CHARGES

DIRAC QUANTIZATION $\mathbb{Z} \subset \mathbb{R}$

c_1 (LINE-BUNDLE) - NATURAL FOR K-THEORY

IN STRING THEORIES MANY MORE EXAMPLES

VARIOUS VARIANTS OF K-THEORY APPEAR TO BE NEEDED

$K, K_G, KO, KR, \ldots$

[MOORE
DOUGLAS
WITTEN
⋮]

AND "TWISTED" VERSIONS

cf. TWISTED COHOMOLOGY

(LOCAL COEFFICIENTS)

BUT MORE SUBTLE

CONSIDER <u>SIMPLEST</u> CASE

RECALL COHOMOLOGY WITH LOCAL COEFFTS GIVEN BY ACTION OF π_1 ON COEFFT GROUP. FOR INTEGER COEFFTS ONLY HAVE "SIGN" ACTION $\pi_1 \to Z_2$ GIVEN BY ELEMENT α OF $H^1(X, Z_2)$ OR BY DOUBLE COVERING $X_\alpha \to X$

[EX. X NON-ORIENTABLE MANIFOLD, X_α ITS ORIENTED COVER]

$\Rightarrow \quad H^*_\alpha(X, Z)$ $\qquad\qquad$ $Z_2 = O(1)$

IF L_α IS REAL LINE-BDLE DEFINED BY α. THEN

$$H^q_\alpha(X, Z) \cong H^{q+1}(L_\alpha)$$

(TWISTED SUSPENSION ISOMORPHISM) RELATED TO POINCARÉ DUALITY FOR NON-ORIENTABLE MFDS. (COMPACT SUPPORTS)

BY ANALOGY <u>DEFINE</u> $K^q_\alpha(X) = K^{q+1}(L_\alpha)$

(q mod 2)

NOTE EXACT TRIANGLE

$$\begin{tikzcd} K^*(X) \arrow[rr] & & K^*(X_\alpha) \arrow[dl] \\ & K^*_\alpha(X) \arrow[ul, "\delta"] & \end{tikzcd}$$

USE PAIR

(I_α, X_α)

$I_\alpha \subset L_\alpha$

(UNIT INTERVAL)

NOTE $K^*(X) = K^*_{Z_2}(X_\alpha)$

FOR <u>ANY Z_2-SPACE</u> Y, DEFINE

NEW THEORY $K^*_\pm(Y)$ BY EXACT

TRIANGLE

$$\begin{tikzcd} K^*_{Z_2}(Y) \arrow[rr] & & K^*(Y) \arrow[dl] \\ & K^*_\pm(Y) \arrow[ul, "\delta"] & \end{tikzcd}$$

$$K^*(Y) = K^*_{Z_2}(Y \times \{\pm 1\})$$

EXACT TRIANGLE IS $K^*_{Z_2}$ APPLIED TO PAIR $(Y \times [-1,1], Y \times \{\pm 1\})$

HENCE $K^q_{\pm}(Y) = K^{q+1}_{Z_2}(Y \times [-1,1], Y \times \{\pm 1\})$

$$= K^{q+1}_{Z_2}(Y \times R) \quad (*)$$

(Z_2 ACTS BY ± 1 ON R) (q mod 2)

GROUPS $K_{\pm}(Y)$ INTRODUCED BY WITTEN AS HOME FOR CHARGES IN CONNECTION WITH "ORIENTIFOLDS"

(*) DEFINITION PROPOSED BY M. HOPKINS

(1) Z_2 ACTION FREE $\rightarrow$ TWISTED K

(2) Z_2 ACTION TRIVIAL $K^0_{\pm} = K^1$ ✓

$$0 \to K^*_{\pm}(Y) \xrightarrow{\delta} K^*_{Z_2}(Y) \to K^*(Y) \to 0$$

$$0 \to K^1(Y) \to R(Z_2) \otimes K^0(Y) \to K^0(Y) \to 0$$

FREDHOLM OPERATORS

F BOUNDED OPERATOR IN HILBERT SPACE

FREDHOLM DIM(KER F) FINITE p

DIM(COKER F) = DIM(KER F^*) FINITE q

INDEX = $p - q$ DEFORMATION INVARIANT

$\mathcal{F}$ SPACE OF ALL FREDHOLM OPERATORS

(NORM TOPOLOGY)

CONTINUOUS MAP $f: X \to \mathcal{F}$ [FAMILY OF

FREDHOLM OPERATORS PARAMETRIZED BY X]

"INDEX f" $\in K(X)$ CAN BE DEFINED

(Ker f) - (Coker f)

THEOREM $[X, \mathcal{F}] \xrightarrow{\sim} K(X)$

HOMOTOPY CLASSES

EXAMPLE DIRAC OPERATOR ON COMPACT SPIN MANIFOLD X OF EVEN DIM

$$D : S^+ \longrightarrow S^- \qquad D^* : S^- \longrightarrow S^+$$

INDEX D = NUMBER OF ZERO-MODES IN S^+

− " " " IN S^-

[TECHNICALLY $F = \dfrac{D}{\sqrt{1+D^*D}}$ IS BOUNDED

BUT F AND D HAVE SAME KERNEL]

EX. OF FAMILY
$$\begin{array}{c} Y \\ \downarrow \\ X \end{array}$$
FIBRATION WITH

FIBRES COMPACT, SPIN, EVEN DIM

$x \in X$ D_x DIRAC ALONG FIBRE OVER x

IS FAMILY HAS INDEX $\in K(X)$

REFINEMENTS

REAL CASE : $\mathcal{F}_R$ FREDHOLM OPERATORS IN REAL HILBERT SPACE

$$[X, \mathcal{F}_R] \cong KO(X)$$

EQUIVARIANT CASE

G COMPACT LIE GROUP

$\mathcal{F}_G$ OPERATORS ON $H \otimes L^2(G)$

$$[X, \mathcal{F}_G]_G \cong K_G(X)$$

G-MAPS

MADE FROM G-VECTOR BUNDLES

$K_G(X)$ RELEVANT K-GROUP FOR ORBIFOLDS

? <u>FREDHOLM OPERATOR DEFINITION OF $K_{\pm}$</u>

$F \to F^*$ INVOLUTION ON $\mathfrak{F}$

INDUCES -1 ON $K(X) = [X, \mathfrak{F}]$

CASE OF DOUBLE COVER $X_\alpha \to X$

ASSOCIATED BUNDLE $\mathfrak{F}_\alpha$ OVER X

FIBRE $\mathfrak{F}$

$\underline{K^0_\alpha(X) = \{X, \mathfrak{F}_\alpha\}}$ (HOMOTOPY CLASSES OF SECTIONS)

"TWISTED" K-THEORY

<u>SUGGESTS</u> - FOR GENERAL CASE OF Y WITH INVOLUTION

$\boxed{?}$ $K_{\pm}(Y) = [Y, \mathfrak{F}]_{Z_2}$

(WHERE Z_2 ACTS ON $\mathfrak{F}$ BY $F \to F^*$)

CHECK FOR TRIVIAL INVOLUTION

? $K^0_{\pm}(Y) = [Y, \hat{\mathcal{F}}]$

$\hat{\mathcal{F}} \subset \mathcal{F}$ SELF-ADJOINT OPERATORS

$\hat{\mathcal{F}}$ HAS 3 COMPONENTS

$\left.\begin{array}{ll} \hat{\mathcal{F}}_+ : & \text{ESSENTIALLY POSITIVE} \\ \hat{\mathcal{F}}_- : & \quad\text{"} \qquad \text{NEGATIVE} \end{array}\right\}$ CONTRACTIBLE

$\hat{\mathcal{F}}_*$ — INTERESTING COMPONENT

$[Y, \hat{\mathcal{F}}_*] \cong K^1(Y)$

$\boxed{\text{Ex. } Y = S^1 \quad \text{"SPECTRAL FLOW"}}$

$\Rightarrow$? CORRECT IF WE USE $\hat{\mathcal{F}}_*$

INSTEAD OF $\hat{\mathcal{F}}$

ACHIEVE THIS BY DOUBLING H + USING STABLE HOMOTOPY EQUIVALENCE

FIX OPERATOR J ON H WITH

$J^2 = 1$ (± 1 EIGENVALUES WITH ∞ MULTIPLICITY)

EMBED $\mathcal{F}(H)$ IN $\mathcal{F}(H \oplus H)$ BY

$$F \to \begin{pmatrix} F & 0 \\ 0 & J \end{pmatrix}$$

DEFINE 2 MAPS $f_1, f_2 : Y \to \mathcal{F}(H)$

<u>STABLY</u> EQUIVALENT IF THE INDUCED MAPS $Y \to \mathcal{F}(H \oplus H)$ ARE EQUIVALENT

[NOTE $\hat{\mathcal{F}}(H)$ GOES INTO $\hat{\mathcal{F}}_*(H \oplus H)$]

THEN $$\boxed{K_{\pm}(Y) = [Y, \mathcal{F}]^{s}_{\mathbb{Z}_2}}$$

(S : STABLE EQUIV.)

HOW TO GET FROM HOPKINS DEF. TO FREDHOLM OPERATOR DEF. ?

HOPKINS $\quad K^0_{\pm}(Y) = K^1_{\mathbb{Z}_2}(Y \times \mathbb{R})$

$\boxed{\begin{array}{l} R^{2,0},\ R^{0,2} \\ \Rightarrow K^{*}_{\mathbb{Z}_2}(Y) \text{ PERIODICITY} \end{array}}$

$= K^0_{\mathbb{Z}_2}(Y \times \mathbb{R}^{1,1})$

$\mathbb{R}^{1,1} = \mathbb{R}^2$ WITH INVOLUTION $(x_1, x_2) \to (-x_1, x_2)$

$K(\mathbb{R}^2) = K(S^2, \infty) = \mathrm{Ker}\{K(S^2) \to K(\infty)\}$

SO ESSENTIALLY WE WANT A MAP

$$K_{\mathbb{Z}_2}(Y \times S^2) \to [Y, \mathcal{F}]^{s}_{\mathbb{Z}_2}$$

COUPLE VECTOR BUNDLE ON $Y \times S^2$

WITH DIRAC OPERATOR ON S^2

THIS FLIPS TO ITS ADJOINT UNDER REFLECTION IN EQUATOR OF S^2

NOTE RELATION TO BOTT PERIODICITY

(FOR $R^{2,0}$, $R^{0,2}$ $\mathbb{Z}_2$ IS ORIENTATION PRESERVING)

FURTHER TWISTED K-THEORIES

$\beta \in H^3(X,\mathbb{Z}) \rightarrow$ TWISTED THEORY K_β

ONE DEF. $\beta \rightarrow$ FIBRE BDLE $\mathcal{F}_\beta$ OVER X

TAKE $K_\beta(X) = \{X, \mathcal{F}_\beta\}$

[NS 3-FORM FIELD H $\quad H^3(X,\mathbb{Z})$]

(COMBINE WITH $K_\pm$)

ALSO G-EQUIVARIANT VERSIONS

APPLY TO X = G (CONJUGATION ACTION)

$\Rightarrow$ RELATION WITH VERLINDE ALGEBRA

(FREED. HOPKINS. TELEMAN)

EVEN "BIGGER" TWISTS [HOPKINS MROWKA]

COME FROM CONSIDERING MULTIPLICATIVE GROUP OF UNITS OF K-THEORY : $(\pm 1) \times \tilde{K}$

$H^2 \times [X, BSU]$

1st twist
$\alpha \in H^1(\ ,\mathbb{Z}_2)$

2nd twist
$\beta \in H^3(\ ,\mathbb{Z})$

"Big" twists

NEED TO KNOW STRUCTURE OF MULT. GROUP

EXPONENTIAL ISOMORPHISMS FOR λ-RINGS
QUART. J. Math. Oxford (1971) (ATIYAH-SEGAL)

p-adic completion

PHYSICAL APPLICATION OF BIG TWISTS ???