Mastering $N = 1$

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- The "ultimate mirror hypothesis"
- $N = 1$ and the master equation
- $N = 1$ and the master field
- What about the Kähler potential ?

Starting with the work of Seiberg, there has been a lot of success in recent years in "solving" fairly general $N = 1$ supersymmetric field theories. There are two very general aspects of this:

- Systematically exploiting the consequences of supersymmetry for "protected" quantities, such as holomorphy of the superpotential.
- Settling for qualitative results for unprotected quantities, such as the Kähler potential.

In $N = 1$ field theory, this suffices to get exact results on the set of supersymmetric vacua and domain walls between them.

Is there any fundamental obstacle to carrying out such a program for all of string and M theory, to get a description of **all** $N = 1$ supersymmetric vacua ?

Obviously doing this is beyond our present capabilities, but perhaps it is not as far beyond as we think.

The simplest description we might imagine for the problem of $N = 1$ compactification would be purely geometric. What this would mean can be illustrated by the following claim, which is not obviously false:

All $d = 4$, $N = 2$ string/M vacua are dual to those obtained (by continuation from) type IIb on Calabi-Yau.

In type IIb, the prepotential is geometric: it can be computed at $g_s = 0$ by solving PDE's. Mirror symmetry, S-duality, etc. allow reducing other constructions to this case.

A slightly more general claim of the same type would be that one must add "generalized Calabi-Yau's" (for example, with discrete torsion), which can be defined using geometric methods.

Such a claim (if true) does not in itself give the list of possible compactifications, nor would it make the dual constructions useless. But it would set the problem of listing $N = 2$ compactifications in a very clear framework, and give a very clear target for some more fundamental formulation to reproduce.

There are many geometric constructions of $N = 1$ (heterotic; branes on CY_3; flux on CY_3; F theory on CY_4; M on G_2...) but all known at present which lead to quasi-realistic theories require adding instanton corrections to get exact results.

Nevertheless, we can frame the "ultimate mirror symmetry" hypothesis: there exists a class of object (say CY fourfolds with X structure), defined by geometric methods (say solving PDE's or making finite "combinations" of such solutions), which is in one-to-one correspondence with $N = 1$ compactifications of string/M theory on the quantum level.

Is there evidence for or against this hypothesis ?

By now, many geometric duals which sum up nonperturbative effects are known, especially if we restrict attention to the field theory limit and/or the large N limit, and this might be considered evidence in favor. Impressive progress in this direction has recently been made (Dijkgraaf-Ooguri-Vafa; Nekrasov).

A simple example (Gopakumar-Vafa) is the claim that D5-branes wrapping a $(-1,-1)$ $\mathbb{P}^1$ curve are dual to the deformed conifold with flux on S^3.

Assuming this local construction works globally, then a compactification using only D5 and O5 (e.g. see Acharya et al), would be dual to a flux compactification, for which the superpotential is classical.

Let us see what is behind this to judge how far it could go. The starting point is type IIb on a CY_3 M with an orientifold and B-type (holomorphic) D-branes, a very general class of models. One then has an exact formula for the classical superpotential:

$$W = \int \Omega \wedge \Big((\bar{A}\bar{\partial}\bar{A} + \frac{2}{3}\bar{A}^3) + (F + \tau H) \Big) .$$

This combines the holomorphic Chern-Simons action and the flux (GVW) superpotential. Here Ω is the holo. 3-form on M, $\bar{\partial} + \bar{A}$ the holomorphic gauge connection (for branes wrapping M), F and H the quantized RR and NS 3-form fluxes, and $\tau = C^0 + ie^{-D}$ the complex dilaton.

This formula requires some interpretation but leads to all known type II classical superpotentials.

For type I theory (branes wrapping M), it is interpreted as follows: $\bar{A}$ is a ten dimensional gauge field, which can be regarded as a collection of infinitely many four dimensional fields. In any given background, finitely many are light. The four-dimensional superpotential is obtained by integrating out the others.

For branes wrapping lower dimensional cycles, one can either reduce in higher dimensions, or build these branes as brane-antibrane bound states and integrate out. Let us discuss the second procedure as it is not as well known.

The simplest example is a $\mathbb{C}^3/\mathbb{Z}_K$ orbifold, whose resolution is a noncompact CY_3 M. The world-volume theory of branes on M can be derived by projection of $N = 4$ SYM. Let $\mathbb{Z}_K$ act as $z^i \to e^{2\pi i a_i/K} z^i$, then there is a cubic superpotential W given by

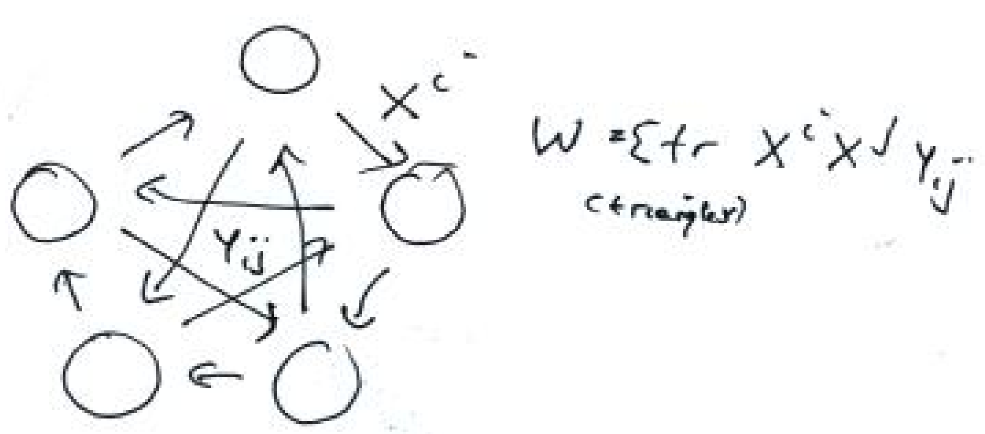
$$\sum_n \epsilon_{ijk} \operatorname{tr} X^i_{n,n+a_i} X^j_{n+a_i,n+a_i+a_j} X^k_{n-a_k,n}.$$

Supersymmetric configurations of these quiver theories correspond to (compact) branes on M carrying holomorphic bundles, and this superpotential is the reduction of the holomorphic Chern-Simons action to the light states. In this example, one can think of this as substituting $\bar{A} = \sum_{i,n} \bar{A}^{(i,n)} X^i_{n,n+a_i}$ into the cubic $\bar{A}^3$ term and integrating.

In the orbifold example, the exact W is cubic, and one can get it by truncating massive modes. But in general, one must integrate them out: tree diagrams in holo CS theory lead to higher order terms.

The general theory of this can be understood using the theory of A_∞ algebras, as has been discussed by Merkulov, Lazaroiu, Tomasiello, and others. Integrating out requires choosing a gauge, which depends on Kähler data (e.g. $0 = g^{i\bar{j}} D_i \bar{A}_{\bar{j}}$). Naively, this spoils the independence of the superpotential on Kähler data ("decoupling"). However, different choices of gauge lead to W related by A_∞ quasi-isomorphism, a generalization of the notion of field redefinition.

Explicit holo CS perturbation theory is complicated, but one can achieve the same result by doing simpler linear sigma model computations. An example studied in DGJT is the quintic $Q \subset \mathbb{P}^4$. The linear sigma model realizes this as a hypersurface in the resolved orbifold $\mathbb{C}^5/\mathbb{Z}^5$, an orbifold theory with 5 fractional branes and antibranes. Mathematically, these are bundles which can be restricted to the quintic, and a large subset of branes on the quintic can be obtained from them as bound states.



In this case, the cubic orbifold superpotential is just a leading approximation: the world-sheet theory is

$$S = \int d^4\theta |Z|^2 + \int d^2\theta f_{ijklm} Z^i Z^j Z^k Z^l Z^m .$$

W obtains an infinite series of corrections which can be computed by **perturbing in the world-sheet superpotential**, i.e. by computations in free $\mathbb{C}^5$ CFT. In DGJT the first correction was computed and checked against geometric results.

$$W = \epsilon_{abcde} \mathrm{tr}\ X^a X^b Y^{cde} + \frac{4\pi^5}{5} \epsilon_{abcde} f_{ijklm} \mathrm{tr}\ X^i X^j Y^{abc} X^k X^l Y^{dem} + \ldots$$

Again, the natural setting for these computations is A_∞ algebras and deformation theory. The superpotential can be regarded as the generating function for structure constants of an A_∞ algebra,

$$\int a_0 \cdot m_k(a_1, \ldots, a_k) = \frac{\partial}{\partial t_0} \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_2} W(t)$$

The A_∞ axioms are encoded in the "classical master equation" of the BV formalism,

$$0 = \{\hat{W}, \hat{W}\}$$

Starting with the cubic W, one can find the space of possible A_∞ deformations up to field redefinition, by cohomological techniques.

The allowed deformations turn out to be precisely the deformations f_{ijklm} of the defining equation for the quintic, and these ideas lead to a **proof** that the resulting category of branes is $D(\text{Coh } Q)$ (Seidel; Douglas and Seidel).

As in previous work, the derived category $D(\text{Coh } Q)$ enters because the question of which supersymmetric configurations are "branes" and which are "antibranes" depends on Kähler moduli. Any result which is independent of Kähler moduli, such as the superpotential for any sufficiently large set of brane configurations, is best formulated in this language. More concrete results can be obtained by choosing bases to get families of Seiberg dual theories.

An application (in preliminary stages): tuning the number of generations. In traditional heterotic string compactification, one seeks bundles V with rank $r = 3, 4, 5$, $c_1(V) = 0$, $c_2(V) = c_2(TM)$, $c_3(V) = 2N_{gen} = 6$.

Using the above ideas, one can construct brane configurations with specified Chern classes, and which can be checked to correspond to bundles in certain cases. Essentially, one takes r D6's carrying a copy of the tangent bundle, and makes bound states with D0's.

For $M = Q$ and the Beilinson basis, this translates into $N_{gen} = 100 - 5n$ with multiplicities $N_i =$

$$(5 - n\ 15 - 2n\ 35 - 3n\ 65 - 4n\ 102 - 5n).$$

One can show that such configurations exist for $n \leq 5$ since the expected dimension $D(N) = 222 - 15n > 0$. They are generically sheaves on $\mathbb{P}^4$ with point-like singularities off of Q, so should restrict to bundles on Q. For $n > 5$ one gets some $N_i < 0$, but by change of basis (Seiberg duality) one can lower N_{gen} further. One can also generalize by allowing M5's, etc.

These techniques can be applied to general LG orbifolds, *i.e.* hypersurfaces in $\mathbb{WP}_{w_1,\cdots,w_5}$. Existing methods can get N_{gen} divisible by $\gcd(m_w)$, where m_w is the multiplicity of the weight $w = w_i$.

This suggests that three generation models might be rather common.

Although not strictly geometric, the derived category is close enough to count as "geometric" for our previous discussion. One also has a criterion for the physical branes (Π-stability) which incorporates all world-sheet instantons, yet is formulated in purely geometric terms. In this sense, we can also count these results as evidence in favor of the "ultimate mirror hypothesis."

The appearance of A_∞ and BV is because these computations fit into the general context of **topological open string field theory**. Its "effective action" is the superpotential, which gives the precise sense in which this is a "protected" quantity. If we could quantize this theory, we could compute exact superpotentials.

This brings us to Vafa and collaborators, who have achieved impressive results along these lines. Perhaps the simplest example (Dijkgraaf-Vafa) is a system of D5's wrapped on a collection of "semi-rigid" curves, *i.e.* a family of curves with normal bundle $\mathcal{O} \oplus \mathcal{O}(-2)$, deformed into isolated curves. This corresponds to deformed holo CS theory:

$$W = \int \mathrm{tr} \;\; \phi^0 \bar{\partial} \phi^1 + \omega w(\phi_0).$$

This is a 2d bc system and everything is computable; in fact it reduces to a one matrix integral.

At large N, this integral has a "master field," the saddle point value of ϕ^0. This encodes the deformation of the original geometry, to the dual S^3/flux geometry.

At finite N, the theory is quantum but still exactly solvable, and produces an exact field theory superpotential.

There are many nontrivial ingredients in these claims:

- The $5+1$ world-volume theory can be reduced to $d = 2$ in the naive way.
- Gauge theory instanton effects are reproduced by what naively appears to be a perturbative open string computation. This has been explained in the simplest example as coming from an overall volume factor, but still seems mysterious.
- Closed strings are not present in these examples (they are not expected to decouple [BCOV]).

Clearly it is too early to say what the real lessons are from this. But is there evidence here for or against the "ultimate mirror hypothesis" ?

Here, the geometric description follows from the existence of a master field for large N. For finite N, one must work with quantized holo CS theory, in which known results are not "geometric" in the simple sense we want.

We want finite N branes for a true string compactification. So, the key question would seem to be whether the superpotential is given by large N results for this case – very important if true.

Either way, these are powerful results – can they generalize to compactifications?

In general, holomorphic Chern-Simons theory is six dimensional. Even if one considers lower dimensional branes wrapped on cycles (such as the D5), if we work on a compact CY, winding modes (or T-duality) come in and make the theory six dimensional. It is hard to believe that lower dimensional reductions exist in the stringy regime. But, perhaps some combination of these arguments to get results in field theory limits, with holomorphy to continue the results into the stringy regime, could lead to stringy exact results on compact CY.

Another issue for compact CY is that closed strings are dynamical, so these moduli should be integrated over. We return to this later.

Let us suppose we manage to compute interesting exact superpotentials in string theory (of course there are known examples such as the flux superpotential). Can we go on to find all supersymmetric vacua?

In general, the problem of finding vacua in string theory must be addressed in $N = 1$ supergravity. In this theory, the supersymmetric vacua depend on the choice of Kähler potential. They are solutions of

$$0 = D_i W = \partial_i W + (\partial_i K) W.$$

So far as anybody knows, K is not protected by supersymmetry, so we have a problem.

The usual response to this problem is to only consider Minkowski vacua, with $W = DW = 0$. The problem of finding these does not depend on K.

There are reasons not to be satisfied with this answer, however. The main one is that we do not live in an $N = 1$ supersymmetric vacuum. Rather, one is at best looking for approximately supersymmetric vacua with breaking at a low scale M_{susy}.
One can still use $N = 1$ techniques to study the high energy dynamics, but then all one knows about the vacuum energy is that it is $O(M_{susy}^4)$. So supersymmetric AdS vacua are *a priori* just as relevant.

There are also formal reasons not to be satisfied just with the Minkowski vacua. The conditions $W = DW = 0$ are more equations than unknowns. This makes them difficult to study – solutions are nongeneric but often appear anyways, in a way which depends sensitively on W.

The equations $DW = 0$ are better behaved and in fact the number of critical points of a holomorphic function W would be expected to be a topological invariant. This raises the possibility that one could find a topological formula for the total number of $N = 1$ vacua in a sector of configuration space, but only by counting AdS vacua as well as Minkowski.

However, since K is not holomorphic, DW is not a holomorphic one-form, and it is easy to construct examples in which the number of vacua changes upon varying K. This is easy to see if $W = 1$ (do a Kähler-Weyl transformation), so that $D_i W = \partial_i K$. Consider one chiral superfield $\phi = x + iy$; the ansatz

$$K = f(x) + y^2$$

will satisfy $g_{i\bar{j}} = \partial_i \bar{\partial}_j K > 0$ if $f'' > -2$, and by varying f one can create pairs of vacua at $f'(x) = y = 0$.

So, there cannot be a topological formula for the number of vacua *per se*.

However, if varying the data can only create pairs of vacua, one can try to define an index, in which this cancels out.

If we are counting critical points of a function K, this is the standard Morse index, which weighs critical points by $\text{sgn } \det \partial_i \partial_j K$. This is constrained by the homology of the space, as is well known. If the configuration space were a compact manifold, the topological formula for this index is the Euler number.

This is all one can say if $W \neq 0$, but the interest of the problem is that W can have zeroes and poles, and in principle one has **more** topological information from supergravity: namely, W is not a function, but a section of a negative line bundle over configuration space.

For general W, susy vacua are critical points of $e^K W$. This is neither real nor a function and Morse theory does not directly apply. Critical points of $\Lambda = -3e^K |W|^2$ include all points $W = 0$.

However, one can define an index as follows. Suppose the vacua v are isolated, then

$$I = \sum_v \text{sgn } \det M_{ij}(v)$$

where $M_{ij}(v)$ is the supergravity fermion mass matrix (for fermions in chiral multiplets), expanded around the vacuum v, with four dimensional space-time taken as Minkowski or AdS depending on Λ. This weighs Minkowski vacua with $+1$, and AdS vacua as sgn $\det(D_i D_j W/W)$ (the same as the Morse sign).

One can define the contributions of nonisolated vacua, following standard lines in the two cases (deforming K splits AdS vacua; a moduli space of Minkowski vacua contributes its Euler character). The result is an index which is invariant under reasonable variations of K, again essentially by standard arguments (since such variations do not mix Minkowski and AdS vacua).

As with the field theory Witten index, this index contains useful information about the existence of vacua, and might be used in studying $N = 1$ duality (two dual theories must have the same index).

If we make unrealistic simplifications (W does not have poles, the configuration space is compact, etc.), one can compute the index using the standard formula for counting zeroes of a section,

$$I = \int_M c_n(\mathcal{L} \otimes T^* M),$$

where M is the a configuration space, and W is a section of a line bundle $\mathcal{L}$ over M. To some extent one can relax these simplifications, but as yet not enough to treat examples from string theory.

In any case, the index I is perhaps the most basic quantity which characterizes a region in $N = 1$ configuration space, and is computable in principle in terms of "protected" quantities.

In our earlier discussion of quantized holomorphic Chern-Simons theory, we mentioned that the closed string moduli are in fact dynamical (governed by the Kodaira-Spencer theory of Bershadsky et al). If we integrate over them, we cannot regard the superpotential as a function (or section) on this configuration space.

Rather than compute a superpotential, it is tempting to instead define the theory so that its partition function counts supersymmetric vacua, in the sense we just defined.

$$I = \int [DA][d\phi]e^{W}$$

evaluated with (presumably) a choice of CY_3 and topological sector for the gauge field (determined by a choice of orientifold projection).

If these numbers are finite, they should be quite interesting to compute.