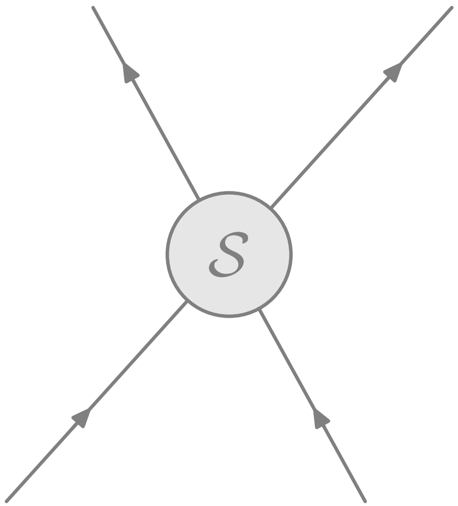
Strong/Weak Interpolation in the Spectrum of AdS/CFT



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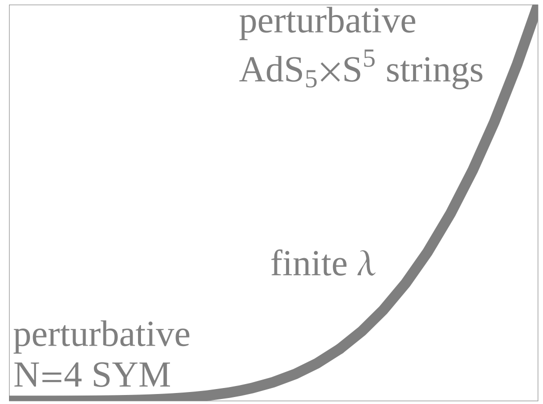


Strings 2007
Madrid, July 26, 2007

Work by NB, Eden, Hernández, López, McLoughlin, Roiban, Staudacher.
References: `hep-th/0609044,0610251;0705.0321`

Introduction

Talk about thoroughly testing AdS/CFT by comparing spectra.
Smooth interpolation between $\mathcal{N} = 4$ SYM and strings on $AdS_5 \times S^5$?
Proposal for the exact asymptotic planar spectrum of a 4D gauge theory.

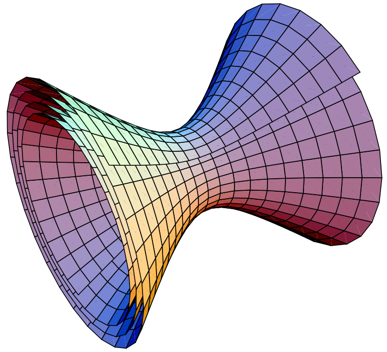
Outline:

- Review of AdS/CFT spectrum and Bethe equations.
- Interpolating scattering phase for AdS/CFT.
- Verifications of the proposal.

Spectrum

Spectrum of AdS/CFT

String Theory: $AdS_5 \times S^5$ background



States: Solutions X of classical equations of motion
plus quantum corrections.

Energy: Charge E_X for translation along AdS-time.

Gauge Theory: Conformal $\mathcal{N} = 4$ SYM

States: Local operators. Local, gauge-inv. combinations of the fields, e.g.

$$\mathcal{O} = \operatorname{Tr} \mathit{\Phi}_1 \mathit{\Phi}_2 (\mathcal{D}_1 \mathcal{D}_2 \mathit{\Phi}_2)(\mathcal{D}_1 \mathcal{F}_{24}) + \ldots .$$

Energy: Scaling dimensions, e.g. two-point function in conformal theory

$$\big\langle \mathcal{O}(x)\, \mathcal{O}(y) \big\rangle = C|x-y|^{-2D_{\mathcal{O}}(\lambda)}.$$

AdS/CFT: String energies and gauge dimensions match, $E(\lambda) = D(\lambda)$?!

Planar Limit: here cylindrical worldsheet/single-trace operators only.

Strong/Weak Duality

Problem: Strong/weak duality.

- Perturbative regime of gauge theory at $\lambda \approx 0$.

$$D(\lambda) = D_0 + \lambda D_1 + \lambda^2 D_2 + \ldots$$

D_ℓ: Contribution at ℓ (gauge) loops. Limit: 3 or 4 (or 5?!) loops.

- Perturbative regime of strings at $\lambda \to \infty$.

$$E(\lambda) = \sqrt{\lambda}\, E_0 + E_1 + E_2/\sqrt{\lambda} + \ldots$$

E_ℓ: Contribution at ℓ (world-sheet) loops. Limit: 1 or 2 loops.

Tests impossible unless quantities are known at finite λ.

Cannot compare, not even approximately.

Integrability helps.

Asymptotic Bethe Equations

Integrability in AdS/CFT . . . [Lipatov ICTP 1997] [Mandal Suryanarayana Wadia] [Minahan Zarembo] [NB Kristjansen Staudacher] [Bena Polchinski Roiban] [NB, Staudacher hep-th/0307042] [Dolan Nappi Witten] [NB hep-th/0310252] [Serban Staudacher] [Kazakov, Marshakov Minahan, Zarembo] [Arutyunov Frolov Staudacher] [Berkovits hep-th/0411170] [Schäfer-Nameki Zamaklar, Zarembo] [Zwiebel hep-th/0511109] . . .

. . . leads to Bethe equations. Complete asymptotic BE [NB, Staudacher hep-th/0504190]

coupling constant

$$g^2 = \frac{\lambda}{16\pi^2}$$

relations between u and $x^\pm$

$$u = x^+ + \frac{1}{x^+} - \frac{i}{2g} = x^- + \frac{1}{x^-} + \frac{i}{2g}$$

magnon momentum and energy

$$e^{ip} = \frac{x^+}{x^-}, \qquad e = -igx^+ + igx^- - \tfrac{1}{2}$$

total energy

$$E = L + \tfrac{1}{2}N + \tfrac{1}{2}\dot{N} + 2\sum_{j=1}^{K}\left(\frac{ig}{x_j^+} - \frac{ig}{x_j^-}\right)$$

local charges

$$q_r(x^\pm) = \frac{1}{r-1}\left(\frac{i}{(x^+)^{r-1}} - \frac{i}{(x^-)^{r-1}}\right), \qquad Q_r = \sum_{j=1}^{K} q_r(x_j^\pm)$$

Bethe equations

$$1 = \prod_{j=1}^{K}\frac{x_j^+}{x_j^-}$$

$$1 = \prod_{\substack{j=1\\j\neq k}}^{\dot{M}}\frac{\dot{w}_k - \dot{w}_j - ig^{-1}}{\dot{w}_k - \dot{w}_j + ig^{-1}}\prod_{j=1}^{\dot{N}}\frac{\dot{w}_k - \dot{y}_j - 1/\dot{y}_j + \frac{i}{2}g^{-1}}{\dot{w}_k - \dot{y}_j - 1/\dot{y}_j - \frac{i}{2}g^{-1}}$$

$$1 = \prod_{j=1}^{\dot{M}}\frac{\dot{y}_k + 1/\dot{y}_k - \dot{w}_j + \frac{i}{2}g^{-1}}{\dot{y}_k + 1/\dot{y}_k - \dot{w}_j - \frac{i}{2}g^{-1}}\prod_{j=1}^{K}\frac{\dot{y}_k - x_j^+}{\dot{y}_k - x_j^-}$$

$$1 = \left(\frac{x_k^-}{x_k^+}\right)^L\prod_{\substack{j=1\\j\neq k}}^{K}\left(\frac{u_k - u_j + ig^{-1}}{u_k - u_j - ig^{-1}}\,\sigma_{12}^2\right)\prod_{j=1}^{\dot{N}}\frac{x_k^- - \dot{y}_j}{x_k^+ - \dot{y}_j}\prod_{j=1}^{N}\frac{x_k^- - y_j}{x_k^+ - y_j}$$

$$1 = \prod_{j=1}^{M}\frac{y_k + 1/y_k - w_j + \frac{i}{2}g^{-1}}{y_k + 1/y_k - w_j - \frac{i}{2}g^{-1}}\prod_{j=1}^{K}\frac{y_k - x_j^+}{y_k - x_j^-}$$

$$1 = \prod_{\substack{j=1\\j\neq k}}^{M}\frac{w_k - w_j - ig^{-1}}{w_k - w_j + ig^{-1}}\prod_{j=1}^{N}\frac{w_k - y_j - 1/y_j + \frac{i}{2}g^{-1}}{w_k - y_j - 1/y_j - \frac{i}{2}g^{-1}}$$

Follow from S-matrix picture and symmetry. [Staudacher hep-th/0412188] [NB hep-th/0511082] [Arutyunov, Frolov Plefka, Zamaklar]

Determined up to overall phase factor $\sigma(p_1, p_2; \lambda) = \exp i\theta(p_1, p_2; \lambda)$.

Magic Phase
Phase Magic

Crossing Symmetry

Missing piece: Phase factor $\sigma(p_1, p_2)$ unconstrained by symmetry/YBE.

- Gauge theory: σ_{12} trivial at first three orders at $\lambda \approx 0$. [Serban Staudacher]
  Trivial phase required for correct BMN scaling.
- String theory: σ_{12} non-trivial at $\lambda \approx \infty$. [Arutyunov Frolov Staudacher]
  NLO known, tested & derived. [Hernández López] [Freyhult Kristjansen] [Gromov Vieira]

Need more constraints:

String S-matrix crossing symmetric? Janik's equation [Janik hep-th/0603038]

$$\sigma_{12}\sigma_{\bar{1}2} = f_{12}.$$

$f_{12} \neq f_{\bar{1}2}$: Superficially cannot solve due to symmetry under $1 \to \bar{1}$.
Branch cuts lead to non-trivial antipode map $1 \to \bar{1} \to \bar{\bar{1}} \neq 1$.

- Trivial phase from gauge theory not consistent with crossing symmetry.
- NLO string phase obeys crossing symmetry! [Arutyunov Frolov]

Phase at Strong Coupling

Decompose phase $\theta(p_1, p_2; g)$ into "modes" $c_{r,s}(g)$ ($g = \sqrt{\lambda}/4\pi$) [NB, Klose hep-th/0510124]

$$\theta(p_1, p_2; g) = \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} c_{r,s}(g) \big(q_r(p_1) q_s(p_2) - q_r(p_2) q_s(p_1)\big).$$

Perturbative proposal for crossing symmetric phase [NB hep-th/0606214] [NB Hernández López]

$$c_{r,r+2k+1} = \sum_{n=0}^{\infty} \frac{(r-1)(r+2k)\,\zeta(n)}{(-2\pi)^n g^{n-1}\,\Gamma(n-1)} \frac{\Gamma(r+k+\frac{1}{2}n-1)\,\Gamma(k+\frac{1}{2}n)}{\Gamma(r+k-\frac{1}{2}n+1)\,\Gamma(k-\frac{1}{2}n+2)}.$$

Consistent with known classical and one-loop results.

- One-loop contribution: Solves odd/hard part of crossing relation.
- Even loop orders: Solves even/easy part of crossing relation.
- Odd loop orders: Solve homogeneous crossing relation. Not unique!

Expansion at $\lambda = \infty$ asymptotic. Essential singularity. How to sum up?

Analytic Continuation of Sorts

How to obtain weak-coupling expansion from asymptotic strong coupling?

Consider something we can handle:

$$f(g) = -\frac{1/g}{1-1/g} = \frac{1}{1-g}\,.$$

Expansions at strong and weak "coupling"

$$f(g) = \sum_{n=1}^{\infty} b_n g^{-n}, \qquad f(g) = \sum_{n=0}^{\infty} a_n g^{+n},$$

with coefficients $b_n = -1$ and $a_n = +1$.

Analytic continuation in the perturbative order [NB, Eden Staudacher]

$$a_{+n} = -b_{-n}.$$

Phase at Weak Coupling

Analytic continuation in loop orders: What about [NB, Eden Staudacher]

$$c_{r,s}(g) = \sum_{n=0}^{\infty} g^{1-n} c_{r,s}^{(n)}, \qquad -\sum_{n=1}^{\infty} g^{1+n} c_{r,s}^{(-n)} \overset{?!}{\to} c_{r,s}(g)$$

Several appealing features of this proposal:

- Contributions only at integer powers of λ.
- Leading order of $c_{r,s}(g)$ compatible with Feynman diagrams. [NB, Klose hep-th/0510124]
- Changes dimensions starting at four loops $c_{2,3}(g) = 4\zeta(3)g^3 + \ldots$.
- Breaks BMN scaling/spinning strings at weak coupling.
- Preserves transcendentality of twist-two and cusp/universal dimension.
- Series convergent for $|\lambda| \leq \pi^2$.

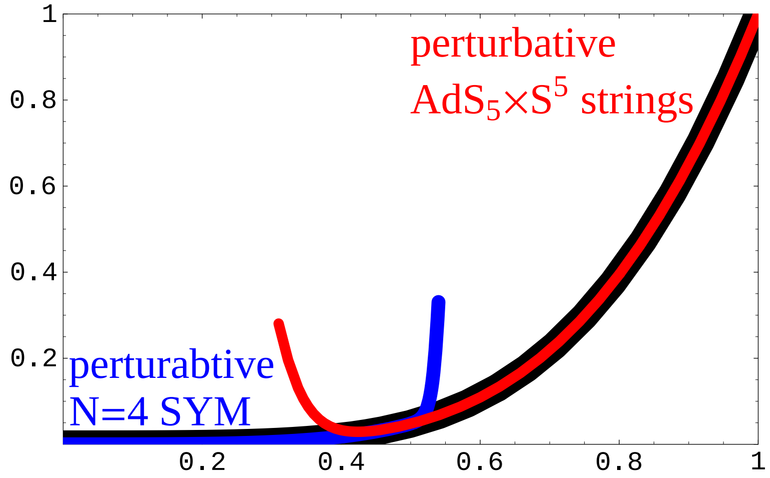
$$\frac{c_{r,r+2k+1}(g)}{(r-1)(r+2k)} \overset{?}{=} \sum_{n=1}^{\infty} \frac{2\,\zeta(2n+1)g^{2n+1}}{(-1)^{1+r+n}(2n+1)} \binom{2n+1}{n+r+k} \binom{2n+1}{n+k}.$$

Comparison

Does the phase interpolate between gauge and string theory?! E.g. $c_{23}(\lambda)$

weak-coupling expansion

asymptotic strong-coupling expansion



$$x = \frac{\sqrt{\lambda}}{\sqrt{\lambda}+\pi}, \qquad y = \frac{c_{23}}{c_{23}+1}.$$

Can derive asymptotic strong coupling expansion from weak coupling

- at NLO by representing $\zeta(n)$ through sum, [NB, Eden Staudacher]
- analytically at all orders (Mellin-Barnes & change of contour). [Kotikov Lipatov]

Useful integral expression for complete phase [Belitsky hep-th/0703058] [Dorey Hofman Maldacena]

$$\oint_1 \frac{dz_1}{2\pi i(x_1-z_1)} \oint_1 \frac{dz_2}{2\pi i(x_2-z_2)}\, i \log\Gamma\big(1+ig(z_1+1/z_1-z_2-1/z_2)\big).$$

Laurent Expansion

Why did the continuation in loop order work?! [NB, Ferretti unpublished]

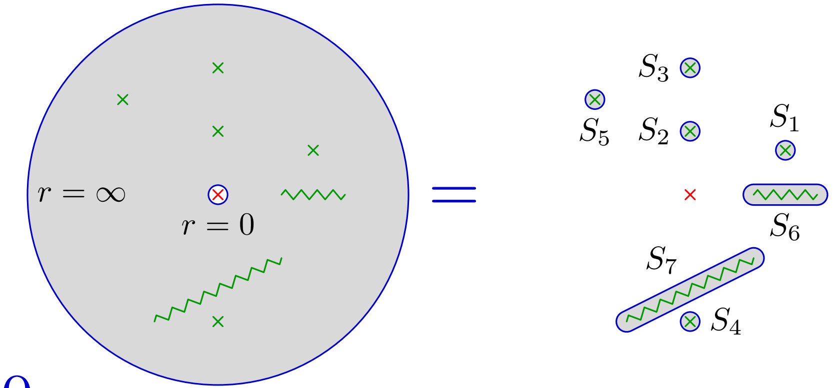
Consider a holomorphic function $f(z)$ without singularities at $z = 0, \infty$.

Laurent expansion for circle of radius r

$$f(z) \overset{|z|\approx r}{=} \sum_{n=-\infty}^{+\infty} c_n^{(r)} z^n, \qquad c_n^{(r)} = \frac{1}{2\pi i} \oint_r z^{n-1} f(z)\, dz.$$

Consider $c_n = c_n^{(\infty)} - c_n^{(0)}$

$$c_n = \sum_{S \in \mathrm{sing}(f)} \frac{1}{2\pi i} \oint_S z^{n-1} f(z)\, dz.$$



Weak-coupling $a_n = c_n^{(0)} = -c_n$ for $n \geq 0$.

Strong-coupling $b_n = c_{-n}^{(\infty)} = c_{-n}$ for $n > 0$.

Analytic continuation in n plausible, even though phase singular at $\lambda = \infty$.

Testing the Proposal

Gluon Scattering Amplitudes

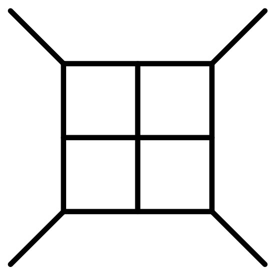
Use gluon scattering to verify conjecture at weak coupling.

Four-gluon scattering amplitude obeys "iteration" relation [Anastasiou, Bern Dixon, Kosower][Bern Dixon Smirnov]

$$A(p,\lambda) \simeq A^{(0)}(p)\exp\left(2D_{\mathrm{uni}}(\lambda)M^{(1)}(p)\right).$$

Tour de Fours calculation [Bern, Czakon, Dixon Kosower, Smirnov]

- Four dimensions,
- Four supersymmetries,
- Four legs,
- Four loops.



Cusp dimension D_{uni} evaluated using unitarity methods [Bern, Czakon, Dixon Kosower, Smirnov][Cachazo Spradlin Volovich]

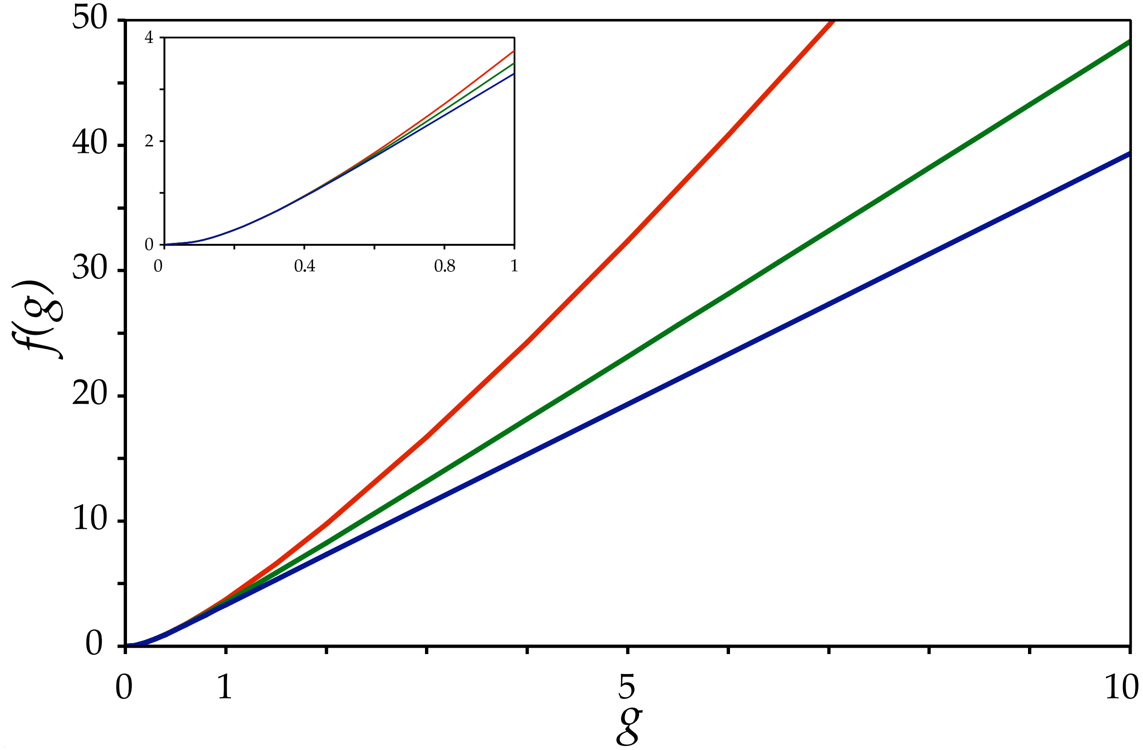
$$\pi^2 D_{\mathrm{uni}}(\lambda) = \frac{\lambda}{2} - \frac{\lambda^2}{96} + \frac{11\lambda^3}{23040} - \left(\frac{73}{2580480} + \frac{\zeta(3)^2}{1024\pi^6}\right)\lambda^4 \pm \ldots .$$

Agrees with 4-loop prediction from Bethe equations. [Eden Staudacher][NB, Eden Staudacher]

Numerical Evaluation

Can now consider universal dimension at finite coupling. [Benna, Benvenuti Klebanov, Scardicchio]

Numerical evaluation of universal dimension:



Plot from [Benna, Benvenuti Klebanov, Scardicchio]

red: without phase

green: half of phase

blue: full phase

- String perturbation theory: $E_{\text{uni}} = \sqrt{\lambda}/\pi - 3\log 2/\pi + \ldots$ [Gubser Klebanov Polyakov][Frolov Tseytlin]
- Numerical evaluation agrees with string theory at large λ. [Benna, Benvenuti Klebanov, Scardicchio]
- Sqrt-singularity at $\lambda = -\pi^2$: $\sqrt{\lambda + \pi^2}$. [NB, Eden Staudacher][Benna, Benvenuti Klebanov, Scardicchio]

String Worldsheet Expansion

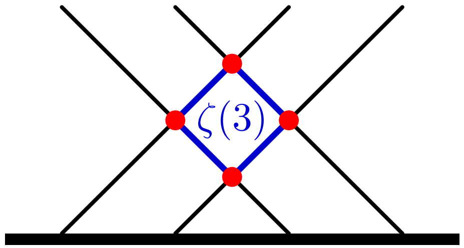
Analytic comparisons of Bethe equations to string theory at $\lambda \approx \infty$.

- Certain near flat space limit with $\sqrt[4]{\lambda}$-behaviour.
  Perturbation series reorders, need to resum all orders. [Maldacena Swanson]
- Flat space limit phase agrees at two loops. [Klose,McLoughlin Minahan,Zarembo]
- Classical string result $E_{\text{uni}} = \sqrt{\lambda}/\pi + \ldots$ [Kotikov Lipatov] [Alday, Arutyunov Benna, Eden, Klebanov] [Kostov Serban Volin] [Beccaria De Angelis Forini]
- One-loop string result $E_{\text{uni}} = \ldots - 3\log 2/\pi + \ldots$ [Casteill Kristjansen]
- Generic agreement at one loop. [Gromov Vieira]
- Two loops in full sigma model plagued by unresolved divergences. [Roiban Tirziu Tseytlin]

Four Loops and $\zeta(3)$

Despite undisputed success of phase, some questions remain:

- Where does $\zeta(3)$ in the leading phase $c_{2,3}(g) = 4\zeta(3)g^3$ come from?
  Not from mixing. Bulk of some Feynman diagram?



$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}\,.$$

- Is the cusp dimension really universal?
  Independent of twist for infinite spin? Order of limits?
  Four-gluon scattering most immediately related to twist two.
  Twist two at four loops is beyond asymptotic regime (wrapping).
- Can't we just compute it to be sure?!
  E.g. dispersion relation can be partially computed.
  $\mathfrak{su}(2)$ sector ideal setup: Gluons & fermions internal; scalars external.

Long-Range Hamiltonian

A full computation of $\mathfrak{su}(2)$ Hamiltonian from field theory is hopeless.
Want to use only known data; not assume integrability.

To constrain Hamiltonian, use [NB McLoughlin Roiban]

- protected vacuum energies,
- magnon dispersion relation,
- two-magnon S-matrix.

Scattering of n magnons in $\mathfrak{su}(2)$ sector

- fixes $2n$-loop spectrum.
- 2-magnon scattering sufficient at 4 loops.

Hamiltonian determined. Legend:

- Spin permutation symbols: $\{\ldots\}$ [NB Kristjansen Staudacher]
- phase factor: $\beta_{r,s}$ [NB, Klose hep-th/0510124]
- similarity transformations: ϵ_k (unphysical)

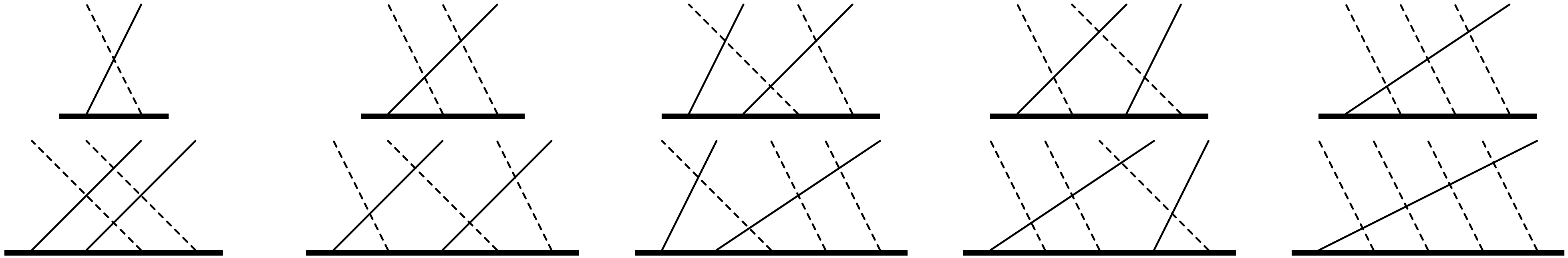
$$
\begin{aligned}
\mathcal{H}_0 &= +\{\} \\
\mathcal{H}_2 &= +2\{\} - 2\{1\} \\
\mathcal{H}_4 &= -8\{\} + 12\{1\} - 2\big(\{1,2\} + \{2,1\}\big) \\
\mathcal{H}_6 &= +60\{\} - 104\{1\} + 4\{1,3\} + 24\big(\{1,2\} + \{2,1\}\big) \\
&\quad - 4i\epsilon_2\{1,3,2\} + 4i\epsilon_2\{2,1,3\} - 4\big(\{1,2,3\} + \{3,2,1\}\big) \\
\mathcal{H}_8 &= +\big(-560 - 4\beta_{2,3}\big)\{\} \\
&\quad + \big(+1072 + 12\beta_{2,3} + 8\epsilon_{3a}\big)\{1\} \\
&\quad + \big(-84 - 6\beta_{2,3} - 4\epsilon_{3a}\big)\{1,3\} \\
&\quad - 4\{1,4\} \\
&\quad + \big(-302 - 4\beta_{2,3} - 8\epsilon_{3a}\big)\big(\{1,2\} + \{2,1\}\big) \\
&\quad + \big(+4\beta_{2,3} + 4\epsilon_{3a} + 2i\epsilon_{3c} - 4i\epsilon_{3d}\big)\{1,3,2\} \\
&\quad + \big(+4\beta_{2,3} + 4\epsilon_{3a} - 2i\epsilon_{3c} + 4i\epsilon_{3d}\big)\{2,1,3\} \\
&\quad + \big(4 - 2i\epsilon_{3c}\big)\big(\{1,2,4\} + \{1,4,3\}\big) \\
&\quad + \big(4 + 2i\epsilon_{3c}\big)\big(\{1,3,4\} + \{2,1,4\}\big) \\
&\quad + \big(+96 + 4\epsilon_{3a}\big)\big(\{1,2,3\} + \{3,2,1\}\big) \\
&\quad + \big(-12 - 2\beta_{2,3} - 4\epsilon_{3a}\big)\{2,1,3,2\} \\
&\quad + \big(+18 + 4\epsilon_{3a}\big)\big(\{1,3,2,4\} + \{2,1,4,3\}\big) \\
&\quad + \big(-8 - 2\epsilon_{3a} - 2i\epsilon_{3b}\big)\big(\{1,2,4,3\} + \{1,4,3,2\}\big) \\
&\quad + \big(-8 - 2\epsilon_{3a} + 2i\epsilon_{3b}\big)\big(\{2,1,3,4\} + \{3,2,1,4\}\big) \\
&\quad - 10\big(\{1,2,3,4\} + \{4,3,2,1\}\big)
\end{aligned}
$$

Feynman Diagrams

Feynman diagrams reshuffle $\mathfrak{su}(2)$ spins. Flavour flow.

- Four-point scalar interactions can permute $\mathfrak{su}(4)$ flavour.
- Gluons can dress diagrams, but not exchange flavour.
- Fermion loops can permute flavour, but in a limited way.

Some diagrams have maximal spin shift property: Scalar interactions only



- One non-zero external momentum sufficient to regularise IR.
- Can be reduced to one-loop bubble and two-loop master integral!
- Need to cancel $1/\epsilon^{4,3,2}$ singularities from subdiagrams.
- Lo and behold: $\beta_{2,3} = 4\zeta(3)$ or $c_{2,3} = 4\zeta(3)g^3 + \ldots$. [NB McLoughlin Roiban]
- $\zeta(3)$ to be expected: $\epsilon^{-4}\Gamma(1+\epsilon) = \ldots - \frac{1}{3}\zeta(3)\epsilon^{-1} + \ldots$.

Asymptotic BE, Wrapping & Limitations

The asymptotic Bethe equations are not exact.

- Coordinate BA based on the assumption of infinite length.
- Interactions that wrap the whole state/string not seen.
- Corrections exponential in L expected.

Gauge Theory:

- Bethe equations for similar Inozemtsev chain are not exact. [Serban Staudacher]
- Mismatch of energies for some virtual states. [NB, Dippel Staudacher]
- Analyticity problems in Baxter equation. [Belitsky unpublished]
- Twist-2 energy continued to negative spin contradicts BFKL. [Kotikov, Lipatov Rej, Staudacher Velizhanin]

String Theory:

- Well-known from 2D sigma models.
- Mismatch for states with large mode numbers $n \longrightarrow \infty$. [Schäfer-Nameki Zamaklar, Zarembo]

Need to find exact equations to generalise asymptotic Bethe equations.

Conclusions

Conclusions

⋆ **Spectrum of AdS/CFT**

- Planar asymptotic spectrum described by Bethe equations.
- Bethe equations fixed by symmetry up to one phase function.

⋆ **Phase Factor**

- Constrained by crossing relation.
- Proposal for interpolating phase factor at strong and weak coupling.
- Full agreement with AdS/CFT! Several predictions tested.
- "Three-loop discrepancy" resolved. No BMN scaling at weak coupling!
- Solution for planar asymptotic spectrum of non-trivial 4D gauge theory.

⋆ **Open Questions**

- Prove integrability for gauge and string theory.
- Find exact equations: Extra d.o.f.? Phase effective quantity? TBA?