Why Supersymmetry is Different

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I view the foundation of string theory as a sort of tripod, with the three supporting legs being perturbative string theory, by which the subject was discovered; the web of nonperturbative dualities, whose consistency gives powerful evidence that the theory exists beyond perturbation theory; and gauge/gravity duality, which gives a nonperturbative definition under certain circumstances.

There is a certain sense in which perturbative string theory is first among equals: if one looks closely, most of what we know about the other two legs of the tripod ultimately requires making contact with something we know from perturbative string theory.

25 years ago, it seemed (to me) that perturbative string theory was sufficiently well understood, but in the meantime so much progress was made in the other two areas that (to me again) the foundation in perturbative stringy theory came to look a little shaky by comparison.

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Let us orient ourselves by starting with bosonic closed strings. Let V be a (1,1) primary state (constructed from matter fields only) that represents a massless graviton (or B-field) mode. V is pure gauge – it is a null state in the language of conformal field theory – if $V = L_{-1}W$ (or $\widetilde{L}_{-1}W$) for some W. (W is a primary of appropriate dimension.)

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$$\int_\Sigma V = \int_\Sigma \partial W = 0.$$

This is the most elementary explanation of gauge-invariance for massless states of the bosonic string.

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In the BRST approach, to compute an S-matrix element, one considers the worldsheet path integral

$$\langle \mathcal{V}_1 \dots \mathcal{V}_n \rangle_{\text{worldsheet}}$$

with an insertion of a product of BRST-invariant vertex operators $\mathcal{V}_1, \dots, \mathcal{V}_n$. To make this path integral nonzero, we need a lot of antighost insertions ($6g - 6 + 2n$ of them) and the dependence of the worldsheet path integral on the antighost insertions gives a differential form $F_{\mathcal{V}_1,\dots,\mathcal{V}_n}$ of top degree on $\mathcal{M}_{g,n}$, the moduli space of Riemann surfaces of genus g with n marked points. The genus g contribution to the scattering amplitude is then

$$\langle \mathcal{V}_1 \mathcal{V}_2 \dots \mathcal{V}_n \rangle_g = \int_{\mathcal{M}_{g,n}} F_{\mathcal{V}_1,\dots,\mathcal{V}_n}.$$

Suppose now that one of the vertex operators is pure gauge, say $\mathcal{V}_1 = \{Q, \mathcal{W}\}$, where the ghost number of $\mathcal{W}$ is 1 less than that of $\mathcal{V}_1$ (and so 1 instead of 2). We consider a worldsheet path integral

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and as the ghost number is 1 less than before, it takes 1 less antighost insertion than before to make this path integral nonzero. Hence it defines a differential form $F_{\mathcal{W}\mathcal{V}_2 \dots \mathcal{V}_n}$ whose degree is 1 less than that of the form $F_{\mathcal{V}_1\mathcal{V}_2 \dots \mathcal{V}_n}$ that has to be integrated to compute the genus g contribution to the S-matrix element.

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$$F_{\mathcal{V}_1\mathcal{V}_2\dots\mathcal{V}_n} = \mathrm{d}F_{\mathcal{W}\mathcal{V}_2\dots\mathcal{V}_n}.$$

The proof of gauge-invariance of the genus g contribution to a scattering amplitude (for any g) is almost immediate:

$$\begin{aligned}\langle \mathcal{V}_1 \mathcal{V}_2 \ldots \mathcal{V}_n \rangle_g &= \int_{\mathcal{M}_{g,n}} F_{\mathcal{V}_1,\ldots,\mathcal{V}_n} = \int_{\mathcal{M}_{g,n}} \mathrm{d}F_{\mathcal{W},\mathcal{V}_2,\ldots,\mathcal{V}_n} \\ &= \int_{\partial\mathcal{M}_{g,n}} F_{\mathcal{W},\mathcal{V}_2,\ldots,\mathcal{V}_n}, \end{aligned} \tag{1}$$

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where $\partial\mathcal{M}_{g,n}$ is the "boundary" of moduli space. In the last step, we integrated by parts and used Stokes's theorem. To complete the proof, we just have to show that the boundary contributions vanish. For the problem we are considering at the moment (decoupling of pure gauge modes in the S-matrix) this poses no great difficulty.

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We can integrate over $\mathcal{M}_{g,n}$ by first integrating over the fiber (i.e. over the position of one given vertex opertar $\mathcal{V}_1$ in a fixed Riemann surface Σ) and then over the base (the remaining moduli of Σ, including positions of other vertex operators). For massless null states, one gets a total derivative already in the first step, but for massive null vectors, only the overall (or final) integral is a total derivative.

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To explain how this happens, let us consider in bosonic closed string theory the gauge parameter $\mathcal{W} = \widetilde{c}\varepsilon \cdot \widetilde{\partial} X e^{ik\cdot X}$. For k non-zero, $\mathcal{V}_1 = \{Q, \mathcal{W}\}$ is a massless null vector, but if we set $k = 0$, $\mathcal{W}$ becomes antiholomorphic and $\mathcal{V}_1 = \{Q, \mathcal{W}\} = 0$.

We can run the same argument as before for decoupling of null vectors, but now, since $\mathcal{V}_1 = 0$, the left hand side is zero for a more trivial reason:

$$0 = \int_{\mathcal{M}_{g,n}} F_{\mathcal{V}_1 \dots \mathcal{V}_n} = \int_{\mathcal{M}_{g,n}} \mathrm{d} F_{\mathcal{W}\mathcal{V}_2 \dots \mathcal{V}_n} = \int_{\partial \mathcal{M}_{g,n}} F_{\mathcal{W}\mathcal{V}_2 \dots \mathcal{V}_n}.$$

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Although the left hand side is trivially zero, once we set $k = 0$, the right hand side is not trivially zero. On the contrary, we get nonzero contributions from the integrals over the different components of $\partial\mathcal{M}_{g,n}$. The fact that these contributions add to zero is a conservation law (conservation of momentum or momentum plus winding, in this case).

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For this, instead of saying that at zero momentum $0 = \{Q, \mathcal{W}\}$, we remove the ghosts and then the usual relation $V = \partial W$ becomes (since $V = 0$ at zero momentum) $0 = \partial W$, in other words W is an (antiholomorphic) conserved current. We do not need fancy arguments on moduli space to get a conservation law because we simply have a conserved current on the worldsheet.

As usual a conserved current leads to a conservation law:

$$0 = \int_{\Sigma} \mathrm{d}^2 z \langle \partial W(\bar{z}, z) \cdot \mathcal{V}_2 \dots \mathcal{V}_n \rangle_{\Sigma} = \sum_{j=2}^{n} \langle \mathcal{V}_2 \dots \oint_{\gamma_i} \mathcal{W} \cdot \mathcal{V}_i \dots \mathcal{V}_n \rangle_{\Sigma},$$

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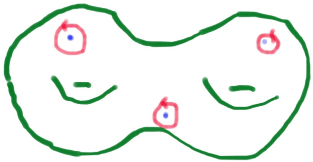

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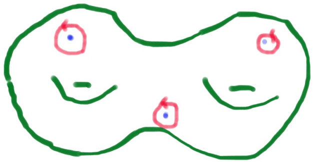

So if $\mathcal{V}_i$ has "charge" q_i, in the sense that $\oint_{\gamma_i} \mathcal{W} \cdot \mathcal{V}_i = 2\pi i q_i \mathcal{V}_i$, then $0 = \sum_i q_i \cdot \langle \mathcal{V}_2 \ldots \mathcal{V}_n \rangle$ and for the correlation function to be nonzero requires a conservation law

$$\sum_i q_i = 0$$

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A vertex operator of the bosonic string can be moved around on the string worldsheet Σ without changing Σ. But in the case of a *massive* null state, this information is not useful in proving gauge invariance. We need a more powerful formalism of integration by parts on moduli space $\mathcal{M}_{g,n}$, not on Σ.

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A Ramond sector vertex operator in superstring theory cannot be moved around on the superstring worldsheet Σ without changing Σ. So we can't prove gauge invariance by integration by parts on Σ. We need a more powerful formalism of integration by parts on the supermoduli space $\mathfrak{M}_{g,n}$, not on Σ.

For those who find this helpful, the fancy way to say this is that the fibration of $\mathcal{M}_{g,n} \to \mathcal{M}_{g,n-1}$ that forgets one puncture

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has no analog for Ramond punctures on a super Riemann surface. So we cannot prove gauge-invariance for Ramond states by integrating over the fibers of such a fibration, as we do for massless (but not massive) gauge invariances of the bosonic string.

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Let us recall what we said before. If $\mathcal{V}_1, \ldots, \mathcal{V}_n$ are BRST-invariant vertex operators of physical states, then the worldsheet path integral

$$\langle \mathcal{V}_1 \ldots \mathcal{V}_n \rangle_{\mathrm{worldsheet}}$$

defines a top-form $F_{\mathcal{V}_1 \ldots \mathcal{V}_n}$ on moduli space. If one of the $\mathcal{V}$'s is BRST-trivial, say $\mathcal{V}_1 = \{Q, \mathcal{W}\}$, then the worldsheet path integral with $\mathcal{V}_1$ replaced by $\mathcal{W}$

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defines a codimension-one form $F_{\mathcal{W}\mathcal{V}_2 \ldots \mathcal{V}_n}$ on moduli space. The relation between these forms is

$$F_{\mathcal{V}_1 \mathcal{V}_2 \ldots \mathcal{V}_n} = \mathrm{d} F_{\mathcal{W}\mathcal{V}_2 \ldots \mathcal{V}_n}.$$

So the proof of decoupling of $\mathcal{V}_1 = \{Q, \mathcal{W}\}$ proceeds as before:

$$\begin{aligned} \langle \mathcal{V}_1 \mathcal{V}_2 \ldots \mathcal{V}_n \rangle_g &= \int_{\mathfrak{M}_{g,n}} F_{\mathcal{V}_1,\ldots,\mathcal{V}_n} = \int_{\mathcal{M}_{g,n}} \mathrm{d}F_{\mathcal{W},\mathcal{V}_2,\ldots,\mathcal{V}_n} \\ &= \int_{\partial\mathfrak{M}_{g,n}} F_{\mathcal{W},\mathcal{V}_2,\ldots,\mathcal{V}_n}, \qquad (2) \end{aligned}$$

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where $\partial\mathcal{M}_{g,n}$ is the "boundary" of moduli space. In the last step, we integrated by parts and used Stokes's theorem. I copied this from a previous slide, except that I changed $\mathcal{M}_{g,n}$ to $\mathfrak{M}_{g,n}$.

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$$\langle \mathcal{V}_1 \mathcal{V}_2 \dots \mathcal{V}_n \rangle_g = \int_{\mathfrak{M}_{g,n}} F_{\mathcal{V}_1,\dots,\mathcal{V}_n} = \int_{\mathcal{M}_{g,n}} \mathrm{d}F_{\mathcal{W},\mathcal{V}_2,\dots,\mathcal{V}_n}$$
$$= \int_{\partial \mathfrak{M}_{g,n}} F_{\mathcal{W},\mathcal{V}_2,\dots,\mathcal{V}_n}, \qquad (2)$$

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For the bosonic string, we only needed this formalism for massive gauge invariances, which are not related to conservation laws, so the proofs of conservation laws were more straightforward. What is different about superstring theory is that since we need this formalism for some of the massless gauge fields – gravitinos – there are conservation laws – spacetime supersymmetry – that really require this formalism.

The gauge generator for a gravitino null state is

$$\mathcal{W} = \exp(ik \cdot X)\zeta^\alpha \mathcal{S}_\alpha$$

where $\mathcal{S}_\alpha$ is the fermion vertex operator of Friedan, Martinec, and Shenker and ζ^α is a c-number solution of the Dirac equation $k \cdot \Gamma\zeta = 0$. If we set $k = 0$, the Dirac equation becomes trivial, so we can forget ζ^α and take $\mathcal{W} = \mathcal{S}_\alpha$ (for some α). Now we have the same formula as the one that proves decoupling of the null vectors except that we are in the special case that the null vector $\mathcal{V}_1 = \{Q, \mathcal{W}\} = \{Q, \mathcal{S}_\alpha\}$ is actually 0:

$$0 = \int_{\mathfrak{M}_{g,n}} F_{\mathcal{V}_1,\dots,\mathcal{V}_n} = \int_{\mathfrak{M}_{g,n}} \mathrm{d}F_{\mathcal{S}_\alpha,\mathcal{V}_2,\dots,\mathcal{V}_n} = \int_{\partial\mathfrak{M}_{g,n}} F_{\mathcal{S}_\alpha,\mathcal{V}_2,\dots,\mathcal{V}_n}.$$

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The supersymmetric Ward identity comes by explicitly evaluating the right hand side as a sum over the components of $\partial\mathfrak{M}_{g,n}$.

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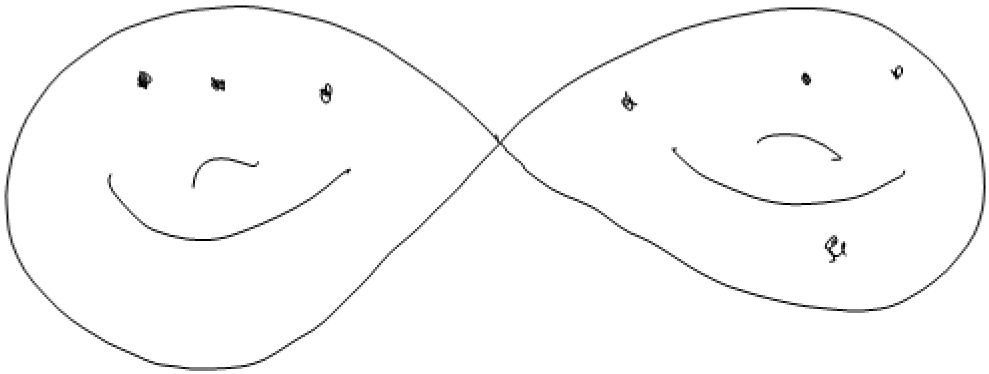

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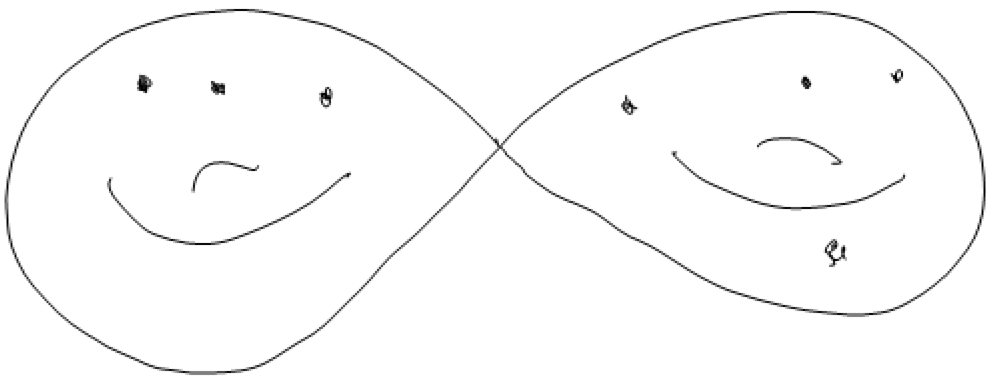

So we get a Ward identity

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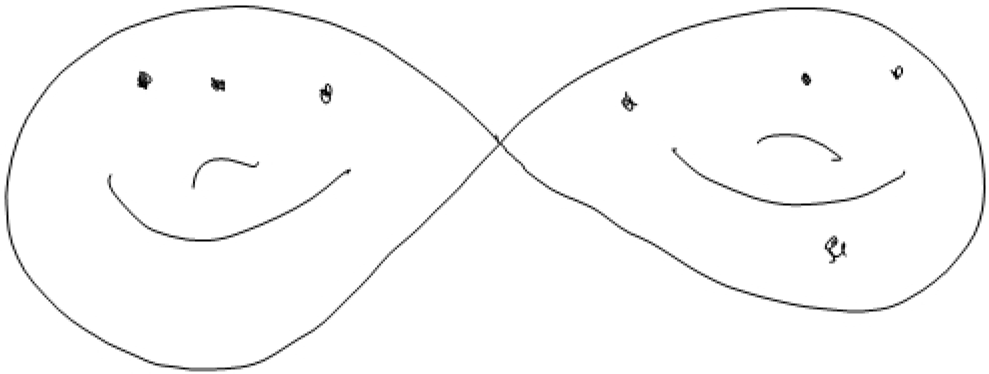

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and this is the identity that will under favorable conditions lead to spacetime supersymmetry. However, most of the $\mathcal{D}_i$ do not contribute. (A necessary condition is that the momentum flowing through the singularity should be generically on-shell.)

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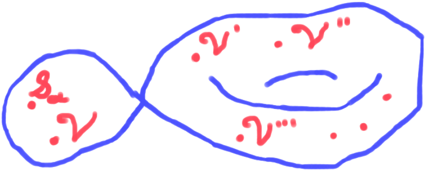

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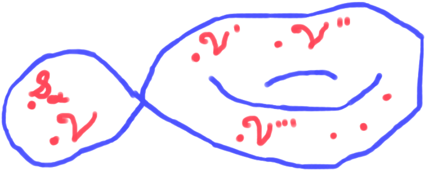

The left part of the worldsheet contains the supercurrent $\mathcal{S}_\alpha$ and precisely one other vertex operator $\mathcal{V}$.

The contribution of this type of component is an S-matrix element obtained by replacing the left part of the worldsheet that contains the product $\mathcal{S}_\alpha \cdot \mathcal{V}$ by an effective operator that couples to the right hand side of the picture. This operator is linear in $\mathcal{S}_\alpha$ and $\mathcal{V}$, so we can call it $\{Q_\alpha, \mathcal{V}\}$, where this formula defines the spacetime supersymmetry generator Q_α.

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$$0 = \sum_i \langle \mathcal{V}_1 \ldots \mathcal{V}_{i-1} \{Q_\alpha, \mathcal{V}_i\} \mathcal{V}_{i+1} \ldots \mathcal{V}_n \rangle = 0.$$

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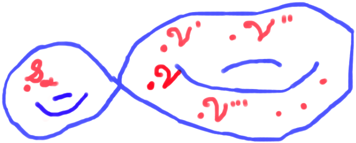

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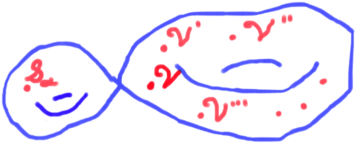

In field theory terms, this contribution involves the matrix element for the supercurrent to create a Goldstone fermion that then couples to $\mathcal{V}_1 \ldots \mathcal{V}_n$. (Such a contribution arises at 1-loop order in the SO(32) heterotic string on a Calabi-Yau; this statement is related to old analyses by Dine-Ichinose-Seiberg and Atick-Dixon-Sen.)

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Apart from thanking the organizers of Strings 2013 for their hard work and hospitality, I have only one more thing to say:

Apart from thanking the organizers of Strings 2013 for their hard work and hospitality, I have only one more thing to say: You are all invited to Strings 2014 in Princeton.

Strings 2014 in Princeton June 23 -- 27

- Princeton University and Inst. for Advanced Study

Richardson Auditorium in Alexander Hall