Matrix Models, Topological Strings, & $\mathcal{N}=1$ Gauge Theory

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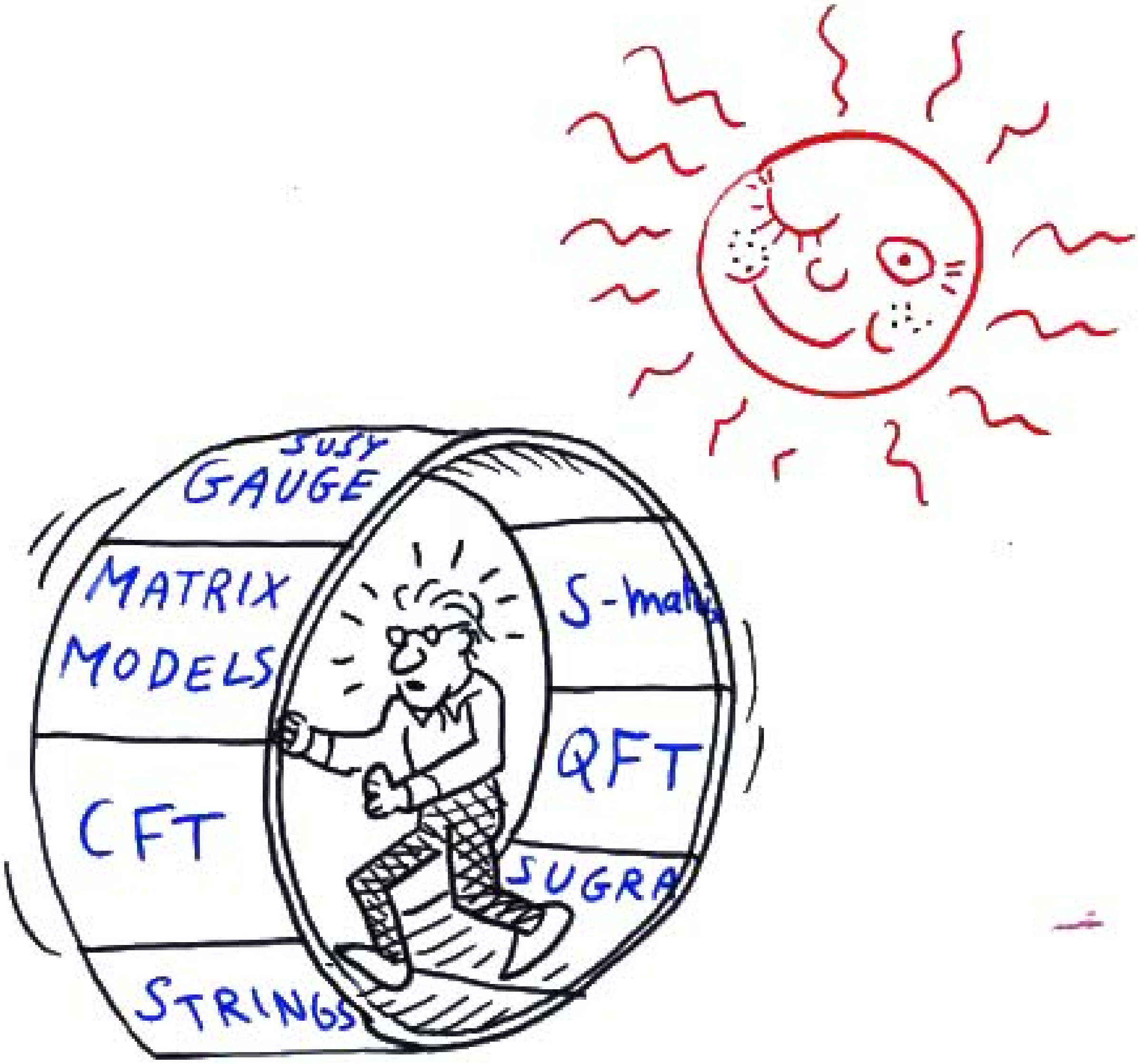
w/ Cumrun Vafa

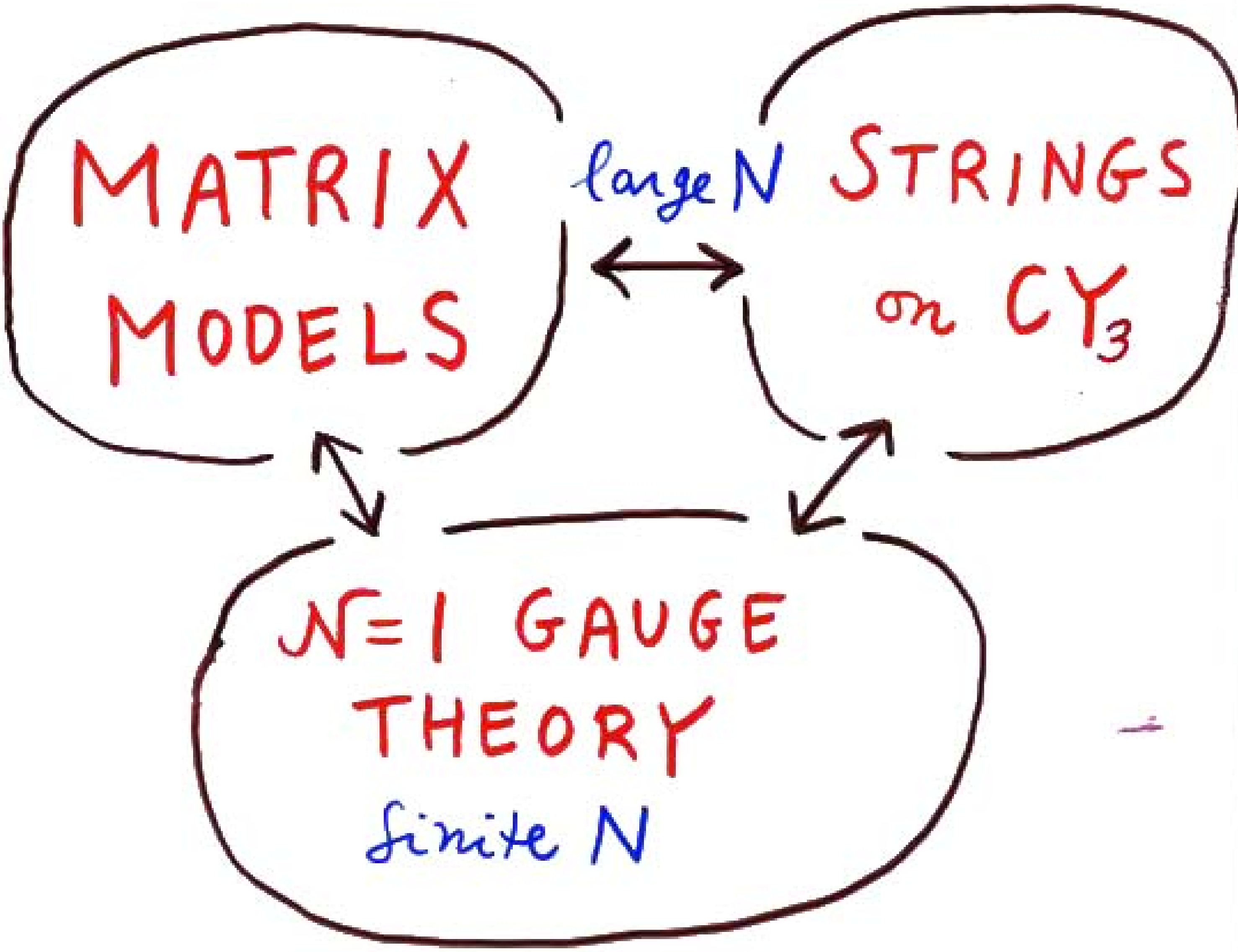
hep-th/0206255
hep-th/0207106
in progress...

Vafa & collaborators

(STRINGS 2002)



cyclic universe


MATRIX MODELS
large N
STRINGS on CY_3
$\mathcal{N}=1$ GAUGE THEORY
finite N

1) Large N MATRIX MODELS

$$\boxed{\int_{N\times N} d\phi \cdot e^{\mathrm{Tr}\, W(\phi)} \underset{\text{saddle pt}}{\sim}} e^{-\sum_{g\geq 0} N^{2-2g} \mathcal{F}_g}$$

$$\mathcal{F}_g = \sum_{\text{graphs}}$$

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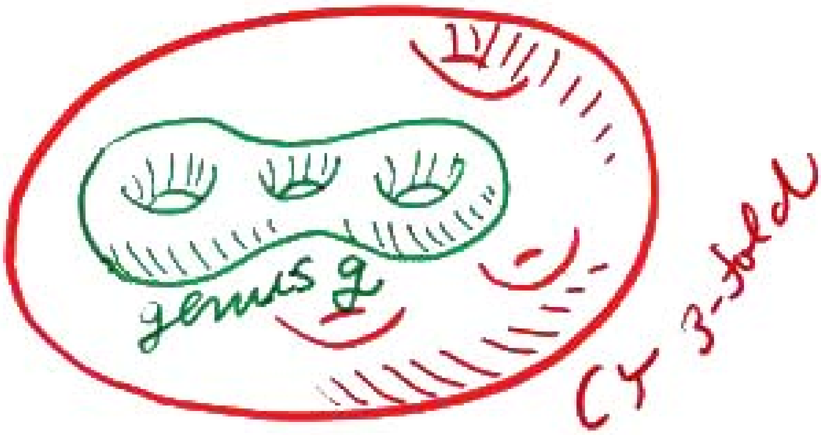
2) CLOSED STRINGS in CY

$\mathcal{F}_g$:



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3) D=4, $\mathcal{N}=1$ GAUGE THEORY

$$\int d^2\theta \; (\text{F-TERMS})$$

$\mathcal{F}_0 \rightarrow W_{eff}(S)$ finite N !

$\mathcal{F}_g \rightarrow$ R^2-terms (grav. induced)

N=1 GAUGE THEORY, adj Φ

U(N)

tree level

$$\int d^4x\, d^2\theta \cdot W(\Phi)$$

effective superpotential

$$W_{eff}(S) \qquad S = \langle T_r \lambda\lambda \rangle$$

gluino condensate

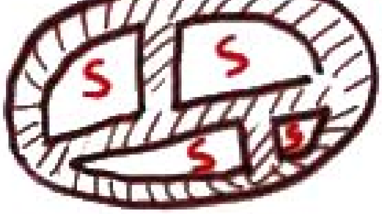
Burgess-Derendinger-Quevedo-Quirós
De la Macorra-Ross

claim: one-loop

$$\boxed{W_{eff} = N\left(S \log(S/\Lambda)\right) - \alpha_{bare} S + \sum_n N a_n S^n}$$

PLANAR diagrams

n holes

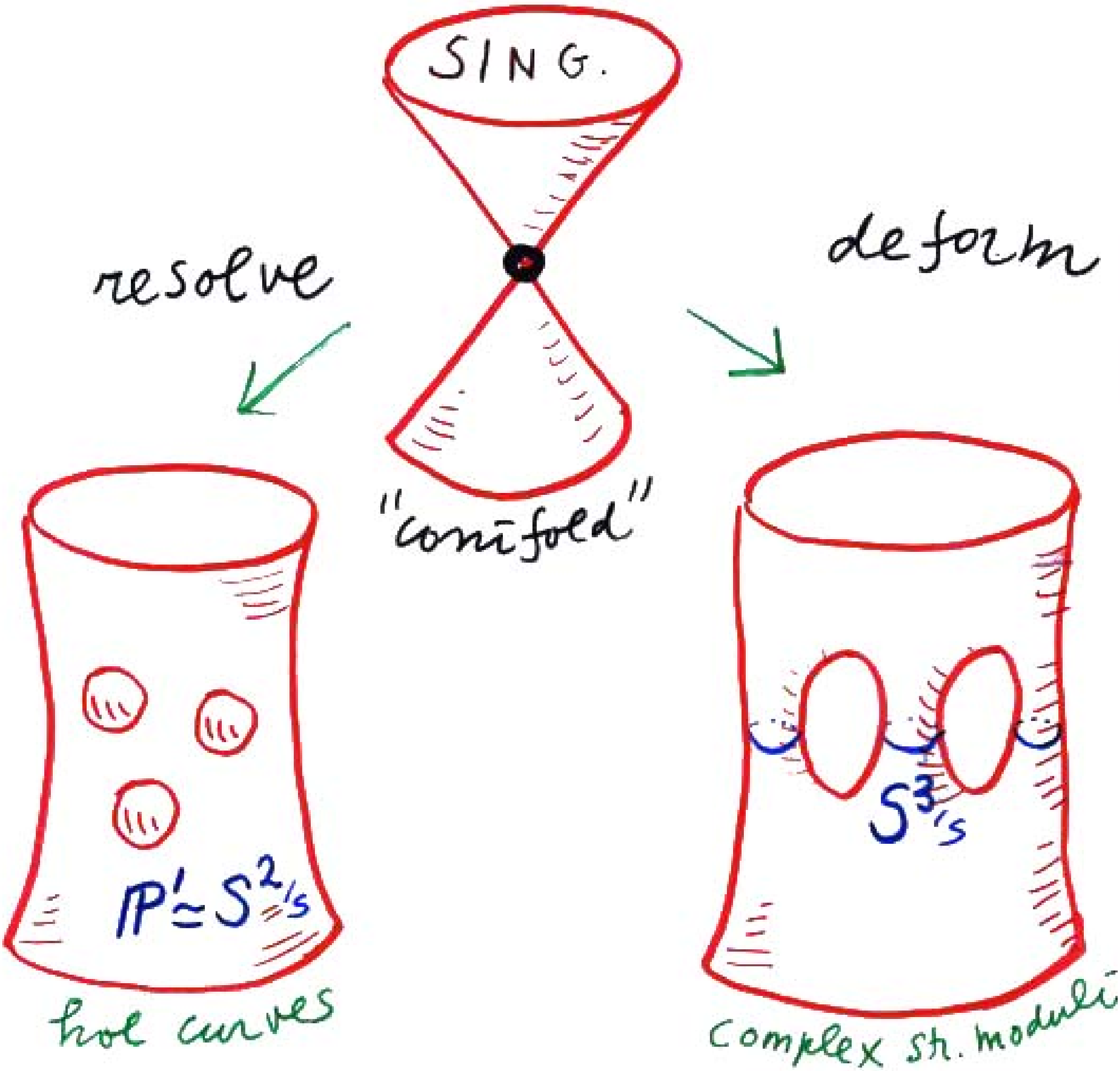


matrix model

$$\int_{N\times N} d\phi\, e^{W(\phi)}$$

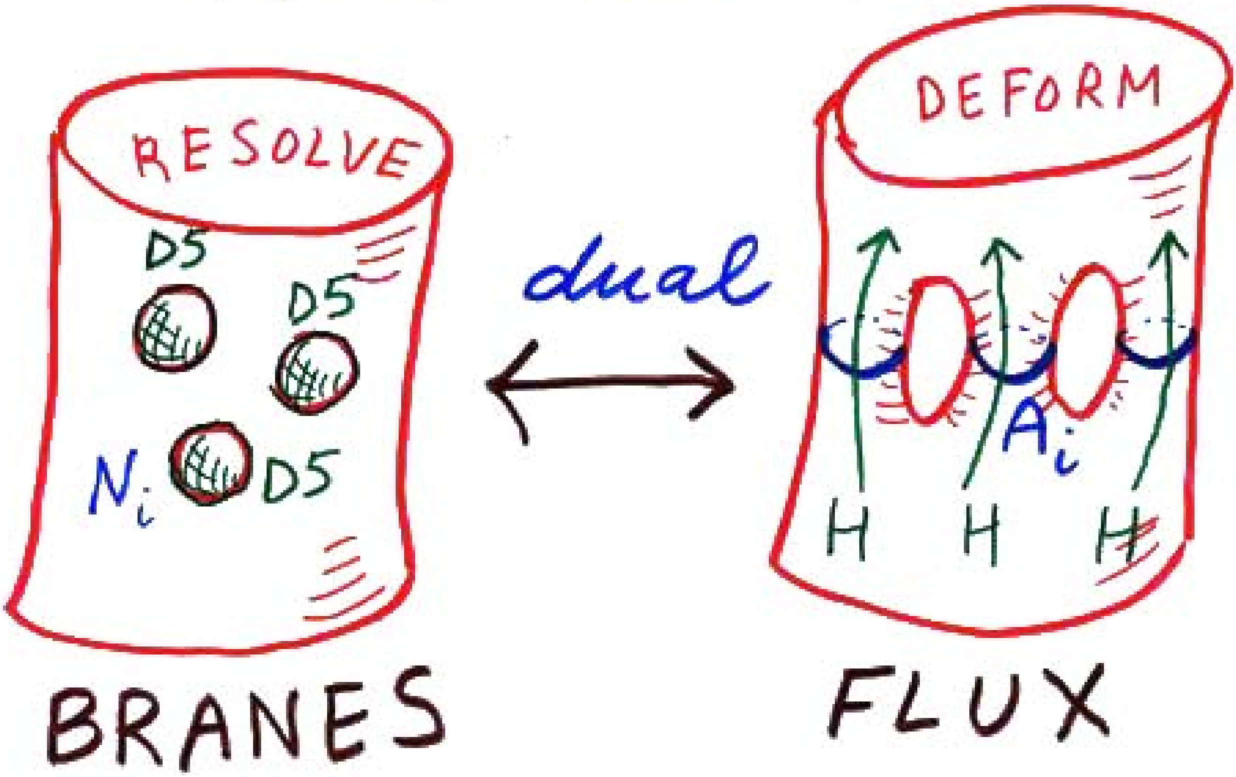
GEOMETRIC TRANSITIONS

IIB String on $CY_3 \times \mathbb{R}^4$



$\mathcal{N}=2$ SUSY

$\mathcal{N}=1$ SUSY



$$\#\, D5/S^2_i = N_i = \oint_{A_i \simeq S^3_i} H_{RR}$$

$$U(N) \rightarrow U(N_1) \times \cdots \times U(N_n)$$

$$\langle \mathrm{Tr}\, \lambda_i \lambda_i \rangle = S_i = \text{moduli} = \oint_{A_i} \Omega$$

gluino condensate

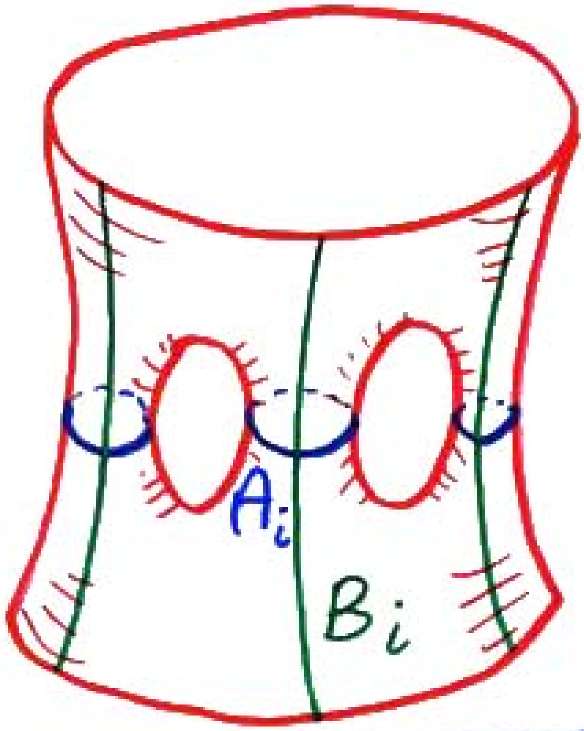
3-form

SUPER POTENTIAL

Gukov Vafa Witten

$$W_{eff}(S) = \int \Omega \wedge H$$

$$= \sum_i N_i \frac{\partial \mathcal{F}_0}{\partial S_i} - \alpha S_i$$

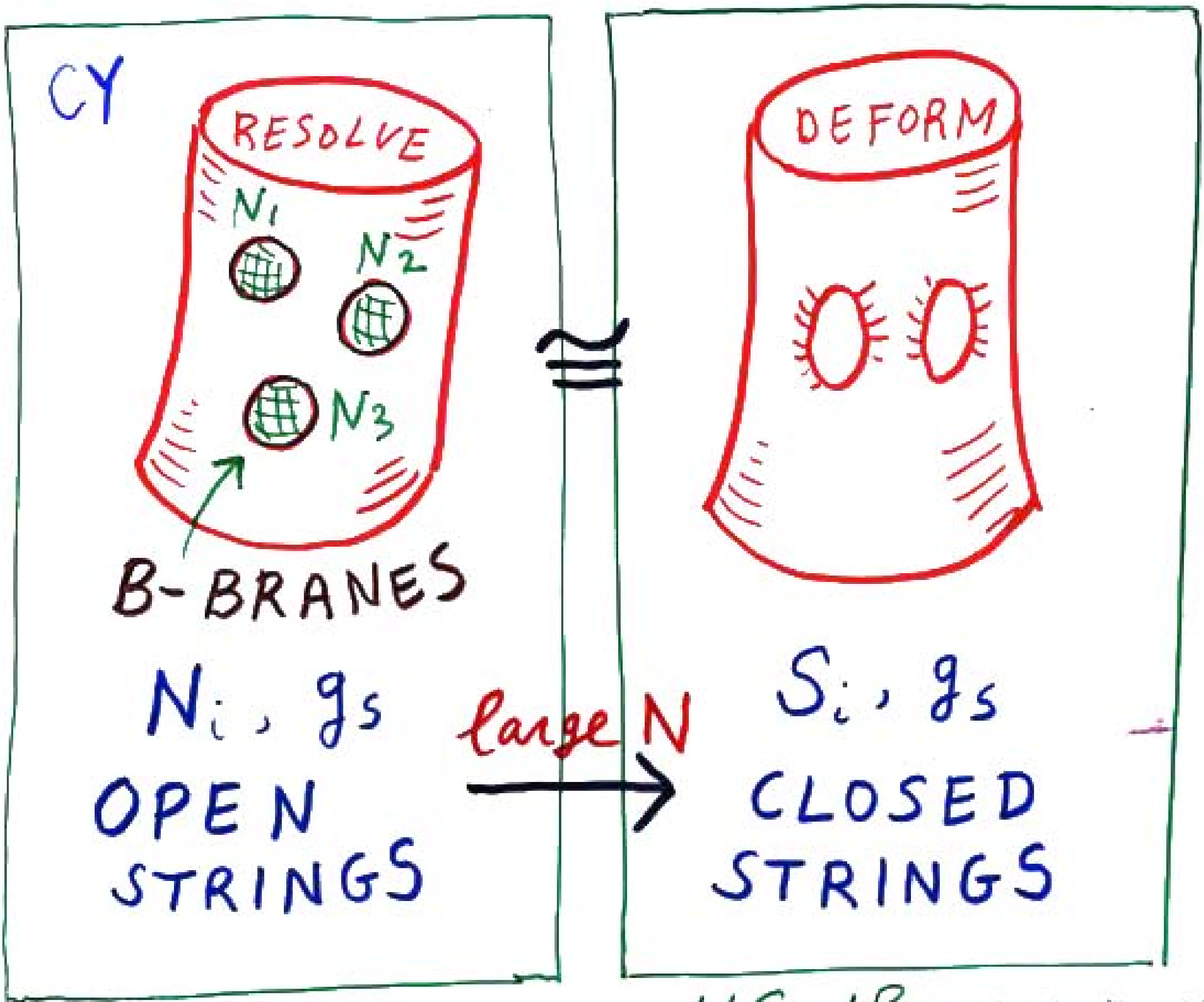


$$\boxed{\begin{aligned} S_i &= \oint_{A_i} \Omega \\ \frac{\partial \mathcal{F}_0}{\partial S_i} &= \int_{B_i} \Omega \end{aligned}}$$

prepotential $\mathcal{F}_0(S)$

SPECIAL GEOMETRY

B TOPOLOGICAL STRINGS



2D theory

KS theory BCOV

't Hooft couplings

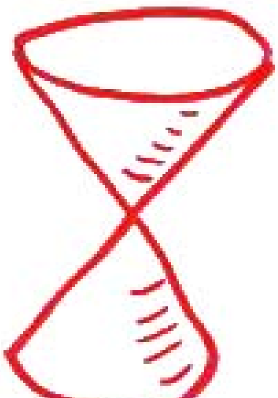
$$g_s N_i = S_i = \text{moduli}$$

(mirror to A-model: Ooguri's talk)

SIMPLE EXAMPLE

Cachazo-Intriligator-Vafa

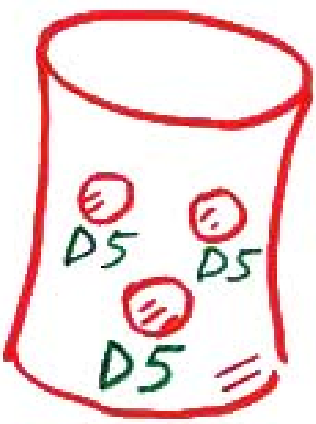
SINGULAR CY_3 $\quad W = x^{n+1} + \ldots$



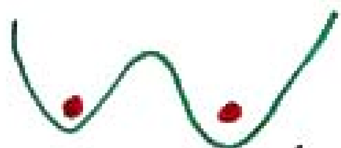
$$u^2 + v^2 + y^2 + W'(x)^2 = 0$$

(conifold $u^2+v^2+y^2+x^2=0$)

RESOLVED :

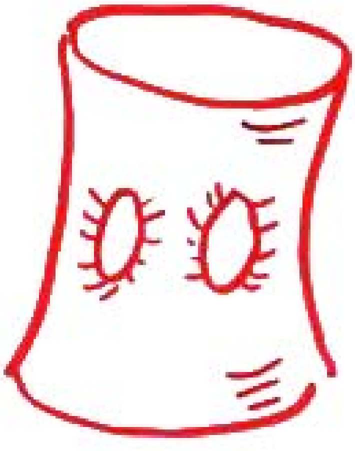


$$\int d^2\theta \cdot \mathrm{Tr}\, W(\Phi)$$



n crit pts

DEFORMED

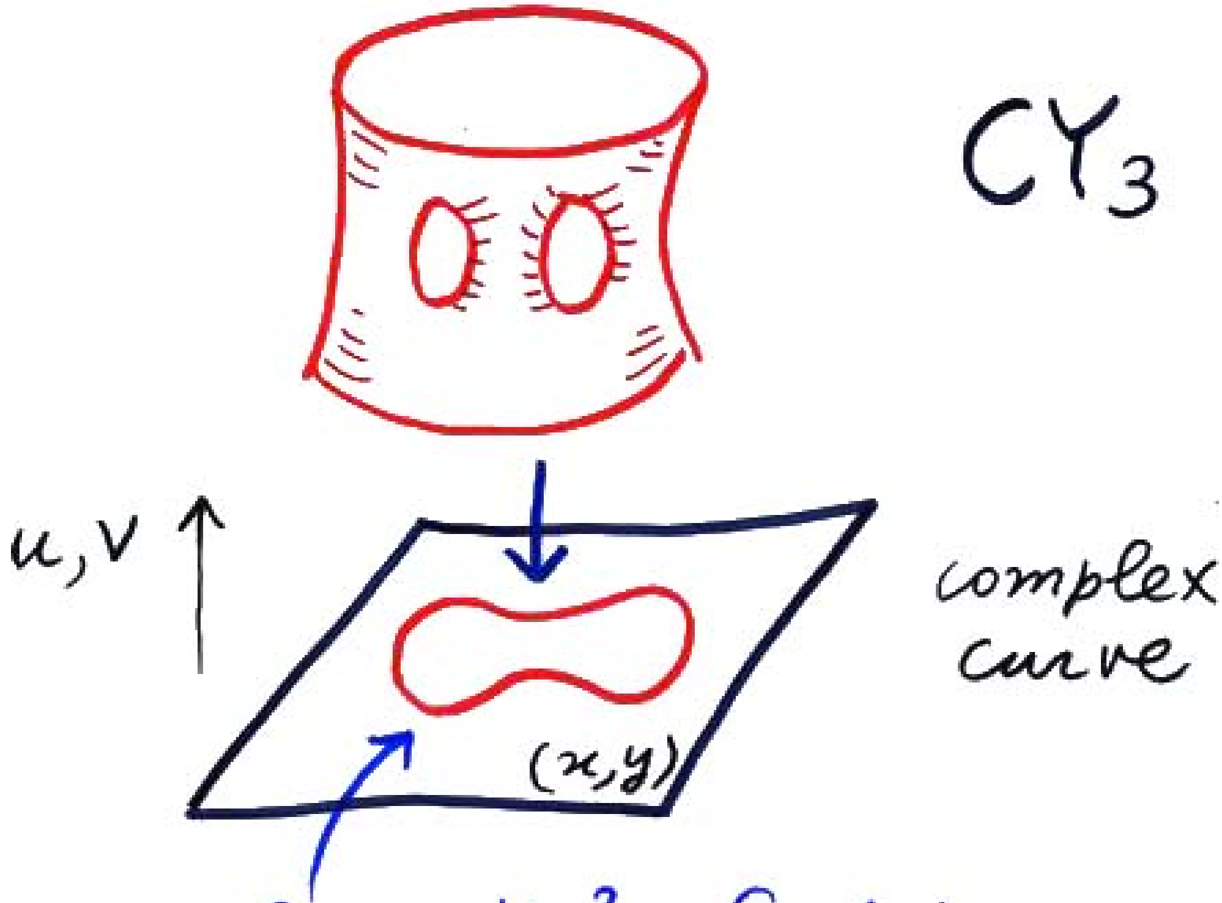


$$u^2 + v^2 + y^2 + W'(x)^2 + f_{n-1}(x) = 0$$

n moduli $\sim S_i$

$(u^2+v^2+y^2+x^2=\mu)$

DIM$^{\underline{al}}$ REDUCTION



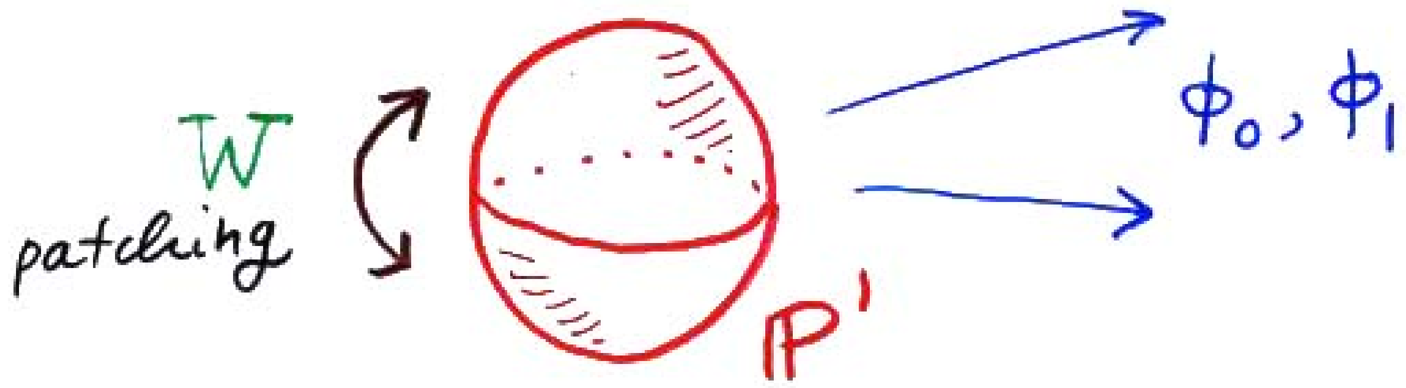
$$y^2 + W'(x)^2 + f_{n-1}(x) = 0$$

$$\Omega = \frac{du\,dv\,dx}{y}$$ 3-form holomorphic

$$\downarrow$$

$$y\,dx$$ 1-form meromorphic

OPEN STRINGS on BRANE



FIELDS: $(A_{\bar{z}}, \phi_0, \phi_1)$ $N \times N$

spin 0, 1 — chiral

hol CS Witten

$$\boxed{\int_{\mathbb{P}^1} d^2 z \, \mathrm{Tr}\left(\phi_0 \bar{D}_A \phi_1 + W(\phi_0)\right)}$$

Kachru-Katz-Lawrence-McGreevy

localize

$$\phi_0(z) = \phi \text{ cnst}$$

$$\boxed{\frac{1}{\mathrm{Vol}\, U(N)} \int_{N \times N} d\phi \cdot e^{-\frac{1}{g_s} \mathrm{Tr}\, W(\phi)}}$$

holomorphic

MATRIX MODEL

EIGENVALUE DYNAMICS

Brézin-Itzykson-Parisi-Zuber

eigenvalues

$$\Phi = U \cdot \begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_N\end{pmatrix} \cdot U^{-1}$$

$$\int \prod_i d\lambda_i \cdot \underbrace{\prod_{i<j}(\lambda_i-\lambda_j)^2}_{\text{JACOBIAN}} \cdot e^{-\frac{1}{g_s}\sum_i W(\lambda_i)}$$

(integrate out U)

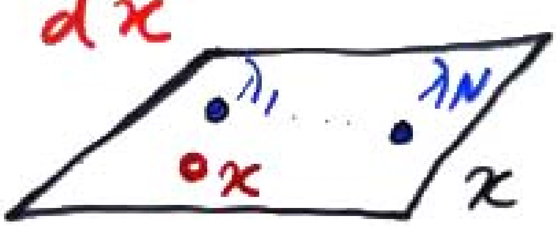
effective action

COULOMB repulsion

$$S = \sum_i W(\lambda_i) - 2g_s \sum_{i<j} \log(\lambda_i - \lambda_j)$$

PROBE EIGENVALUE $\lambda_i = x$

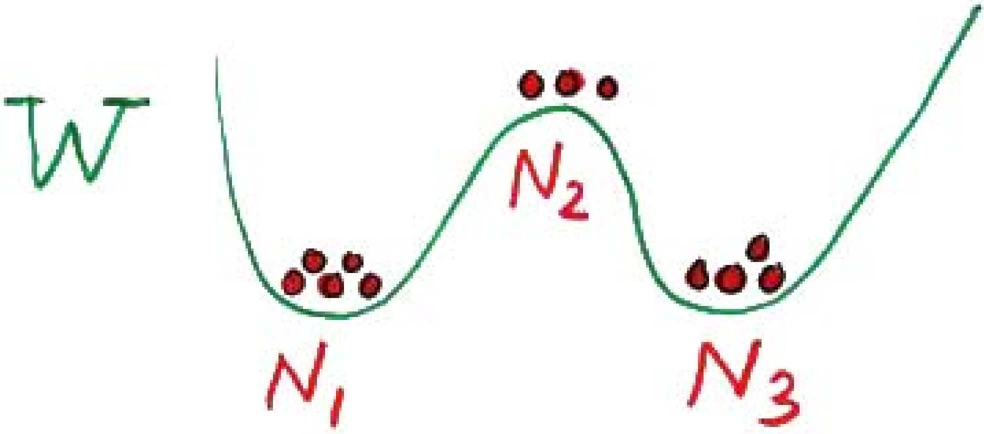
$$y(x) = \frac{dS}{dx} = W'(x) - 2g_s \sum_j \frac{1}{x-\lambda_j}$$



force

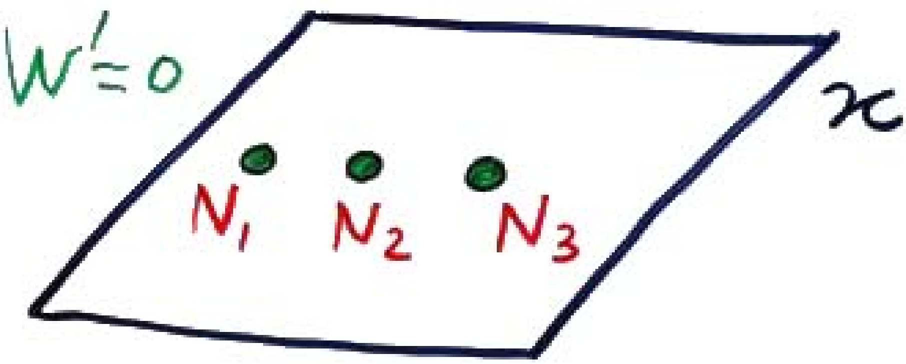
CLASSICAL $\quad g_s = 0$

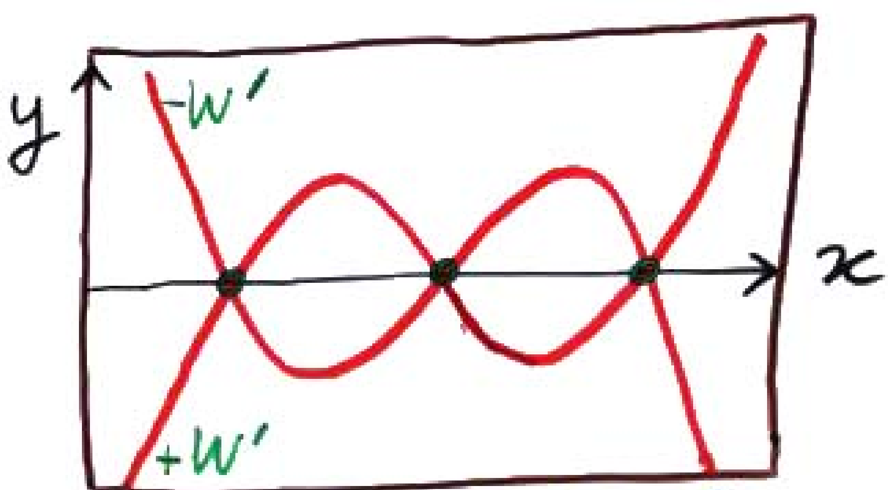
$$y = W'(x) = 0$$



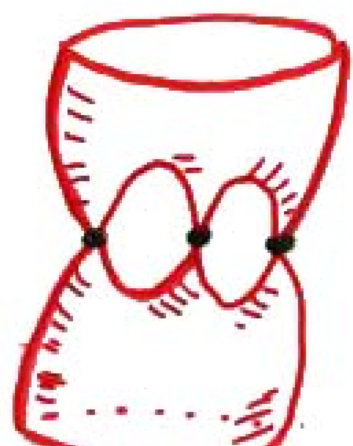
distribute λ_i's

filling numbers $\sum N_i = N$





$$y^2 = W'(x)^2$$



Riemann surface

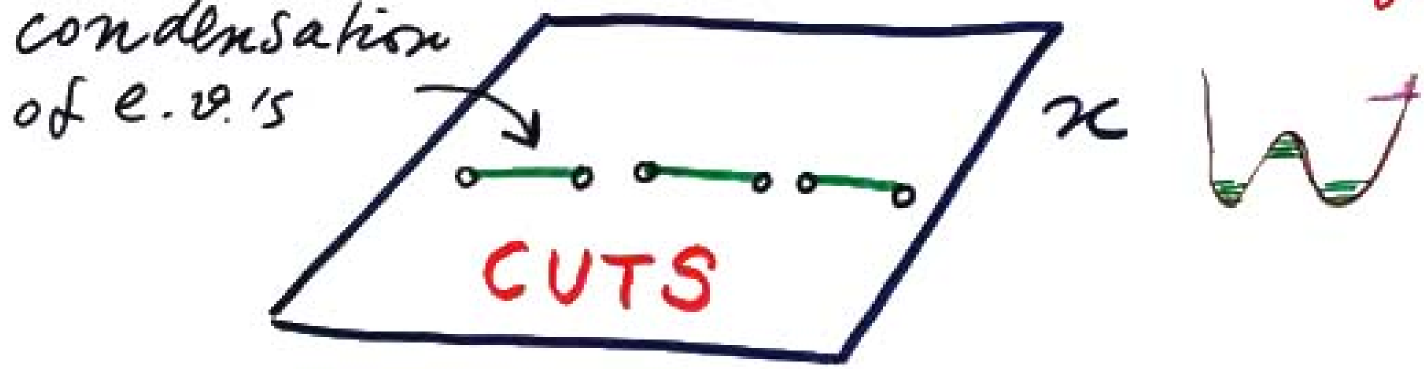
QUANTUM $g_s > 0$

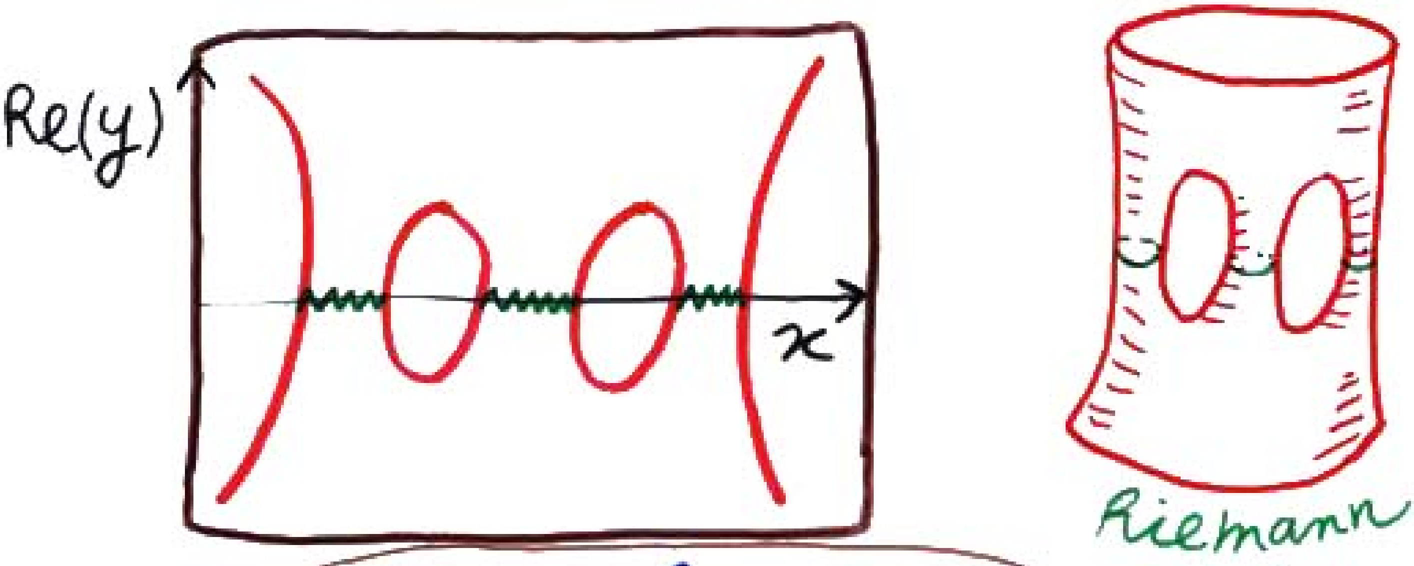
't Hooft limit $N_i \to \infty$

$S_i = g_s N_i$ fixed $\frac{N_i}{N}$ finite

continuous eigenvalue density

$$\rho(x) = \frac{1}{N} \sum \delta(x - \lambda_i) = \text{jump } \frac{y(x)}{g_s}$$



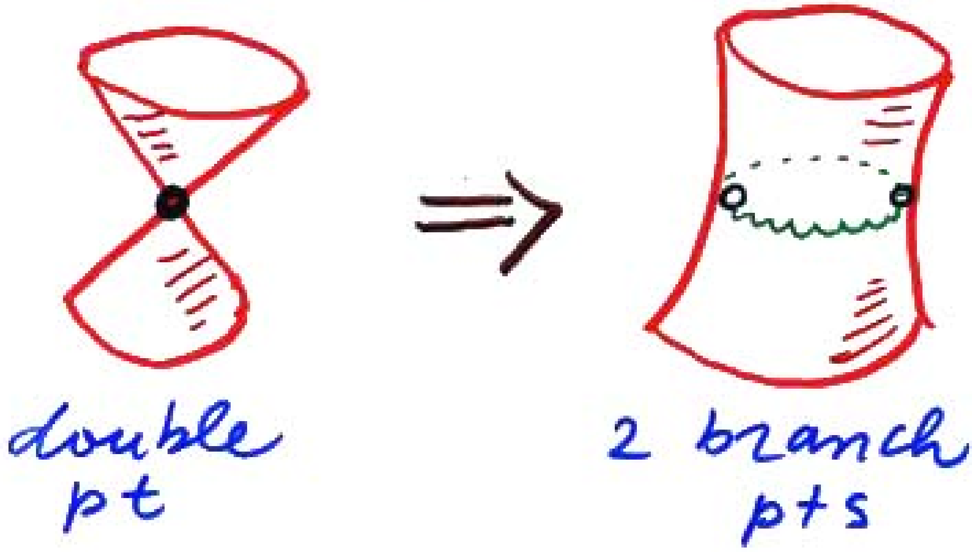


$$y^2 = W'(x)^2 + f_{n-1}(x)$$

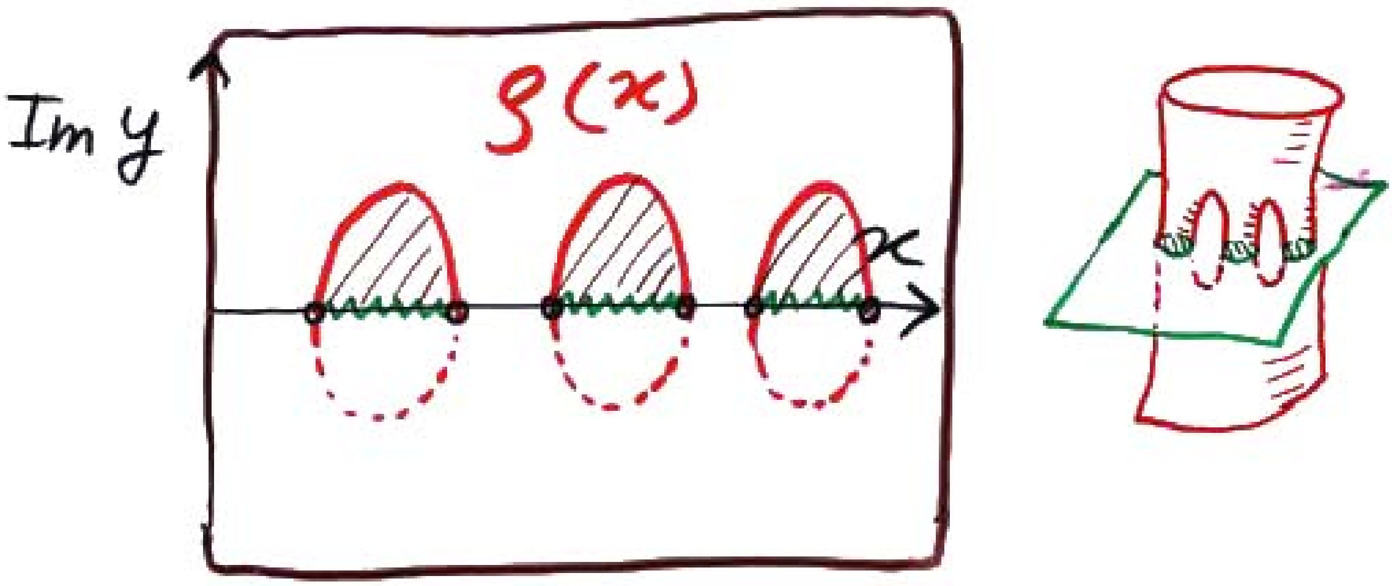
determined by S_i

SPECTRAL CURVE

Quantum resolution of sing.

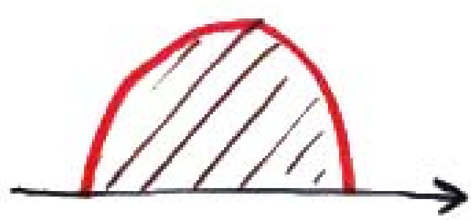


N_i size cut $\sim g_s N_i = S_i$



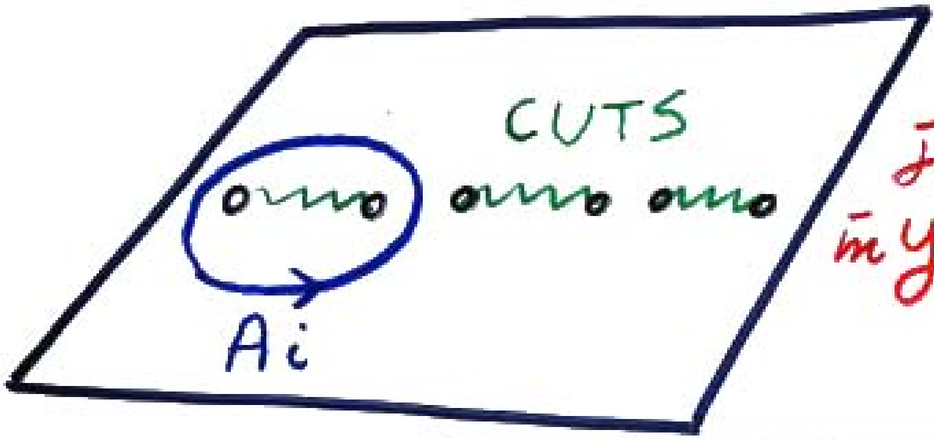
CONIFOLD: gaussian $W = x^2$

$\rho(x) = \sqrt{\mu - x^2}$



WIGNER semi-circle $\int_{N\times N} d\phi \, e^{-\mathrm{Tr}\,\phi^2}$

SPECIAL GEOMETRY & CY_3

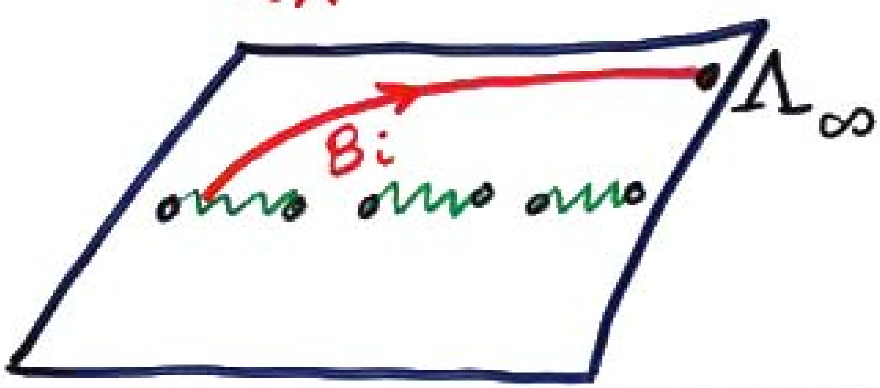


Jump
$\mathrm{Im}\, y(x) = \rho(x) g_s$

$$* \quad \boxed{\frac{1}{2\pi i}\oint_{A_i} y\,dx = g_s N_i = S_i}$$

filling #

$y(x) = \dfrac{dS}{dx}$ force on probe $\lambda_i = x$



$$* \quad \boxed{\int_{B_i} y\,dx = \frac{\partial F_0}{\partial S_i}}$$

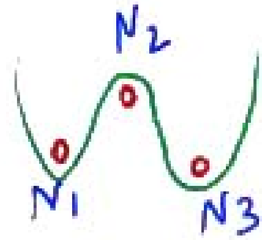
free energy

$$CY_3 : \; u^2 + v^2 + y^2 + W'(x)^2 + f(x) = 0$$

$$\Omega \to y\,dx$$

$\mathcal{N}=1$ GAUGE THEORY

$\mathcal{N}=2 \longrightarrow \int d^2\theta\, W(\phi)$



$U(N) \longrightarrow U(N_1) \times \cdots \times U(N_n)$

$\Downarrow$ confine

gluino condensates $\quad S_1, \ldots, S_n$

EFFECTIVE SUPERPOTENTIAL

CIV

$$W_{eff} = \sum_i \left(N_i \frac{\partial \mathcal{F}_0}{\partial S_i} - \alpha S_i \right)$$

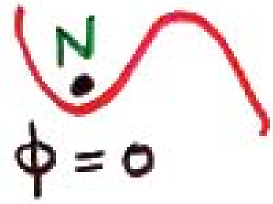
vol $U(N_i)$ in MM

$$= \sum N_i S_i \log S_i/\Lambda + \text{PERTURBATIVE}$$

"one-loop"

only PLANAR diagrams contribute!

e.g. $W = m\phi^2 + g\phi^3$,

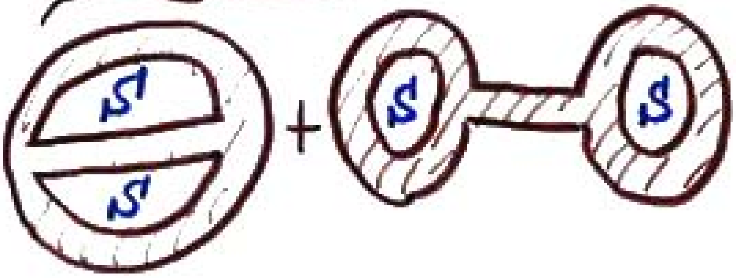


$W_{eff} = N S \log S/\Lambda - \alpha S$

↑ bare coupling

$+ * \frac{g^2}{m^3} S^2 + \dots$

CIV



computed by MATRIX MODEL

SOLVE: $d W_{eff} = 0$

$S \sim e^{-\frac{\alpha}{N\Lambda}}$

fract inst.

$W_{eff} = \sum$ instantons

- PERTURB. $\Rightarrow$ INSTANTONS
  nth order — n instantons (Nekrasov?)
- PLANAR = EXACT
- Large N = CY 3-fold
  "master field"

Douglas

CLAIM: this works in general*

*classical groups, vectorlike rep, massive + abelian vacua.

$D=4,\ \mathcal{N}=1$ gauge theory

$$\int d^2\theta \cdot W(\phi_i)$$

MATRIX MODEL: $\Rightarrow$ 0 dim

$$\boxed{\int d\phi_i \cdot e^{-\frac{1}{g^2}W(\phi_i)}}$$

PLANAR: $W_{eff}(S) = \sum (N_i \partial_i \mathcal{F}_0 - \alpha_i S_i$

NON-PLANAR: gravity induced

e.g. $\mathcal{F}_1(S) \cdot R \wedge R$ 

WHY?

field theory argument

sketch (in progress)

$$\int d^2\theta \cdot W_{eff}(S)$$ "topological" INDEX

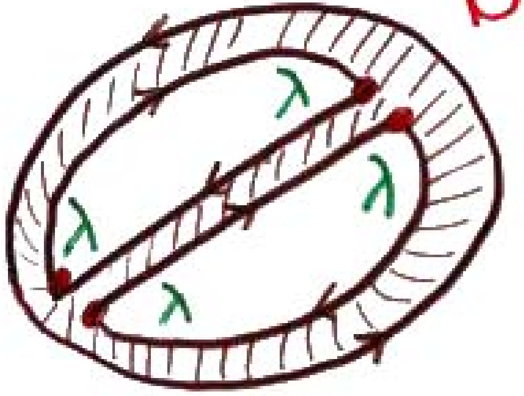
F-term: localize $x^\mu = \theta^\alpha$ const

$$\int d^4x\, d^2\theta\, W_{tree}(\Phi) \longrightarrow W(\Phi)$$

QFT_4 $\qquad$ MM_0

PLANAR? $\lambda\lambda$ / boundary (cut)

S



top strings

BCOV

Berkovits

$\alpha' \to 0$ Schwinger picture QFT

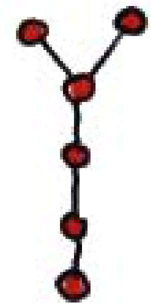
QUIVERS: ADE, $\hat{A}\hat{D}\hat{E}$

$$\boxed{W = \sum_i W_i(\phi_i) + \sum_{\langle i,j \rangle} Q_{ij}\, \phi_j\, Q_{ji}}$$

Cachazo-Katz-Vafa

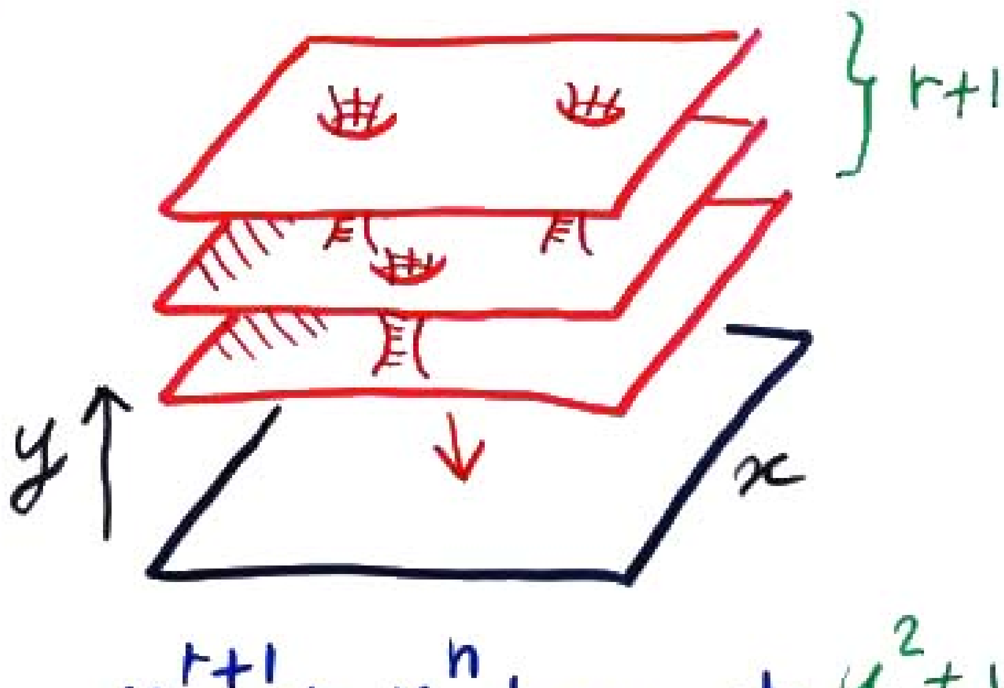
solvable multimatrix models

Kostov
Kharchev et al

strings on 

$$\prod (\lambda_I - \lambda_J)^2 \longrightarrow \prod (\lambda_{i,I} - \lambda_{j,J})^{p_i \cdot p_j}$$

MULTIPLE COVERS



$$y^{r+1} + x^n + \ldots + u^2 + v^2 = 0$$

SEIBERG-like dualities

(same univ. class MM)

(in progress)

MM $\longrightarrow$ CY_3 > curves

e.g. $W = \phi_1^n + \phi_1 \phi_2^2$

$$\downarrow \text{CKV}$$

$$u^2 + F(v, x, y) = 0$$

complex surface

("Laufer geometry")

$N=4 \longrightarrow N=1^*$ (in progress)

$$W = \phi_1 [\phi_2, \phi_3] + m \sum \phi_i^2$$

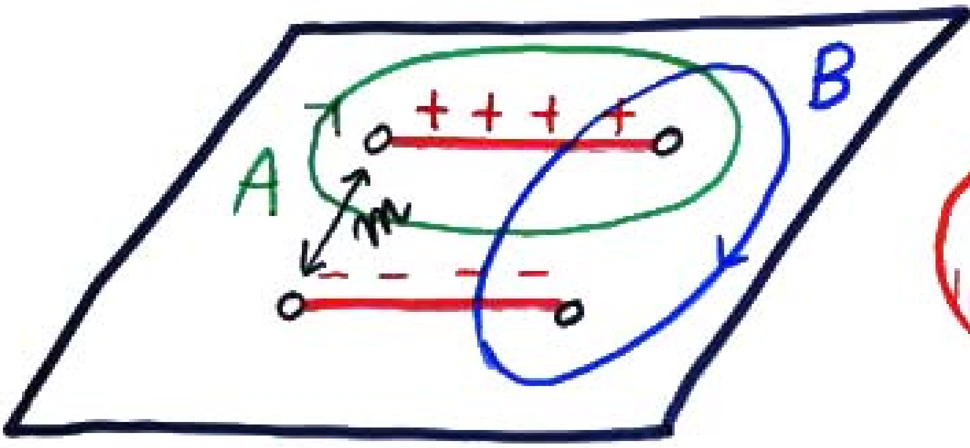
3 MATRIX MODEL

Kazakov
-Kostov
-Nekrasov

integrate out ϕ_2, ϕ_3

$$\int d\phi_1 \, \frac{e^{-T_2 \phi_1^2}}{\det(m + [\phi_1, \cdot])}$$

$\downarrow$ large N



T^2

elliptic curve

S-DUALITY

periods

$$\Pi_A = \oint_A y\,dx$$

$$\Pi_B = \oint_B y\,dx$$

$$W_{eff} = N\,\Pi_B - \tau_0\,\Pi_A$$

$$\Big\downarrow \text{minimize}$$

$$\tau = \frac{\partial \Pi_B}{\partial \Pi_A} = \frac{\tau_0}{N}$$

$$W_{eff} \sim E_2(\tau) + \ldots$$ Dorey

$SL(2,\mathbb{Z})$ perturbative

LEIGH-STRASSLER deformation

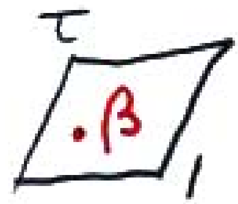
$$[\phi_2, \phi_3] \to \quad \phi_2\phi_3 - q\,\phi_3\phi_2$$

$$q = e^{2\pi i \beta}$$

6 VERTEX MODEL

Kazakov
-Kostov

$$\beta \to \beta + \tau$$



preserve modularity

$$\beta \to \beta/(c\tau + d)$$

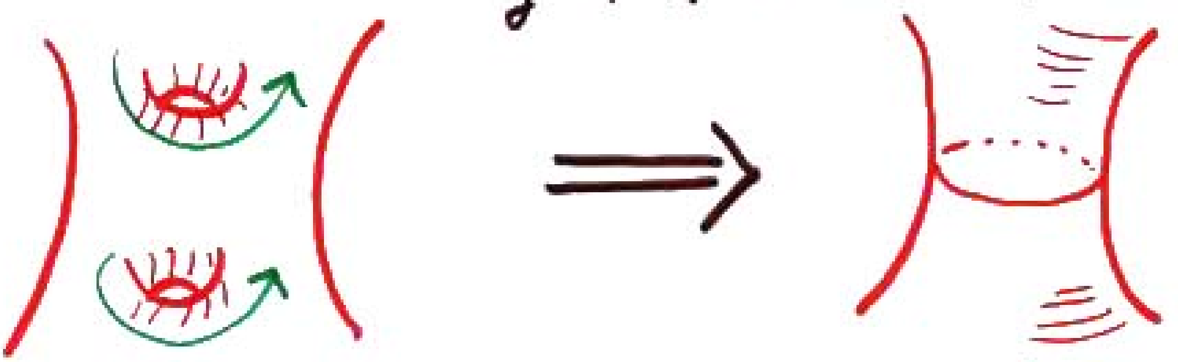
DOUBLE SCALING LIMITS

g_s fixed, $N \longrightarrow \infty$

e.g. $N_1 - N_2$ fixed (cf PP≈)

"OLD" (1989) MATRIX MODELS

$y^2 + x^3 + \dots = 0$ pure gravity



UNITARY MODELS

$\Phi \simeq \Phi + 2\pi$ periodic

$U = e^{i\Phi}$, $W = \cos\Phi$

GROSS-WITTEN $\rightarrow$ SEIBERG-WITTEN

$\int dU \cdot e^{\mathrm{Tr}\, U + U^{-1}}$

one-plaquette

$\mathcal{N} = 2$ $SU(2)$

CONCLUSION:

$$\int_{N\times N} d\phi \cdot e^{\mathrm{Tr}\, W(\phi)} \sim e^{\sum N^{2-2g} \mathcal{F}_g}$$

CLOSED STRINGS

CY_3

HIGHER GENUS

non-linear LOOP EQNS

$$y = \partial\varphi(x)$$

$$\langle \partial\varphi(x)^2 \rangle = W'(x)^2 + f(x)$$

quantum CY

$$\downarrow$$

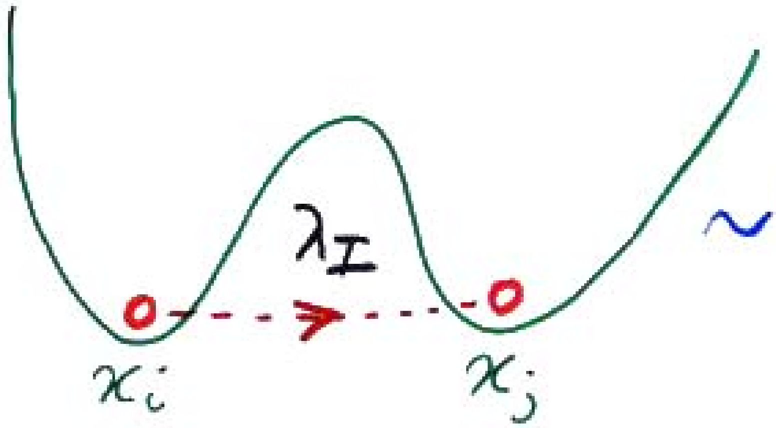
$$\langle T(x) \rangle \text{ regular}$$

VIRASORO constraints

$$\mathbb{L}_n Z = 0 \qquad n \geq -1$$

reparametrize matrix Φ

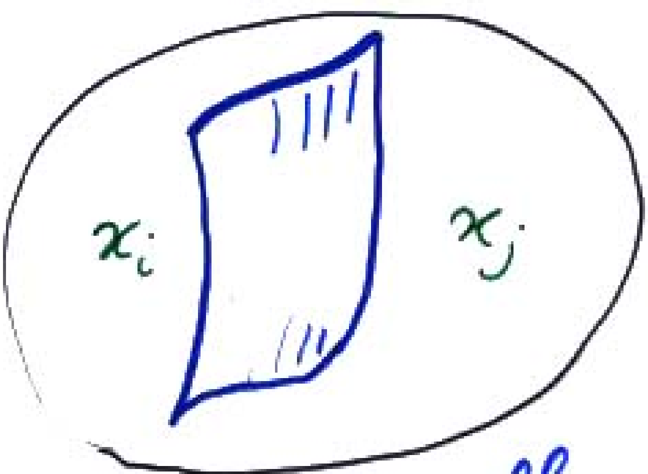
EIGENVALUE TUNNELING $\cong$ DOMAIN WALLS



$\sim e^{-\frac{1}{g_s}\Delta \mathcal{F}}$

Shenker

tension $\sim \frac{1}{g_s}\left(\partial_j \mathcal{F} - \partial_i \mathcal{F}\right) \sim \Delta \bar{W}$



$\mathbb{R}^4$

domain wall