The Entropy of a Hole in Space-Time

Based on:

arXiv:1305.0856, arXiv:1310.4204, arXiv:1406.nnnn

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Related work in:

arXiv:1403.3416 - Myers, Rao, Sugishita
arXiv:1406.4889 - Czech, Dong, Sully
arXiv:1406.4611 - Hubeny

There are many interesting connections between black hole entropy, entanglement entropy and space-time geometry.

Entanglement entropy is usually defined for QFT. It has been suggested that more generally in quantum gravity there is also a notion of entanglement entropy associated to a region and its complement which equals

Bianchi, Myers

$$S_{\rm ent} = \frac{A}{4G}$$

This result is *finite*, as opposed to entanglement entropy in QFT.

Qualitative idea: finiteness is due to built in UV regulator in quantum gravity. UV scale = Planck scale.

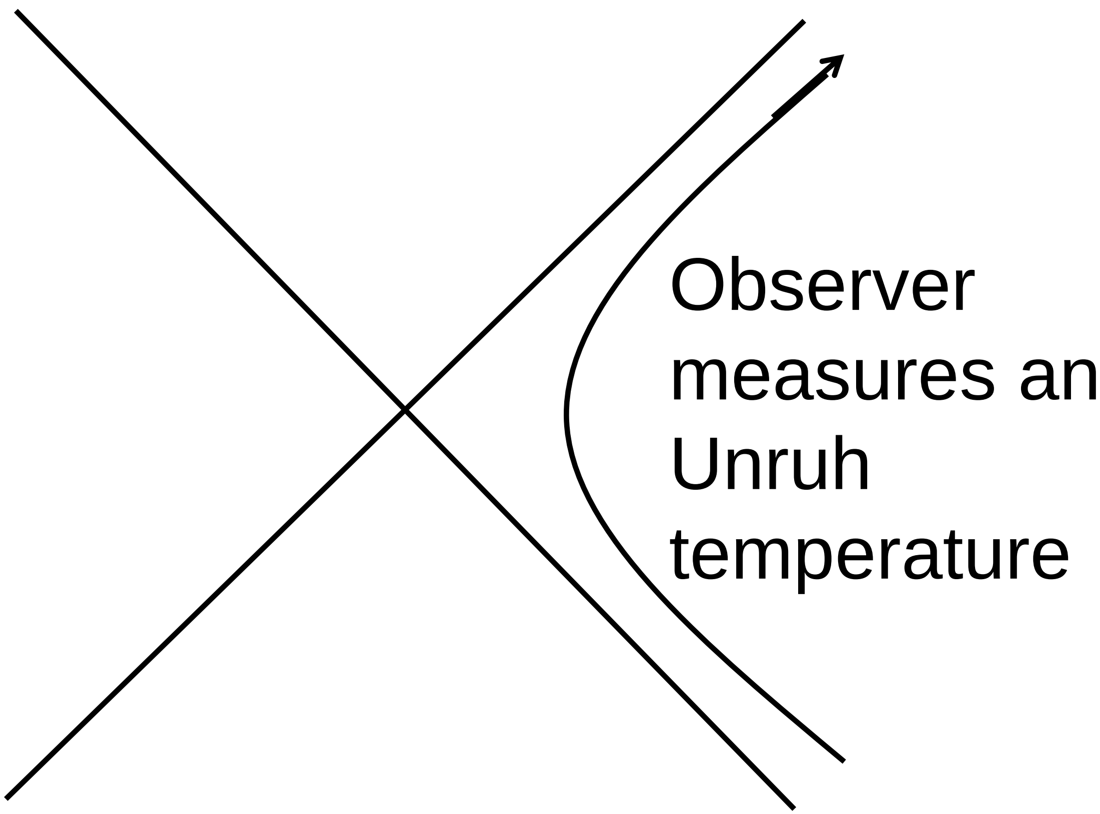
Indeed: $S_{\rm ent} = \frac{A}{\epsilon^{D-2}} + \ldots \sim \frac{A}{4G} \quad {\rm for} \quad \epsilon = \ell_P$

Key question: can we make the link between $\frac{A}{4G}$ and some notion of entanglement entropy more precise?

Would provide an interesting new probe of space-time geometry and the dof of quantum gravity.

We can try to study this question in AdS/CFT.

Idea: use Rindler like philosophy where the entanglement between the left and right half of Minkowski space-time is detected by a accelerated observers who are not in causal contact with one half.



We want to generalize this idea to more complicated situations:

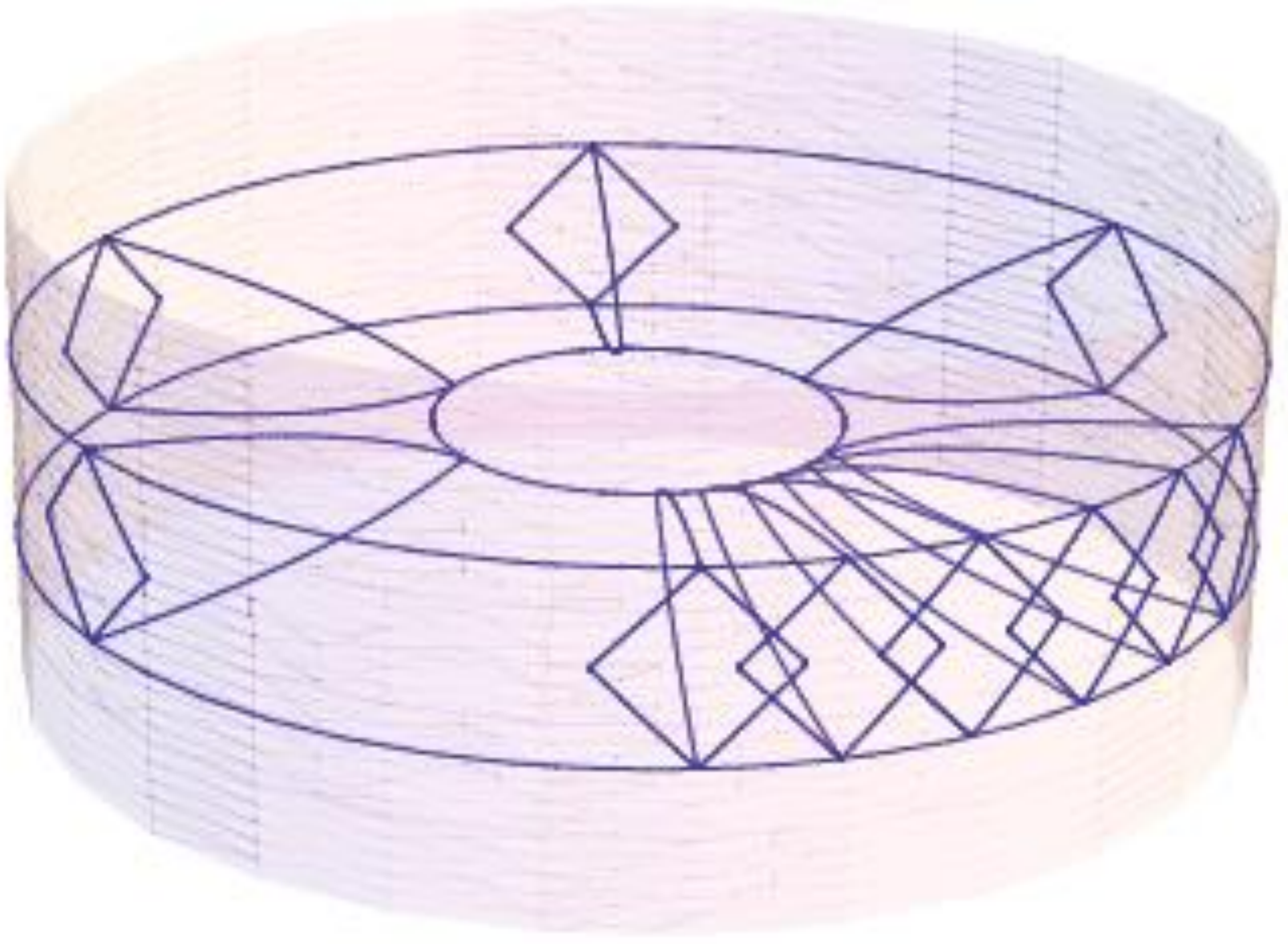
Consider a spatial region A and consider all observers that are causally disconnected from A. These observers must accelerate away from A as in Rindler space.

Individual observers are causally disconnected from a region larger than A. However, all observers together are causally disconnected from precisely A.

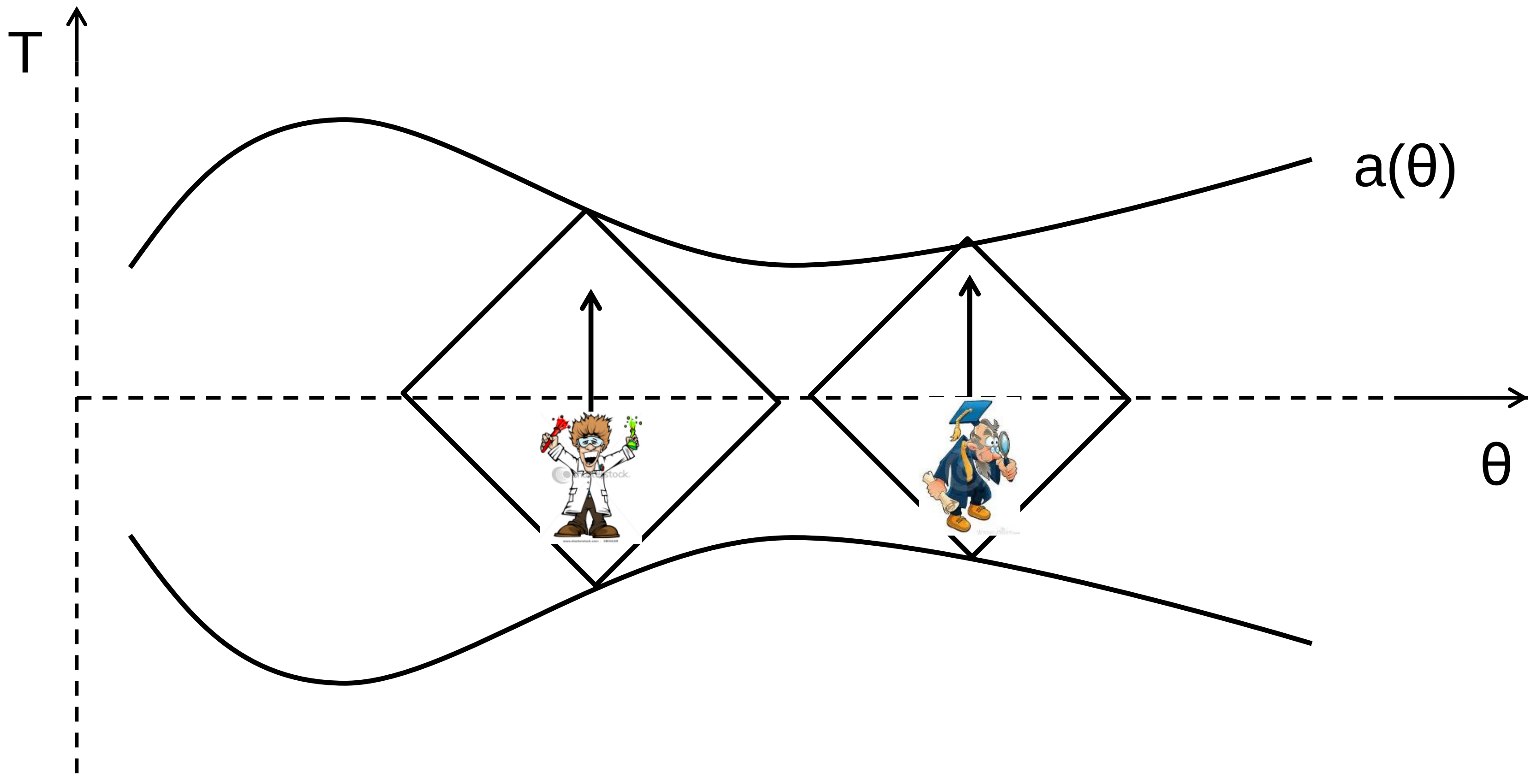
Therefore, this family of observers should effectively see a reduced density matrix where all degrees of freedom associated to A have been traced over. The entropy of this reduced density matrix is a candidate for the entanglement entropy in (quantum) gravity.

How do we associate entropy to a family of observers?

Specialize to a region in AdS3 and consider all observers causally disconnected from this region. These observers connect to a domain on the boundary of AdS3 which covers all of space but not all of time.



Local observers cannot access all information in the field theory, only information inside causal diamonds.



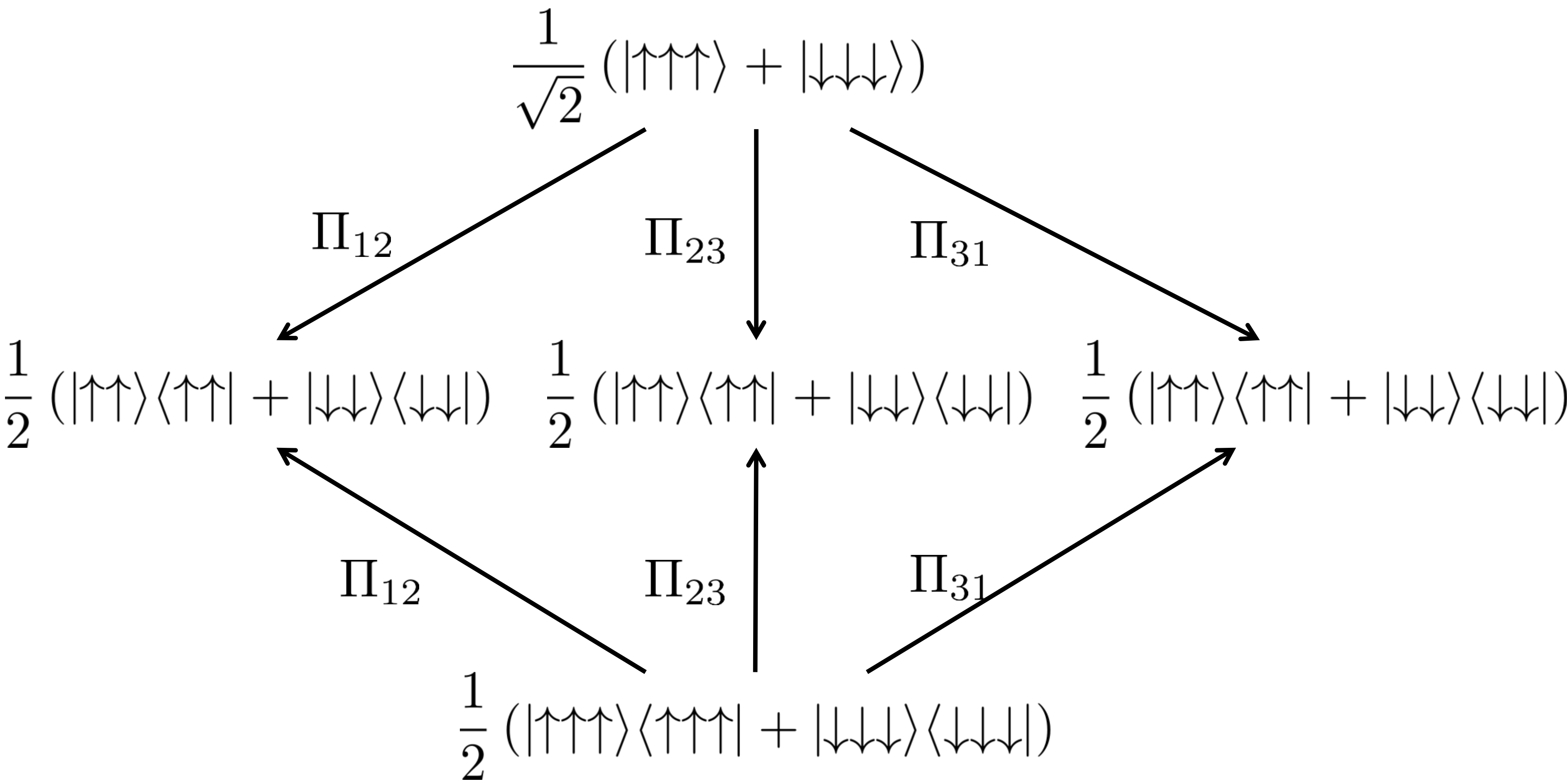
Proposal: in situations like this we can associate a Residual Entropy to the system which measures the lack of knowledge of the state of the full system given the combined information of all local observers.

Can we compute this residual entropy?

Suppose each observer would be able to determine the complete reduced density matrix on the spatial interval the observer can access (unlikely to be actually true)

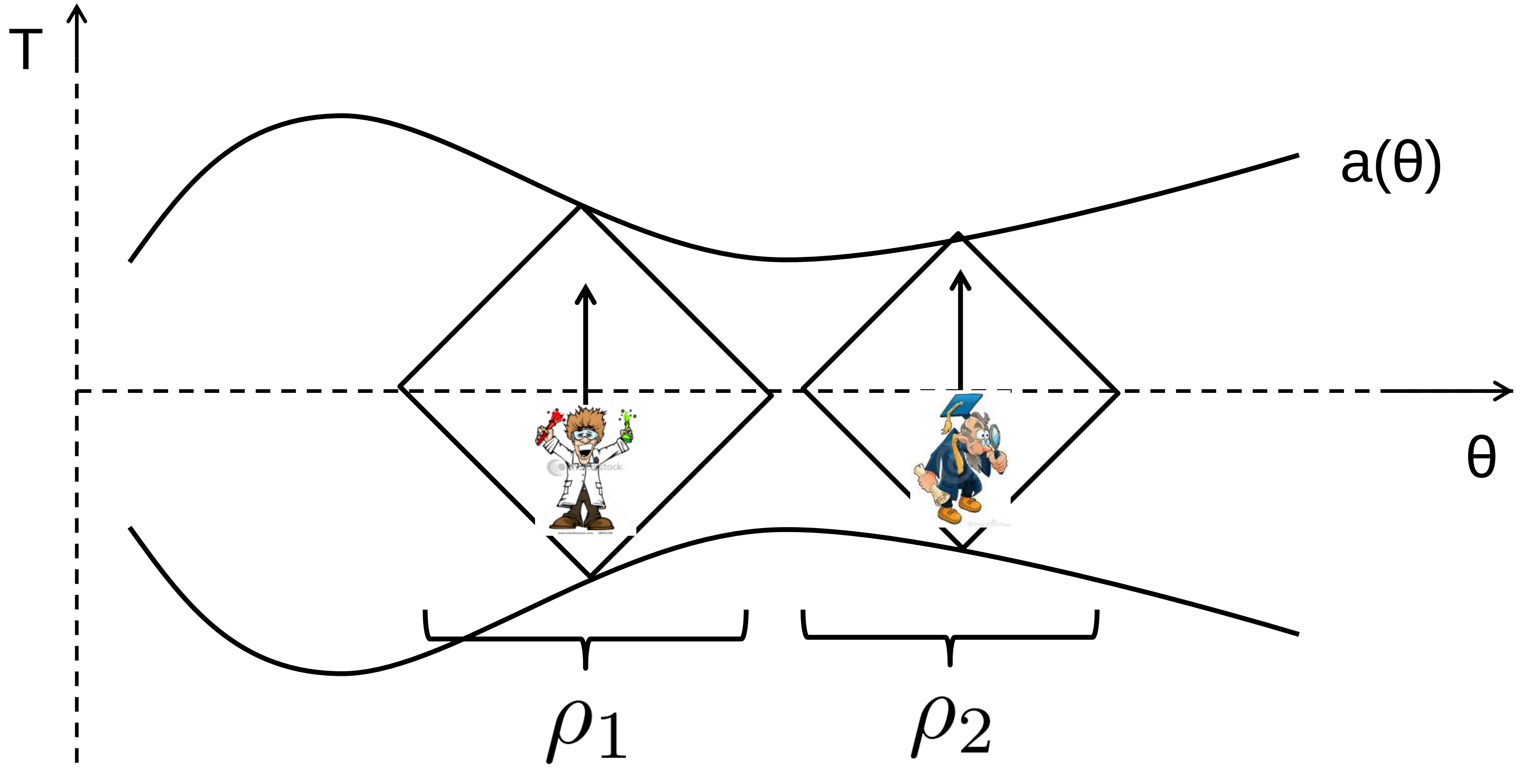
Then the question becomes: given a set of reduced density matrices, what is the maximal entropy the density matrix of the full system can have?

$$S_{\mathrm{RES}} = \max S(\rho) \,|\, \forall i \, \mathrm{Tr}_i \rho = \rho_i$$



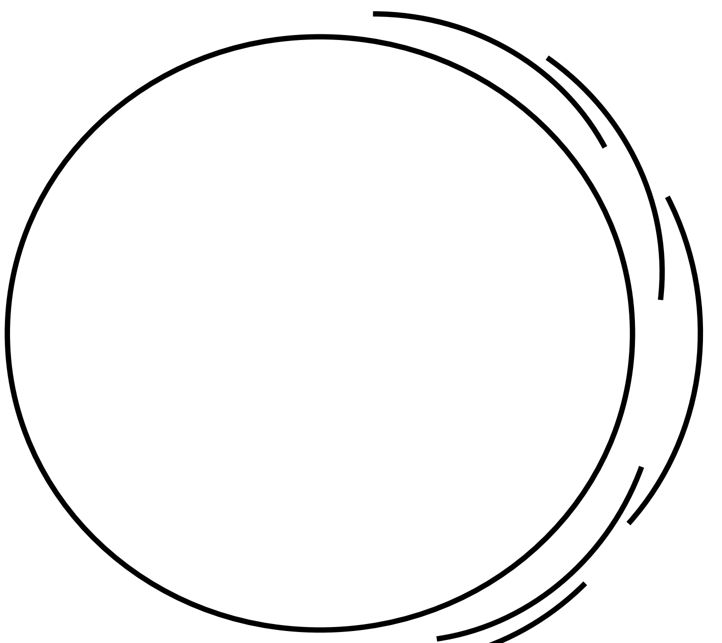
Local observers who can only measure two spin subsystems cannot distinguish the pure state from the mixed state. We would associate Residual Entropy S= log 2 to this system.

Working hypothesis: all local observers can determine the full reduced density matrices associated to the spatial interval they have access to.



Obviously, there are infinitely many local observers and the spatial intervals they have access to will overlap.

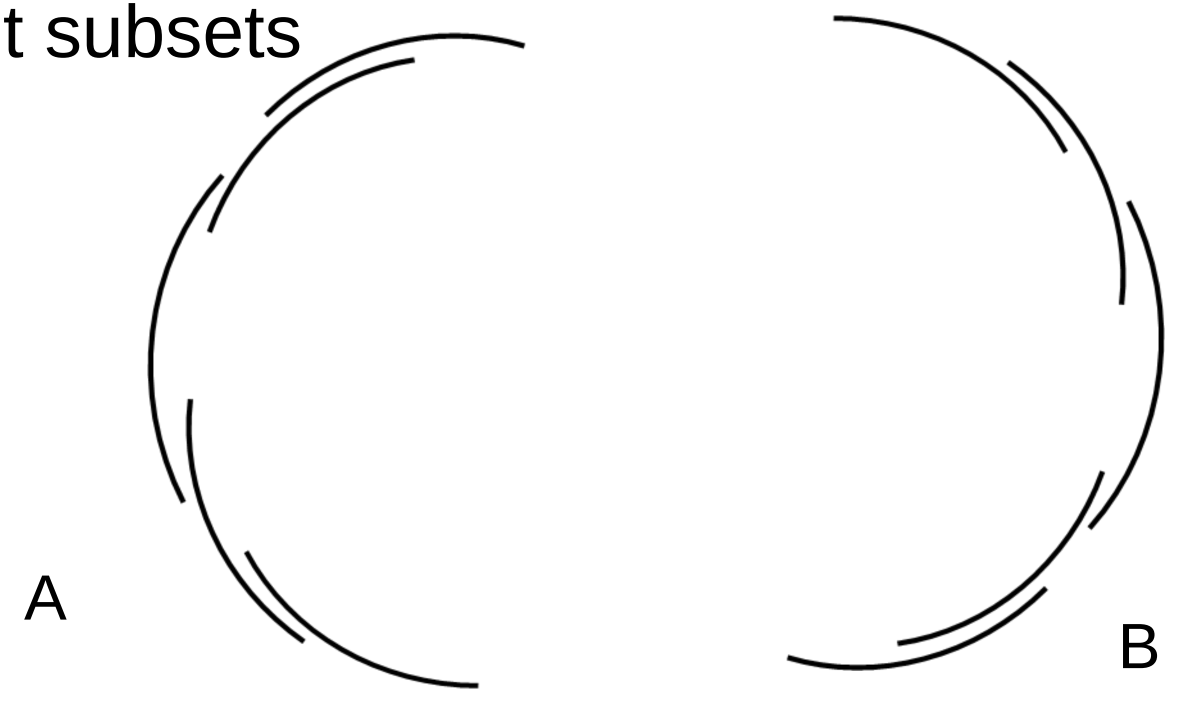
Is there still any Residual Entropy left in this case? If so, can it be computed?



In general, when we have overlapping systems, we can use strong subadditivity to put an upper bound on the entropy of the entire system

$$S(A \cup B) \leq S(A) + S(B) - S(A \cap B)$$

To apply this to the case at hand, we first split the circular system of overlapping intervals in two disjoint subsets



We apply strong subadditivity to these two subsets. Need to do this to avoid the mutual information phase transition.

Next, we peel of intervals one by one of each of the two subsets and iteratively apply strong subadditivity.

This then shows that for a system of overlapping intervals A_i

$$S_{\text{RES}} \leq \sum_i S(A_i) - \sum_i S(A_i \cap A_{i+1})$$

This gives an upper bound for the Residual Entropy.

The quantity

$$\sum_i S(A_i) - \sum_i S(A_i \cap A_{i+1})$$

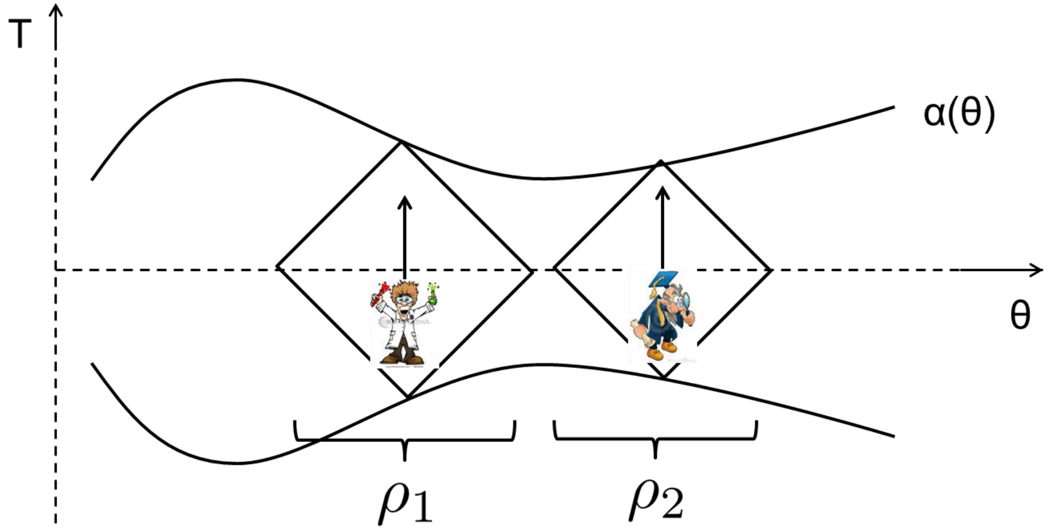
is finite and quite interesting and we will call it differential entropy

Differential entropy can easily be computed in AdS3.

For an interval of length $\Delta\theta$

$$S_{\mathrm{EE}} = \frac{c}{3}\log\left(\frac{\sin(\Delta\theta/2)}{a}\right)$$

where a is the UV cutoff and c the central charge of the CFT.



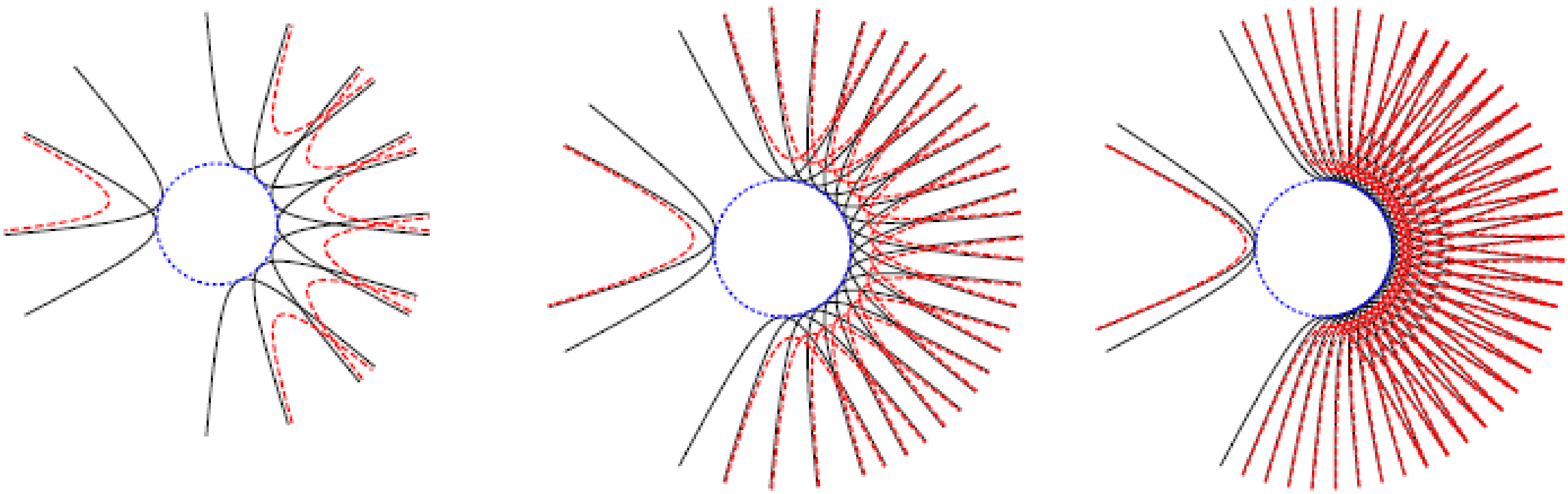
Use this result, take the continuum limit with infinitely many intervals to obtain

$$S_{\mathrm{RES}} \leq \frac{c}{6} \int_0^{2\pi} (1 - \alpha'(\theta)) \cot \alpha(\theta)$$

Take an arbitrary domain with convex boundary in AdS3. By considering light rays can determine shape of boundary geometry. Plug this into the above integral, do some changes of variables and a rather complicated partial integration and one finally obtains

$$\boxed{S_{\mathrm{RES}} \leq S_{\mathrm{DE}} = \frac{A}{4G}}$$

It is quite remarkable that this works, but the reason that it does has a nice geometric interpretation.



Comments:

$$\Delta t \Delta E \geq \hbar$$

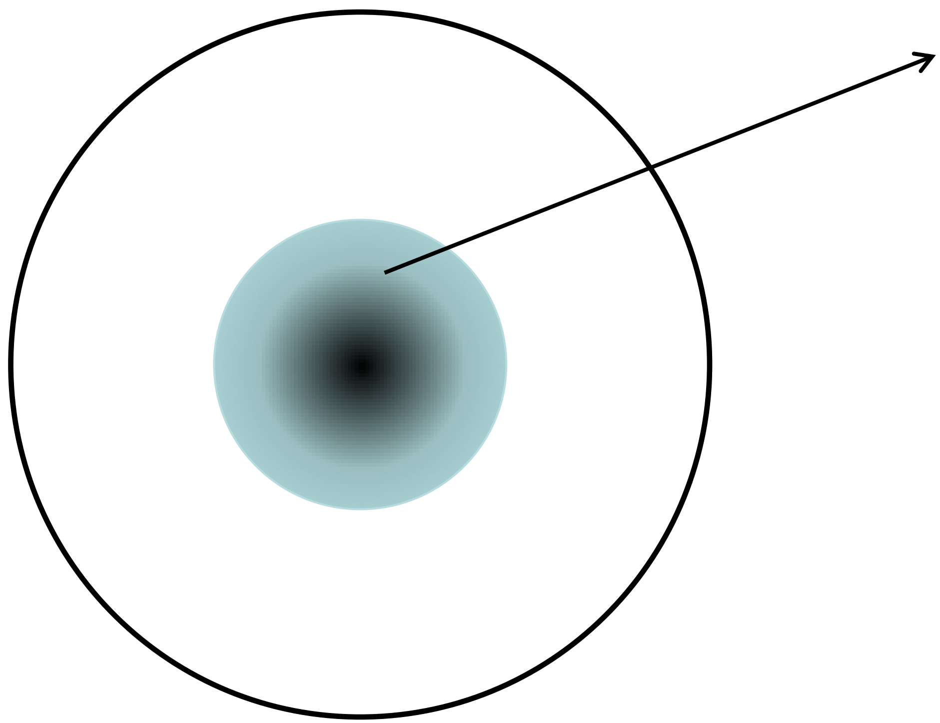
- The notion of residual entropy needs improvement – unlikely that one can access the full density matrix in finite time. Moreover, our working definition generically yields a result strictly smaller than S_{DE} . Alternative bulk definitions are discussed by Hubeny
- Differential entropy does not correspond to standard entanglement entropy in the field theory, so it appears that A/4G is not measuring standard entanglement entropy of quantum gravity degrees of freedom.

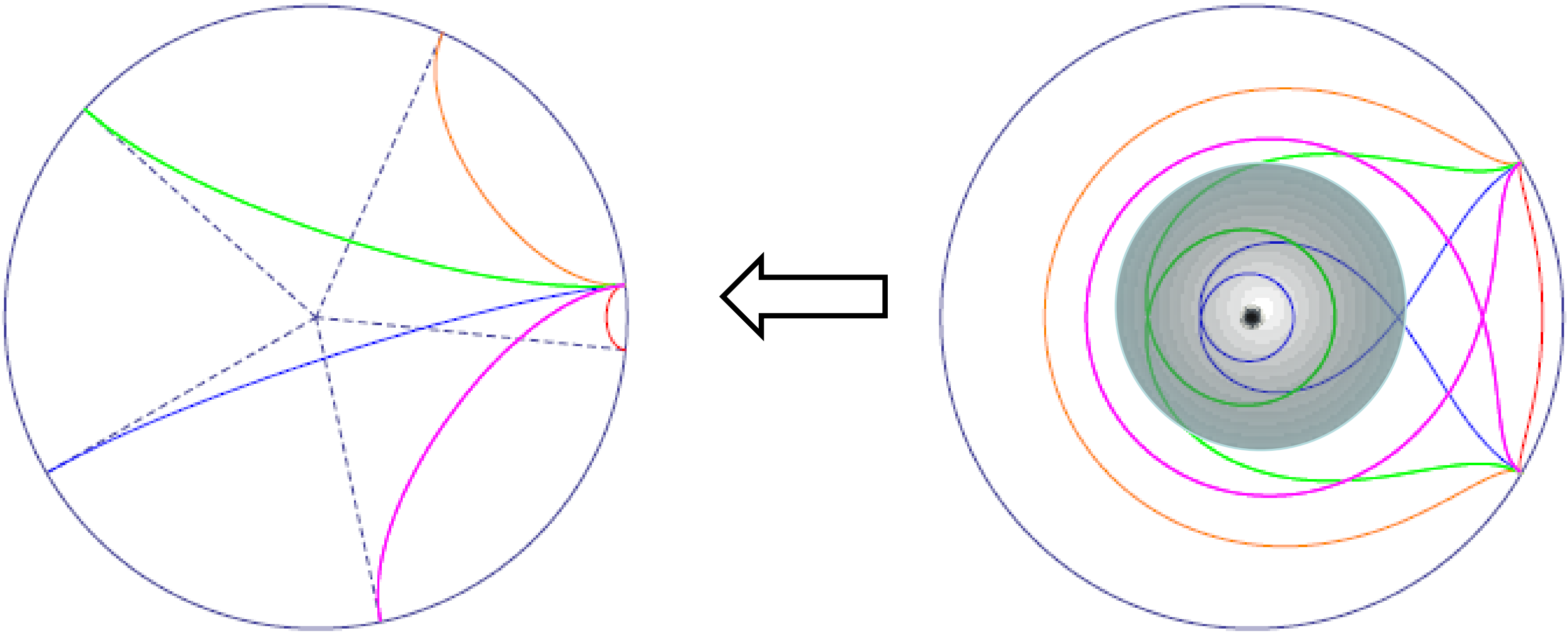
- For certain curves, the boundary strip becomes singular or even ill-defined (cf Hubeny). A suitable generalization of differential entropy still yields the length but it is unclear whether this has an information-theoretic meaning.
- Residual entropy was based on causality and observers, suggesting a role for causal holographic information (Hubeny, Rangamani), but differential entropy on entanglement entropy and geodesics/minimal surfaces. In general CHI≠EE and in more general cases one should use EE and not CHI (Myers, Rao, Sugishita)

Generalizations:

- Higher dimensions (Myers, Rao, Sugishita; Czech, Dong, Sully) works as well – expressions neither generic nor covariant.
- Inclusion of higher derivatives (Myers, Rao, Sugishita)
- Black holes/conical defects? Computations still work, but new ingredients are needed and new features appear.

Conical defect geometry





Regular geodesics in covering space.

Covering space = a "long string" sector of dual CFT.

Long geodesics can penetrate this region

Does the length of these long geodesics have a field theory dual?

This requires us to go to the long string picture and ungauge the Z_n symmetry, compute the entanglement entropy there, and then sum over gauge copies. (Balasubramanian, Chowdhury, Czech, JdB)

Ungauging is often necessary as an intermediate step in defining entanglement entropy in gauge theories. (see e.g. Donnelly; Agon, Headrick, Jafferis, Kasko; Casini, Huerta, Rosabal)

The gauge theory description is valid at the weakly coupled orbifold point, but may survive to strong coupling.

Since the long string contains fractionated (matrix) degrees of freedom, we apparently need entanglement between fractionated degrees of freedom to resolve the deep interior and near horizon regions in AdS.

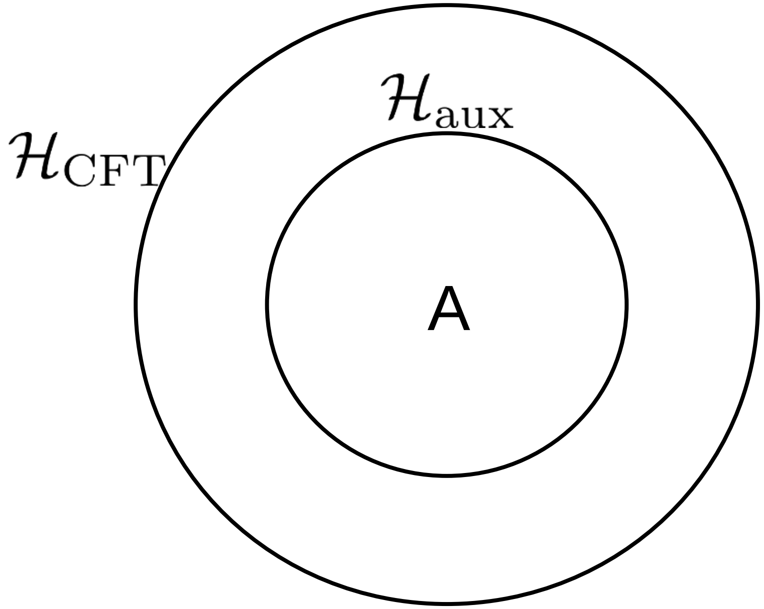
Interpretation of differential entropy/residual entropy?

Suppose it corresponds indeed to the entropy of some density matrix ρ, but there is no evidence that this is the reduced density matrix of some tensor factor in the Hilbert space.

If not, what does it have to do with the entanglement of quantum gravitational degrees of freedom? How do we reconstruct the original vacuum state from ρ if we cannot purify it?

Idea:

In quantum gravity we usually need to associate Hilbert spaces to boundaries of space-time. Think Wheeler-de Witt wavefunctions, Chern-Simons theory, etc.



Now suppose that to the outside we should really associate a state in

$$\mathcal{H}_{\mathrm{CFT}} \otimes \mathcal{H}_{\mathrm{aux}}$$

and to the inside region a state in

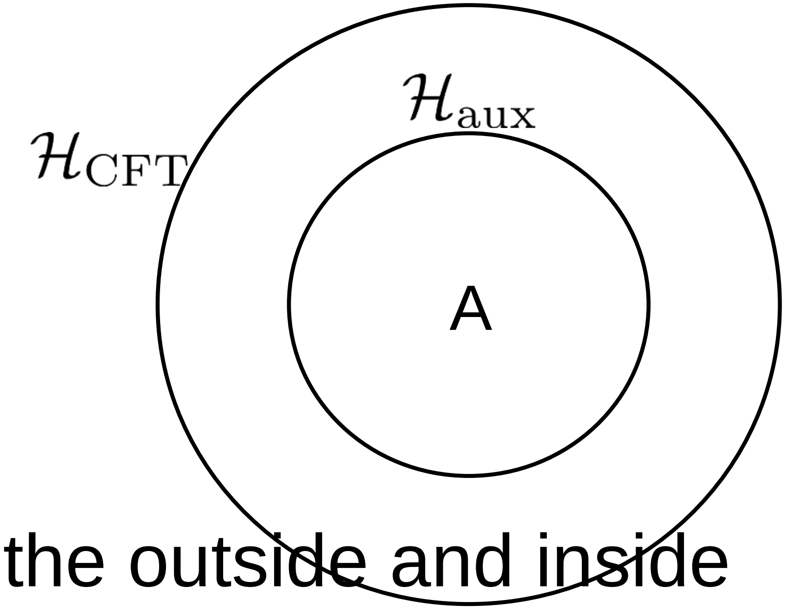
$$\mathcal{H}_{\mathrm{aux}}$$

and that gluing the spacetimes together involves taking an obvious product over $\mathcal{H}_{\mathrm{aux}}$.

Now if we write

$$\rho = \sum_i a_i |i\rangle\langle i|$$

$$|0\rangle_{\rm CFT} = \sum b_i |i\rangle$$



then it is natural to associate to the outside and inside regions the pure states

$$\psi_{\rm out} = \sum_i \sqrt{a_i} |i\rangle_{\rm CFT} \otimes |\tilde{i}\rangle_{\rm aux}$$

$$\psi_{\rm in} = \sum_i \frac{b_i}{\sqrt{a_i}} |\tilde{i}\rangle_{\rm aux}$$

Tracing $\psi_{\rm out}$ over $\mathcal{H}_{\rm aux}$ then yields back ρ.
Gluing $\psi_{\rm out}$ and $\psi_{\rm in}$ together reproduces the vacuum state. Consistent picture!!!

LESSONS FOR QUANTUM GRAVITY?

Our computations suggest that residual entropy is given by a density matrix that involves all degrees of freedom of the field theory. It therefore appears that one cannot localize quantum gravitational degrees of freedom exactly in some finite domain. This inherent non-locality is perhaps key for the peculiar breakdown of effective field theory needed to recover information.

In the BTZ/conical defect case one needs long geodesics, which can perhaps be interpreted by ungauging the orbifold theory. Also, to describe the interior we introduced an auxiliary Hilbert space $\mathcal{H}_{\rm aux}$. Perhaps adding extra gauge degrees of freedom is necessary in order to find a good local description of bulk physics?

Open problems:

- Associate a notion of entropy to families of observers - field theory with finite time duration - vector spaces of observables $\mathcal{A} = \cup_i \mathcal{A}(\mathcal{O}_i)$?
- Does differential entropy have a quantum information theoretic meaning?
- Reconstruct local bulk geometry more directly? Relation to Jacobson’s derivation of Einstein equations?
- All kinds of generalizations.