_{i,j,k,l}^{N} J_{ij,kl}\left(c_{1i}^\dagger c_{1j}^\dagger c_{1k} c_{1l} + c_{2i}^\dagger c_{2j}^\dagger c_{2k} c_{2l} + 8\alpha c_{1i}^\dagger c_{2j}^\dagger c_{2k} c_{1l}\right)$$

[see also S.Sahoo, E. Lantagne-Hurtubise, S. Plugge, M. Franz ’20
E. Lantagne-Hurtubise, S. Sahoo, M. Franz, 20]

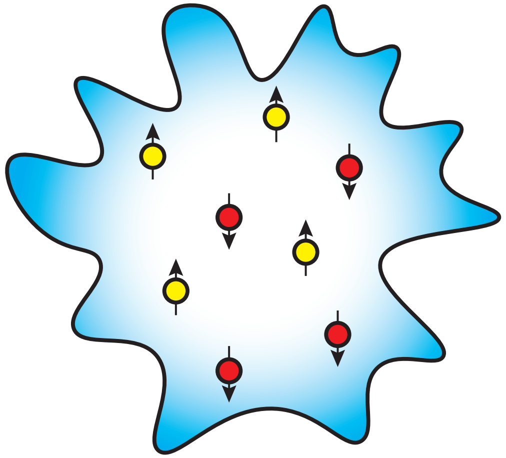
$$\{c_{\sigma i}, c_{\sigma' j}^\dagger\} = \delta_{\sigma\sigma'}\delta_{ij} \qquad \{c_{\sigma i}, c_{\sigma' j}\} = 0$$

- Random couplings $\overline{J_{ij,kl}} = 0 \quad \overline{|J_{ij,kl}|^2} = J^2/(2N)^3$

- The Hamiltonian has $U(1)\times U(1)$ symmetry

$$c_{1i} \to e^{i\phi_1} c_{1i} \qquad Q_1 = \frac{1}{2}\sum_{i=1}^{N}[c_{1i}^\dagger, c_{1i}]$$

$$c_{2i} \to e^{i\phi_2} c_{2i} \qquad Q_2 = \frac{1}{2}\sum_{i=1}^{N}[c_{2i}^\dagger, c_{2i}]$$



- For coupling $\alpha = 1/4$ we find $U(2)$ symmetric Hamiltonian

$$H = \sum_{i,j,k,l}^{N} J_{ij,kl} c_{\sigma i}^\dagger c_{\sigma' j}^\dagger c_{\sigma' k} c_{\sigma l}$$

Operator spectrum

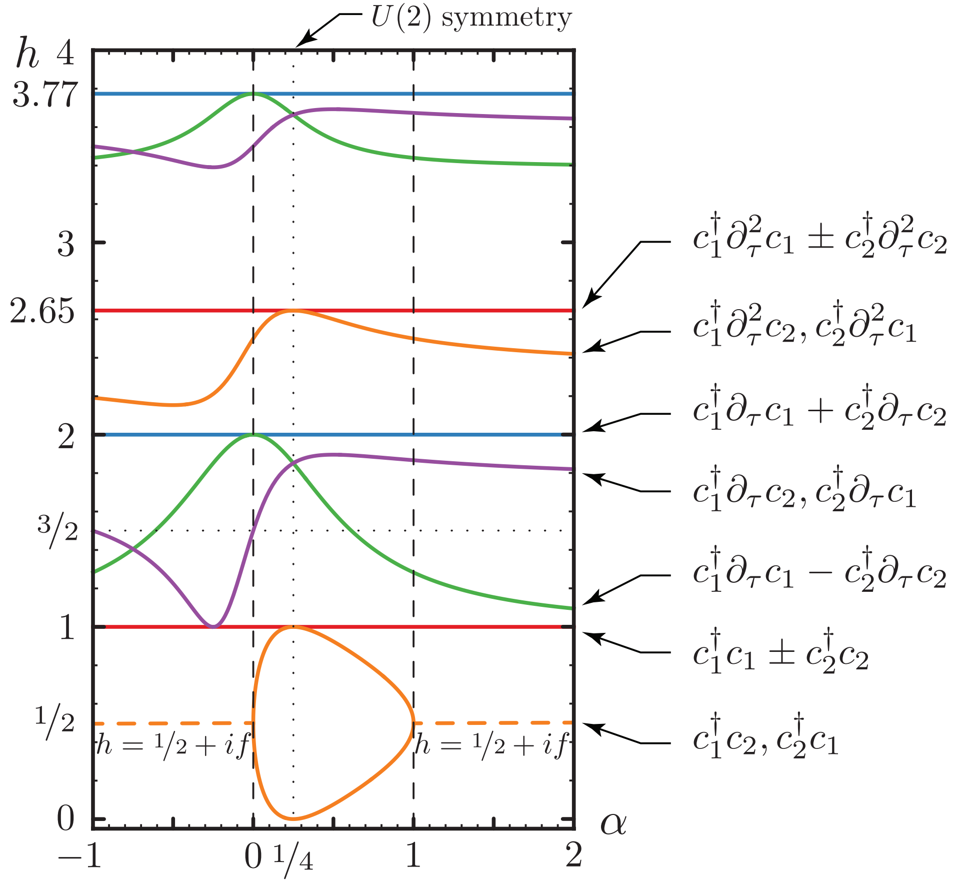
- Hamiltonian: $H = \sum_{i,j,k,l}^{N} J_{ij,kl}\left(c^\dagger_{1i}c^\dagger_{1j}c_{1k}c_{1l} + c^\dagger_{2i}c^\dagger_{2j}c_{2k}c_{2l} + 8\alpha c^\dagger_{1i}c^\dagger_{2j}c_{2k}c_{1l}\right)$

- For $0 \le \alpha \le 1$ the model is similar to the cSYK
  The operators $c_1^\dagger c_2$ and $c_2^\dagger c_1$ are relevant.
  If we perturbed the Hamiltonian by them
  the spectrum will be gapped

- For $\alpha = 1/4$ there is a U(2) global symmetry
  and all quadratic operators are marginal

- For $\alpha < 0$ and $\alpha > 1$
  the operators $c_1^\dagger c_2$ and $c_2^\dagger c_1$
  have complex anomalous
  dimension and acquire vev
  The U(1) symmetry is spontaneously broken

Scaling dimensions



Operator spectrum

- Hamiltonian: $H = \sum_{i,j,k,l}^{N} J_{ij,kl}\Big(c^\dagger_{1i}c^\dagger_{1j}c_{1k}c_{1l} + c^\dagger_{2i}c^\dagger_{2j}c_{2k}c_{2l} + 8\alpha c^\dagger_{1i}c^\dagger_{2j}c_{2k}c_{1l}\Big)$

- For $\sqrt{3/8} < \alpha \leq 1$ there is an operator $O = c_1^\dagger \partial_\tau c_1 - c_2^\dagger \partial_\tau c_2$ with the scaling dimension $1 < h < 3/2$

- This operator is not generated in the effective action, but we can add it. In the Hamiltonian formalism this corresponds to changing commutation relations between fermions

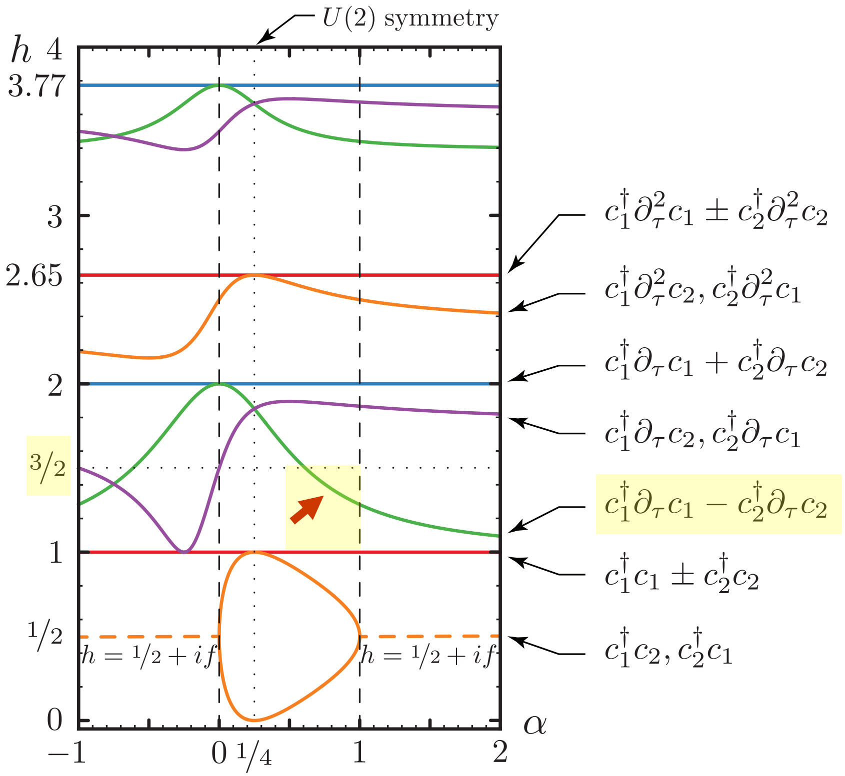
$$\{c^\dagger_{1i}, c_{1j}\} = \frac{1}{1+\xi}\delta_{ij} \qquad \{c^\dagger_{2i}, c_{2j}\} = \frac{1}{1-\xi}\delta_{ij}$$

[A. Milekhin '21]

- In this case the low-energy effective action is not Schwarzian, but

$$S_{\text{non-local}} = -\frac{N\alpha_h}{J^{2h-2}}\int d\tau_1 d\tau_2 \Big(\frac{\phi'(\tau_1)\phi'(\tau_2)}{(\phi(\tau_1)-\phi(\tau_2))^2}\Big)^h$$

Scaling dimensions



[J. Maldacena, D. Stanford, Z. Yang '16, A. Milekhin '21]

Conclusions

- There is a variety of random models with similar large N physics, described by conformal solution

- The operator spectrum of such models can be very different

- In some models the operator spectrum implies instability of the conformal solution and spontaneous symmetry breaking

- In some models there exists an operator with scaling dimension $1 < h < 3/2$ which leads to the non-local IR effective action instead of the Schwarzian action

Thank you for your attention!