Evidence for Ultraviolet Finiteness of $\mathcal{N} = 8$ Supergravity

Strings , Madrid
June 29, 2007

Zvi Bern, UCLA

Z. Bern, N.E.J. Bjerrum-Bohr, D. Dunbar, hep-th/0501137
Z. Bern, L. J. Dixon , R. Roiban, hep-th/0611086
Z. Bern, J.J. Carrasco, L. Dixon, H. Johansson, D. Kosower, R. Roiban hep-th/0702112
Z. Bern, J.J. Carrasco, D. Forde, H. Ita and H. Johansson, to appear

Why $N = 8$ Supergravity?

- UV finiteness of $N = 8$ supergravity would imply a new symmetry or non-trivial dynamical mechanism.
- The discovery of either would have a fundamental impact on our understanding of gravity.
- High degree of supersymmetry makes this the most promising theory to investigate.
- By $N = 8$ we mean ungauged Cremmer-Julia supergravity.

No known superspace or supersymmetry argument prevents divergences from appearing at *some* loop order.

$$\frac{1}{\epsilon} D^n R^4 \leftarrow$$

Potential counterterm predicted by susy power counting

Range of opinions on where this can happen — from 3 to 9 loops, depending on assumptions.

Reasons to Reexamine This

1) The number of *established* counterterms in *any* supergravity theory is zero.

2) Discovery of remarkable cancellations at 1 loop – the "no-triangle hypothesis". ZB, Dixon, Perelstein, Rozowsky
ZB, Bjerrum-Bohr and Dunbar; Bjerrum-Bohr, Dunbar, Ita, Perkins and Risager

3) *Every* explicit loop calculation to date finds N = 8 supergravity has identical power counting as in N = 4 super-Yang-Mills theory, which is UV finite. Green, Schwarz and Brink; ZB, Dixon, Dunbar, Perelstein, Rozowsky; Bjerrum-Bohr, Dunbar, Ita, PerkinsRisager; ZB, Carrasco, Dixon, Johanson, Kosower, Roiban.

4) Very interesting hint from string dualities. Chalmers; Green, Vanhove, Russo
   – Dualities restrict form of effective action. May prevent divergences from appearing in D = 4 supergravity.
   – Difficulties with decoupling of towers of massive states.
   See Russo's talk for latest status

5) Berkovits' string non-renormalization theorems suggest N = 8 supergravity may be finite through 8 loops. No argument beyond this. See Berkovits' talk
Green, Vanhove, Russo

Gravity Feynman Rules

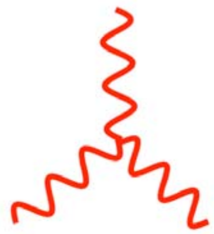
$$\mathcal{L} = \frac{2}{\kappa^2}\sqrt{g}R, \qquad g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

Propagator in de Donder gauge:

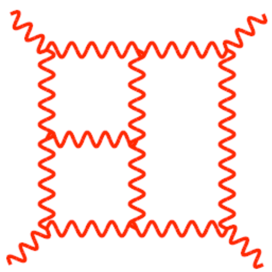
$$P_{\mu\nu;\alpha\beta}(k) = \frac{1}{2}\left[\eta_{\mu\nu}\eta_{\alpha\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \frac{2}{D-2}\eta_{\mu\alpha}\eta_{\nu\beta}\right]\frac{i}{k^2 + i\epsilon}$$

Three vertex has about 100 terms:

$$\begin{aligned}
&G_{3\mu\alpha,\nu\beta,\sigma\gamma}(k_1, k_2, k_3) = \\
&\quad \mathrm{sym}[-\tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \tfrac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) + \tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) \\
&\qquad + P_6(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma}) \\
&\qquad + P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma}) \\
&\qquad + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})]
\end{aligned}$$



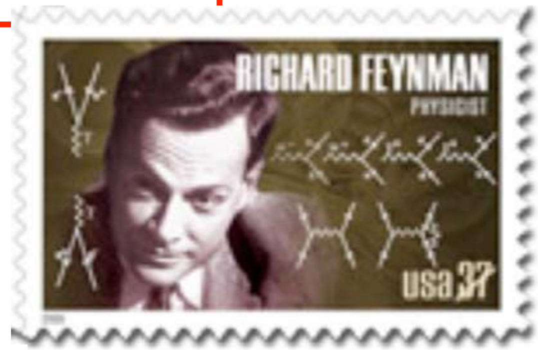
An infinite number of other messy vertices



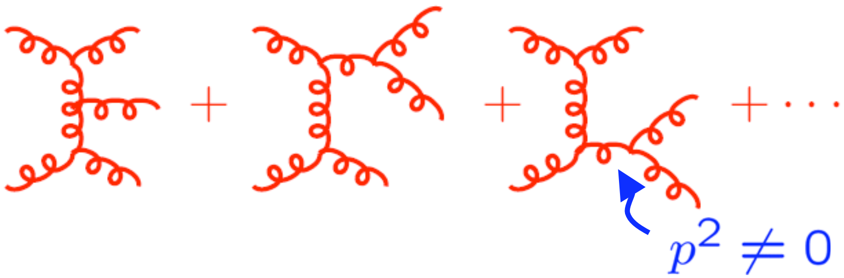
$\sim 10^{20}$ terms

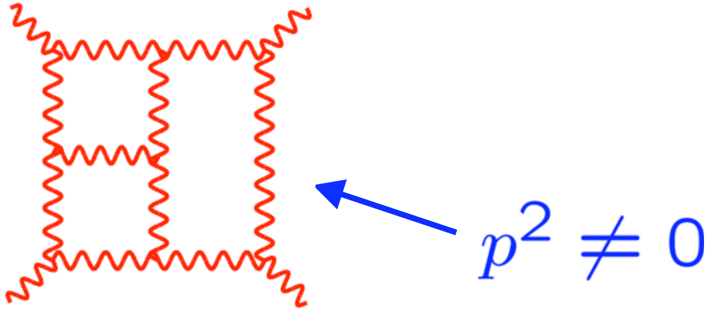
It is "impossible" to calculate

Why are Feynman diagrams clumsy for loop or high-multiplicity processes?



- Vertices and propagators involve gauge-dependent off-shell states. Origin of the complexity.



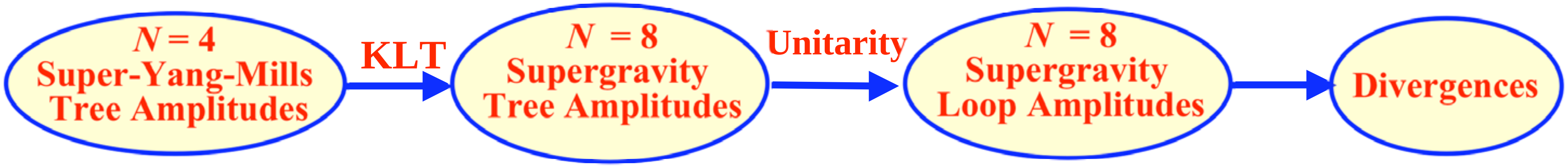


- To get at root cause of the trouble we must rewrite perturbative quantum gravity.

- **All steps should be in terms of gauge invariant on-shell states.** $p^2 = 0$
- **Need on-shell formalism.**

Basic Strategy

ZB, Dixon, Dunbar, Perelstein and Rozowsky (1998)



- Kawai-Lewellen-Tye relations: sum of products of gauge theory tree amplitudes gives gravity tree amplitudes.
- Unitarity method: efficient formalism for perturbatively quantizing gauge and gravity theories. Loop amplitudes from tree amplitudes.

ZB, Dixon, Dunbar, Kosower (1994)

Key features of this approach:

- **Gravity calculations mapped into much simpler gauge theory calculations.**
- **Only on-shell states appear.**

KLT Relations

At *tree level* Kawai, Lewellen and Tye presented a relationship between closed and open string amplitudes.
In field theory limit, relationship is between gravity and gauge theory

$$M_4^{\rm tree}(1,2,3,4) = s_{12} A_4^{\rm tree}(1,2,3,4)\, A_4^{\rm tree}(1,2,4,3)\,,$$

$$\begin{aligned} M_5^{\rm tree}(1,2,3,4,5) &= s_{12}s_{34} A_5^{\rm tree}(1,2,3,4,5) A_5^{\rm tree}(2,1,4,3,5) \\ &\quad + s_{13}s_{24} A_5^{\rm tree}(1,3,2,4,5)\, A_5^{\rm tree}(3,1,4,2,5) \end{aligned}$$

Gravity amplitude

Color stripped gauge theory amplitude

where we have stripped all coupling constants

$$\mathcal{A}_4^{\rm tree} = g^2 \sum\nolimits_{\rm non-cyclic} {\rm Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) A_4^{\rm tree}(1,2,3,4)$$

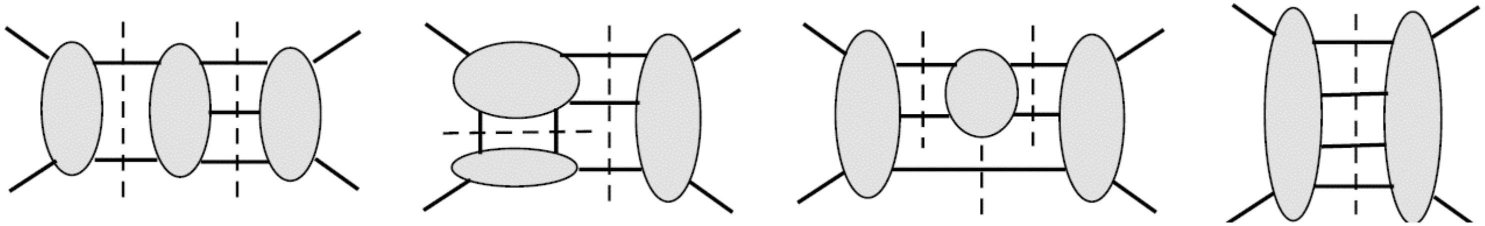
Full gauge theory amplitude

Holds for any external states. See review: gr-qc/0206071

Progress in gauge theory can be imported into gravity theories

Bern, Dixon, Dunbar and Kosower

Onwards to Loops: Unitarity Method



The unitarity method gives us a means for directly going between on-shell tree amplitudes and loop amplitudes

- **Lagrangian not needed.**
- **No Feynman diagrams.**
- **No gauge fixing required.**
- **No unphysical off-shell states.**
- **KLT relations can be used to determine tree amplitudes.**

A number of recent improvements to method, which I won't discuss here

Bern, Dixon and Kosower; Britto, Buchbinder, Cachazo and Feng; Berger, Bern, Dixon, Forde and Kosower; Britto, Feng and Mastrolia; Anastasiou, Britto, Feng; Kunszt, Mastrolia; ZB, Carasco, Johanson, Kosower; Forde

All-loop Resummation in $N = 4$ Super-YM Theory

Obtained using unitarity method

- **Conjecture that planar scattering amplitudes iterate to all loop orders and may be resummable.** Anastasiou, ZB, Dixon, Kosower; ZB, Dixon, Smirnov
- **Explicit form of conjecture determined for MHV amplitudes.**
- **Four-loop cusp anomalous dimension.** ZB, Czakon, Dixon, Kosower, Smirnov

$$\mathcal{A}_n = A_n^{\text{tree}} A_n^{\text{divergent}} \exp\left[\frac{1}{4}\gamma_K F_n^{\text{1-loop}} + C\right]$$

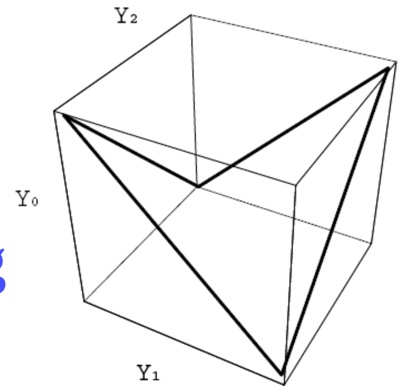
constant independent of kinematics.

all-loop resummed amplitude

IR divergences

cusp anomalous dimension

finite part of one-loop amplitude



Gives a definite prediction for *all* values of coupling given the Beisert, Eden, Staudacher integral equation for the cusp anomalous dimension. See Beisert's talk

In a beautiful paper Alday and Maldacena confirmed this conjecture at strong coupling from AdS string computation.

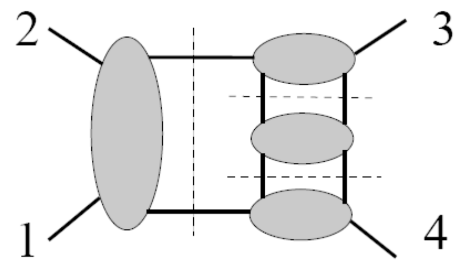
See Maldacena's talk

$N = 8$ Power Counting To All Loop Orders

Z.B., Dixon, Dunbar, Perelstein and Rozowsky

From '98 paper:

- Assumed iterated 2 particle cuts give the generic UV behavior.
- Assumed no cancellations with other uncalculated terms.



No evidence was found that more than 12 powers of loop momenta come out of the integrals.

Result from '98 paper

Elementary power counting gave finiteness condition:

$$\boxed{D < \frac{10}{L} + 2} \qquad (L > 1)$$

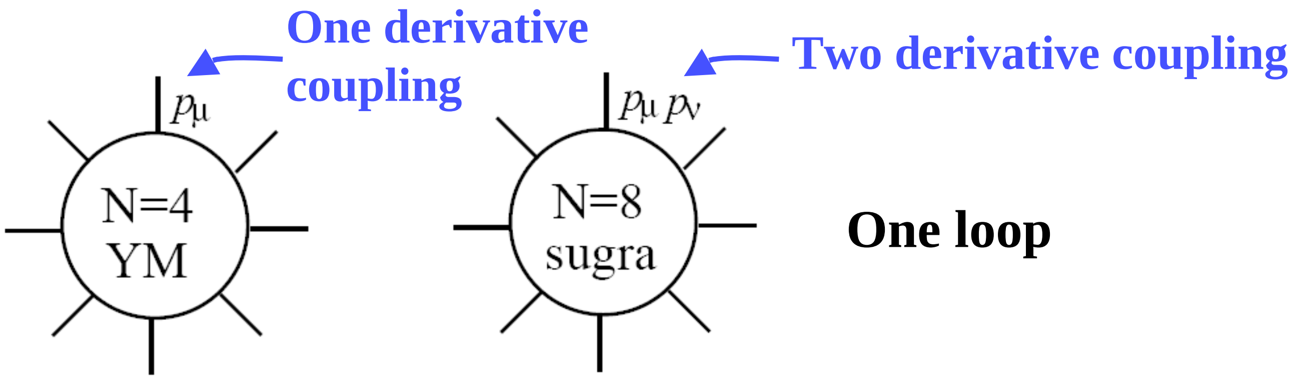
In D = 4 diverges for $L \geq 5$.
L is number of loops.

$D^4 R^4$ **counterterm was expected in D = 4, for** $L = 5$

Additional Cancellations at One Loop

Crucial hint of additional cancellation comes from one loop.

Surprising cancellations not explained by any known susy mechanism are found beyond four points



Two derivative coupling means N = 8 should have a worse power counting relative to N = 4 super-Yang-Mills theory.

- Cancellations observed in MHV amplitudes.

ZB, Dixon, Perelstein Rozowsky (1999)

- "No-triangle hypothesis" — cancellations in all other amplitudes.

ZB, Bjerrum-Bohr and Dunbar (2006)

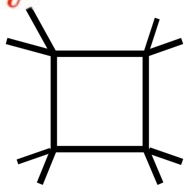
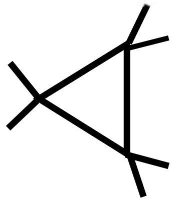
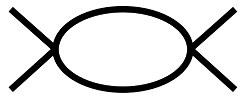
- Confirmed by explicit calculations at 6,7 points.

Bjerrum-Bohr, Dunbar, Ita, Perkins, Risager (2006)

No-Triangle Hypothesis

One-loop D = 4 theorem: Any one loop massless amplitude is a linear combination of scalar box, triangle and bubble integrals with rational coefficients:

$$A_n^{\text{1-loop}} = \sum_i d_i I_4^{(i)} + \sum_i c_i I_3^{(i)} + \sum_i b_i I_2^{(i)}$$

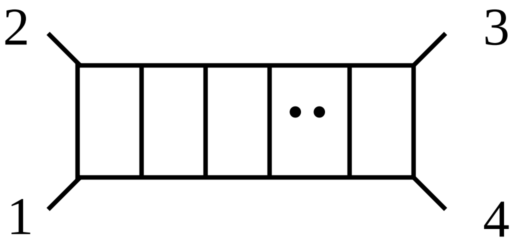




$$\int \frac{d^4 p}{(p^2)^4} \qquad \int \frac{d^4 p}{(p^2)^3} \qquad \int \frac{d^4 p}{(p^2)^2}$$

- **In N = 4 Yang-Mills *only box* integrals appear. No triangle integrals and no bubble integrals.**
- **The “no-triangle hypothesis” is the statement that same holds in N = 8 supergravity.**

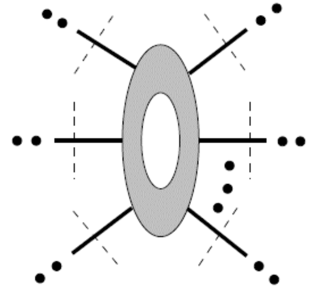
L-Loop Observation

ZB, Dixon, Roiban

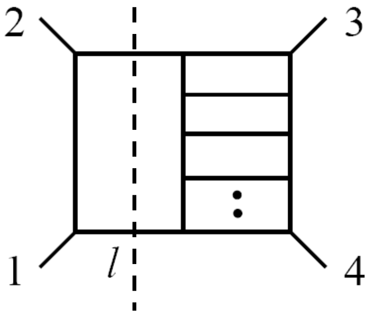


$$[(k_1 + k_2)^2]^{2(L-2)}$$

numerator factor



From 2 particle cut:



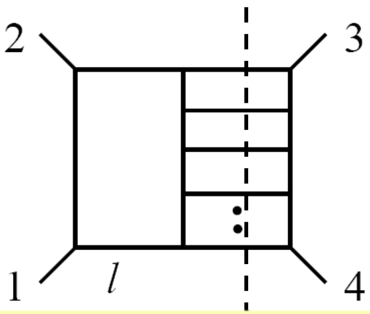
1 in N = 4 YM

$$[(l + k_4)^2]^{2(L-2)}$$

numerator factor

Using generalized unitarity and no-triangle hypothesis *all* one-loop subamplitudes should have power counting of N = 4 Yang-Mills

From L-particle cut:



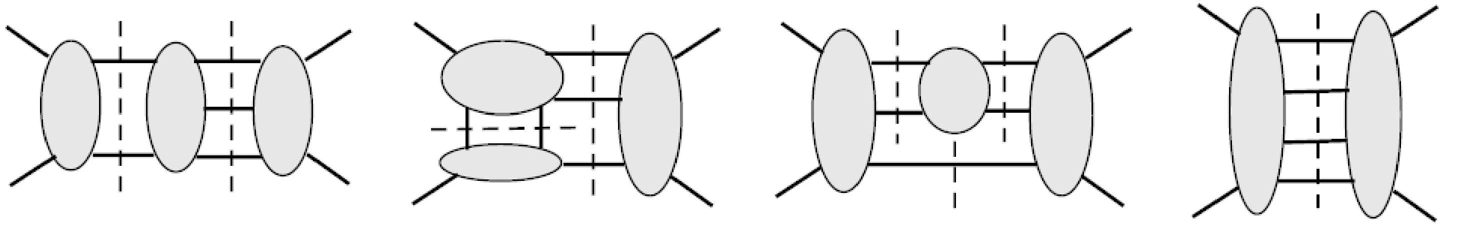
Above numerator violates no-triangle hypothesis. Too many powers of loop momentum.

There must be additional cancellation with other contributions!

Complete Three Loop Calculation

ZB, Carrasco, Dixon,
Johansson, Kosower, Roiban

Besides iterated two-particle cuts need following cuts:



For first cut have:

reduces everything to product of tree amplitudes

$$\sum_{N=8\ \text{states}} M_4^{\text{tree}}(1,2,l_3,l_1) \times M_5^{\text{tree}}(-l_1,-l_3,q_3,q_2,q_1) \times M_5^{\text{tree}}(3,4,-q_1,-q_2,-q_3)$$

Use KLT

$$M_4^{\text{tree}}(1,2,l_3,l_1) = -is_{12}A_4^{\text{tree}}(1,2,l_3,l_1)A_4^{\text{tree}}(2,1,l_3,l_1)$$

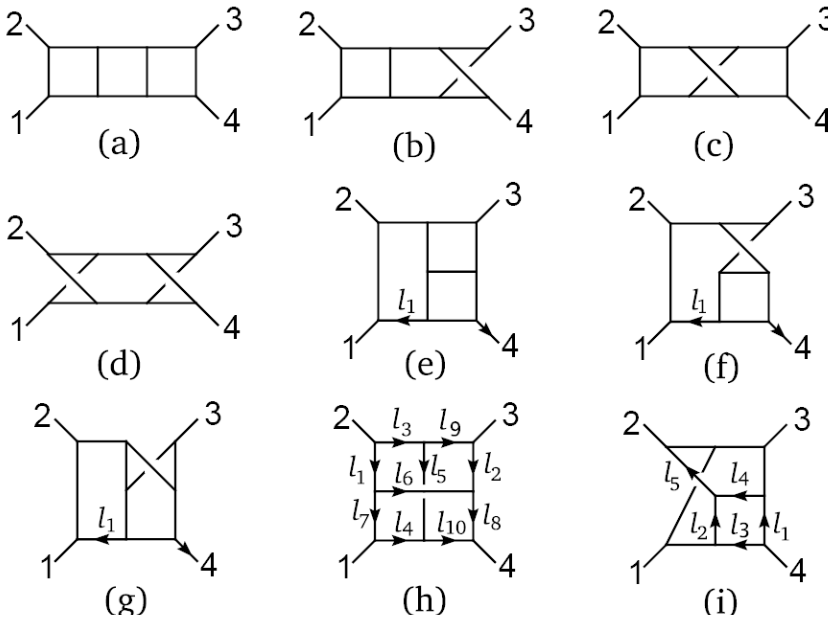
$$M_5^{\text{tree}}(-l_1,-l_3,q_3,q_2,q_1) = i\, s_{l_1q_1} s_{l_3q_3} A_5^{\text{tree}}(-l_1,-l_3,q_3,q_2,q_1)\, A_5^{\text{tree}}(-l_1,q_1,q_3,-l_3,q_2) + \{l_1 \leftrightarrow l_3\}\,,$$

supergravity **super-Yang-Mills**

N = 8 supergravity cuts are sums of products of N = 4 super-Yang-Mills cuts

Complete three loop result

ZB, Carrasco, Dixon, Johansson,
Kosower, Roiban; hep-th/0702112



All obtainable from iterated two-particle cuts, except (h), (i), which are new.

$$l_{i,j}^2 = (l_i + l_j)^2$$
$$s = (k_1 + k_2)^2$$
$$t = (k_1 + k_4)^2$$

Integral $I^{(x)}$	$\mathcal{N}=4$ Super-Yang-Mills	$\mathcal{N}=8$ Supergravity
(a)–(d)	s^2	$[s^2]^2$
(e)–(g)	$s(l_1+k_4)^2$	$[s\,(l_1+k_4)^2]^2$
(h)	$sl_{1,2}^2 + tl_{3,4}^2 - sl_5^2 - tl_6^2 - st$	$(sl_{1,2}^2 + tl_{3,4}^2 - st)^2 - s^2(2(l_{1,2}^2 - t) + l_5^2)l_5^2 - t^2(2(l_{3,4}^2 - s) + l_6^2)l_6^2$ $- s^2(2l_1^2l_8^2 + 2l_2^2l_7^2 + l_1^2l_7^2 + l_2^2l_8^2) - t^2(2l_3^2l_{10}^2 + 2l_4^2l_9^2 + l_3^2l_9^2 + l_4^2l_{10}^2) + 2stl_5^2l_6^2$
(i)	$sl_{1,2}^2 - tl_{3,4}^2 - \frac{1}{3}(s-t)l_5^2$	$(sl_{1,2}^2 - tl_{3,4}^2)^2 - (s^2l_{1,2}^2 + t^2l_{3,4}^2 + \frac{1}{3}stu)l_5^2$

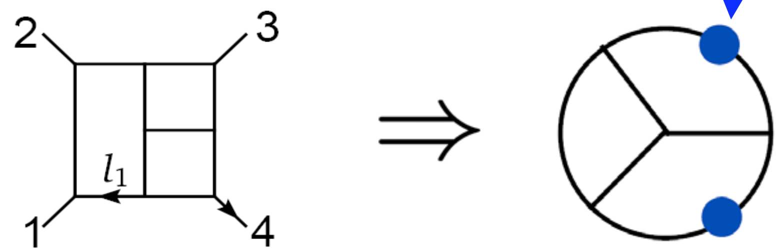
$$M_4^{(3)} = \left(\frac{\kappa}{2}\right)^8 stu M_4^{\mathrm{tree}} \sum_{S_3} \Big[I^{(\mathrm{a})} + I^{(\mathrm{b})} + \tfrac{1}{2}I^{(\mathrm{c})} + \tfrac{1}{4}I^{(\mathrm{d})}$$
$$+ 2I^{(\mathrm{e})} + 2I^{(\mathrm{f})} + 4I^{(\mathrm{g})} + \tfrac{1}{2}I^{(\mathrm{h})} + 2I^{(\mathrm{i})} \Big]$$

Cancellation of Leading Behavior

To check leading UV behavior we can expand in external momenta keeping only leading term.

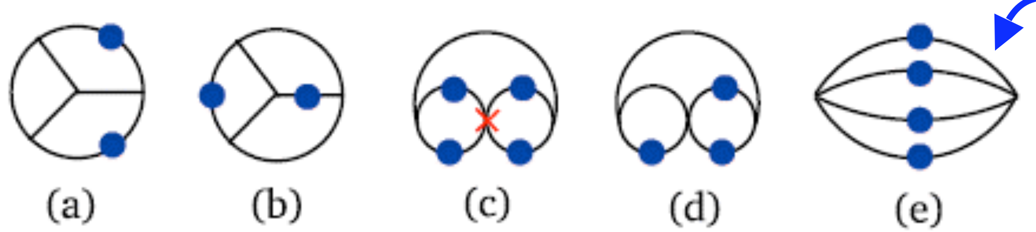
Get vacuum type diagrams:

$$(l_1 + k_4)^4$$



Doubled propagator

Violates NTH



Does not violate NTH but bad power counting

After combining contributions:

The leading UV behavior cancels!!

Finiteness Conditions

Through $L = 3$ loops the correct finiteness condition is ($L > 1$):

"superfinite" in $D = 4$

$$\boxed{D < \frac{6}{L} + 4}$$

- **same as $N = 4$ super-Yang-Mills**
- **bound saturated at $L = 3$**

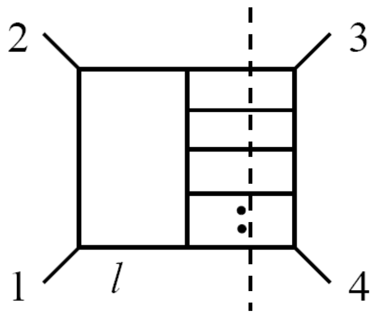
not **the weaker result from iterated two-particle cuts:**

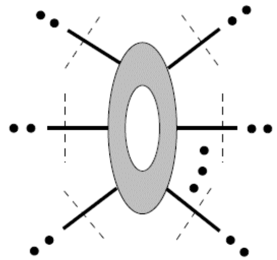
finite in $D = 4$ for $L = 3,4$

$$D < \frac{10}{L} + 2$$

('98 prediction)

Beyond $L = 3$, as already explained, from special cuts we have good reason to believe that the cancellations continue.



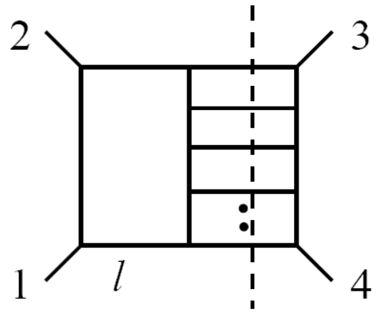


All one-loop subamplitudes should have same UV power-counting as $N = 4$ super-Yang-Mills theory.

Origin of Cancellations?

There does not appear to be a supersymmetry explanation for observed cancellations, especially if they hold to all loop orders, as we have argued.

If it is *not* supersymmetry what might it be?



Tree Cancellations in Pure Gravity

Unitarity method implies all loop cancellations come from tree amplitudes. Can we find tree cancellations?

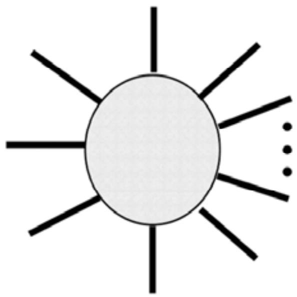
You don't need to look far: proof of BCFW tree-level on-shell recursion relations in gravity relies on the existence such cancellations!

Britto, Cachazo, Feng and Witten;
Bedford, Brandhuber, Spence and Travaglini
Cachazo and Svrcek; Benincasa, Boucher-Veronneau and Cachazo

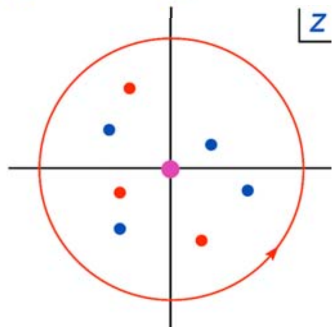
Susy not required

Consider the shifted tree amplitude:

$$k_1^\mu \to k_1^\mu + \frac{z}{2}\langle k_1^- | \gamma^\mu | k_2^- \rangle, \qquad k_2^\mu \to k_2^\mu - \frac{z}{2}\langle k_1^- | \gamma^\mu | k_2^- \rangle,$$

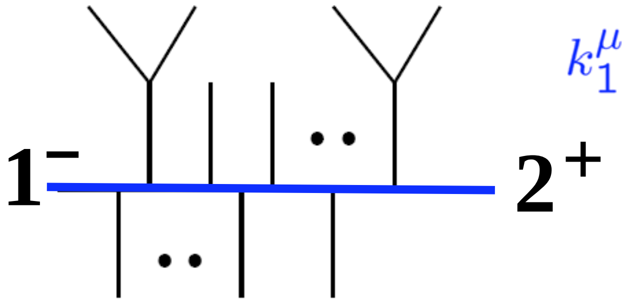


How does $M(z)$ behave as $z \to \infty$?



Proof of BCFW recursion requires $M(z) \to 0$

Tree Cancellations in Pure Gravity



$$k_1^\mu \to k_1^\mu + \frac{z}{2}\langle k_1^- | \gamma^\mu | k_2^- \rangle, \qquad k_2^\mu \to k_2^\mu - \frac{z}{2}\langle k_1^- | \gamma^\mu | k_2^- \rangle$$

m **propagators and** ***m*****+1 vertices between legs 1 and 2**

Yang-Mills scaling: $$z^{m+1} \times \frac{1}{z^m} \times \frac{1}{z^2} \sim \frac{1}{z}$$ **well behaved**

vertices **propagators** **polarizations**

$$z \to \infty$$

gravity scaling: $$z^{2(m+1)} \times \frac{1}{z^m} \times \frac{1}{z^4} \sim z^{m-2}$$ **poorly behaved**

Summing over all Feynman diagrams, correct gravity scaling is:

$$M_n^{\text{tree}}(z) \sim \frac{1}{z^2}$$

Remarkable tree-level cancellations!

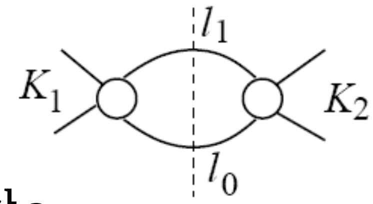
Bedford, Brandhuber, Spence, Travaglini;
Cachazo and Svrcek;
Benincasa, Boucher-Veronneau, Cachazo

z^{n-5} cancels to $\frac{1}{z^2}$

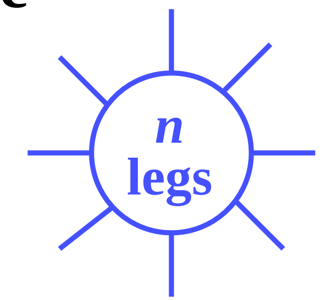
Loop Cancellations in Pure Gravity

ZB, Carrasco, Forde, Ita, Johansson, to appear

Powerful new one-loop integration method due to Forde makes it much easier to track the cancellations. Allows us to link one-loop cancellations to tree-level cancellations.



Observation: Most of the one-loop cancellations observed in N = 8 supergravity leading to "no-triangle hypothesis" are already present in non-supersymmetric gravity. Susy cancellations are on top of these.



$$(l^\mu)^{2n} \to (l^\mu)^{n+4} \times (l^\mu)^{-8}$$

Maximum powers of Loop momenta

Cancellation generic to Einstein gravity

Cancellation from N = 8 susy

Proposal: This continues to higher loops, so that most of the observed N = 8 multi-loop cancellations are *not* due to susy but in fact are generic to gravity theories!

What needs to be done?

- N = 8 four-loop computation. Can we demonstrate that four-loop N = 8 amplitude has the same UV power counting as N = 4 super-Yang-Mills? Certainly feasible (but non-trivial).
- Can we construct a proof of perturbative UV finiteness of N = 8? Perhaps possible using unitarity method – formalism is recursive.
- Investigate higher-loop pure gravity power counting to study cancellations. (It does diverge.) Goroff and Sagnotti; van de Ven
- Link to a twistor string description of N = 8? Abou-Zeid, Hull, Mason
- Can we find other examples with less susy that may be finite?

Guess: N = 6 supergravity theories will be perturbatively finite.

Summary

- Unitarity method gives us a powerful means for studying ultraviolet properties of quantum gravity.
- At four points through three loops, *established* $N = 8$ supergravity has same power counting as $N = 4$ Yang-Mills.
- One-loop $N = 8$ "no-triangle hypothesis" – one-loop cancellations.
- No-triangle hypothesis implies cancellations in a class of terms to *all* loop orders. No known superspace argument gives such cancellations.
- Proposed that most of the observed $N = 8$ cancellations are present in generic gravity theories, with susy cancellations on top of these.

$N = 8$ supergravity may be the first example of a point-like perturbatively UV finite theory of quantum gravity in $D = 4$. Proof is an open challenge.