2d Dualities from 3d Dualities

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Strings 2016

August 4, 2016

based on work with O. Aharony, S. Razamat, and N. Seiberg

Motivation - dualities of supersymmetric QFTs

- Dualities are extremely useful probes to get a deeper understanding of quantum field theory and string theory.
- There is by now a vast web of supersymmetric dualities of QFTs known in various numbers of dimensions and amounts of supersymmetry.

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- Dualities are extremely useful probes to get a deeper understanding of quantum field theory and string theory.
- There is by now a vast web of supersymmetric dualities of QFTs known in various numbers of dimensions and amounts of supersymmetry.
- Examples of exact (conformal) dualities:
  - Mirror symmetry of $2d$ $\mathcal{N}=(2,2)$ Calabi-Yau sigma models
  - $\mathcal{S}$-duality of $4d$ $\mathcal{N}=2$ theories.
- Examples of IR dualities:
  - Hori-Vafa duality- $2d$ $\mathcal{N}=(2,2)$ linear sigma models $\leftrightarrow$ Landau-Ginzburg models.
  - $3d$ mirror symmetry - duality exchanging Higgs and Coulomb branches of IR SCFT of $3d$ $\mathcal{N}=4$ and $\mathcal{N}=2$ gauge theories
  - Seiberg duality of $4d$ $\mathcal{N}=1$ QCD, eg:

    $$SU(N_c) \ + \ N_f \text{ fundamental flavors} \leftrightarrow$$

    $$SU(N_f - N_c) \ + \ N_f \text{ fundamental flavors} + \ {N_f}^2 \ \text{ mesons and } W = q^a M_a^b \tilde{q}_b$$

    also $3d$ $\mathcal{N}=2$ and $2d$ $\mathcal{N}=(2,2)$ versions.
  - Many others

Evidence for dualities

- Many of these dualities were found by taking low energy limits of string theory constructions.
- They can also be found by compactification of higher dimensional theories - *e.g.*, 4*d* $\mathcal{S}$-duality arises by compactification of 6*d* $\mathcal{N} = (2,0)$ theory on a Riemann surface.
- Some tools for testing dualities
  - Matching of quantum moduli space of supersymmetric vacua
  - BPS operators/boundary conditions
  - ’t Hooft anomaly matching
  - Consistency under RG flow
- Also, partition functions on compact manifolds:
  - We can place theories supersymmetrically on a suitable compact curved manifold $\mathcal{M}_d$ and compute the partition function, $\mathcal{Z}(\mathcal{M}_d)$, by localization.
  - This gives a rich, duality invariant observable, which contains much of the information above.

Reduction and duality

- In this talk we will ask the following questions (focusing on the case $d = 3$):
  - How can we describe the compactification of a d-dimensional SQFT to $d - 1$ dimensions?
  - When does a d-dimensional duality imply a duality in $d - 1$ dimensions (and when does it not)?
- In this way we can hope to better organize the web of dualities, and possibly find new ones.

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- In this way we can hope to better organize the web of dualities, and possibly find new ones.
- If two d-dimensional theories are exactly equivalent, then they are clearly equivalent on $\mathbb{R}^{d-1} \times S^1_r$. At low energies compared to $1/r$, we find equivalent $d-1$ dimensional theories.
- However, for IR dualities this is no longer true. If μ_a are relevant parameters which initiate a flow from the UV theory to the IR, we find on each side a family of theories parameterized by:

$$\gamma_a \equiv \mu_a r^{dim(\mu_a)}$$

  - For $\gamma_a \to 0$, we are reducing the UV descriptions, and have a $d-1$-dimensional Lagrangian, but no duality.
  - For $\gamma_a \to \infty$, we are reducing the IR descriptions. Then we have a duality, but may not have a useful Lagrangian description.

Reducing from 3d $\mathcal{N}=2$ to 2d $\mathcal{N}=(2,2)$

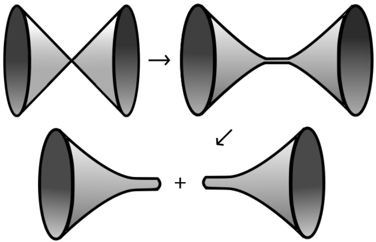
- In this talk we will focus on the case of 3d $\mathcal{N}=2$ gauge theories. Here there are two types of relevant parameters in the UV:

$$\text{gauge couplings } g_{3\,j}^2, \quad \text{real masses } m_a$$

- Correspondingly, we can define two types of parameters for the theory on S^1_r:

$$\gamma_j = g_{3\,j}^2 r, \quad t_a = r m_a$$

- As we'll see, the former control the asymptotic behavior of the target space metric, while the latter give rise to Kahler moduli.
- Another important feature here is the lack of a moduli space of a 2d CFT: instead the states are wavefunctions on the pseudo-moduli space, and there is typically a single superselection sector.
- However, a UV theory with multiple branches may flow to decoupled CFTs, eg, $\mathcal{N}=(4,4)$ $U(1)$ with N_f hypermultiplets.

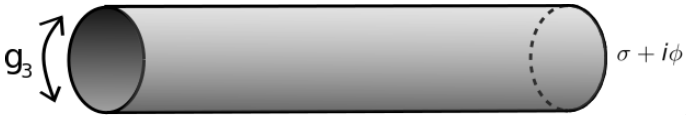


Example: free $U(1)$ gauge theory

- First consider the free $3d$ $\mathcal{N}=2$ $U(1)$ gauge theory, with action:

$$S=\int d^3x\frac{1}{g_3^2}(F_{\mu\nu}^2+(\partial_\mu\sigma)^2-i\lambda^\dagger\gamma^\mu\partial_\mu\lambda)$$

- We can dualize the gauge field to a scalar ϕ by writing $d\phi=\star F$. Then quantization of flux identifies $\phi\sim\phi+g_3$, and the moduli space is a cylinder:



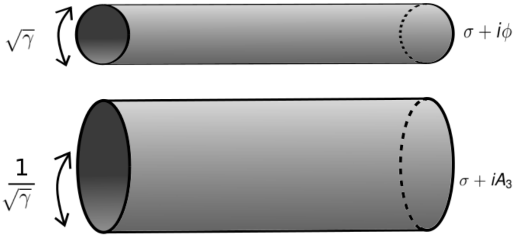
- Next consider placing this theory on $\mathbb{R}^2\times S^1_r$. Then we find a sigma model with cylinder target space, which, in $2d$ normalization, has radius $\sqrt{g_3^2 r}=\sqrt{\gamma}$.

Free $U(1)$ gauge theory (cont'd)

- Alternatively, we describe the $3d$ theory on a circle in terms of a twisted chiral superfield:

$$\Sigma = \sigma + iA_3 + \ldots$$

- Large gauge transformations identify $\Sigma \sim \Sigma + \frac{i}{r}$. In the $2d$ normalization, the radius of the cylinder becomes $\frac{1}{\sqrt{\gamma}}$. Thus we find a T-dual description of the first cylinder.



- Two lessons:
  1. $3d$ EM duality reduces to T-duality. [Aganagic, Karch, Hori, Tong].
  2. The $2d$ theory depends importantly on $\gamma = g_3^2 r$.

Example: $U(1)$ $N_f = 1$

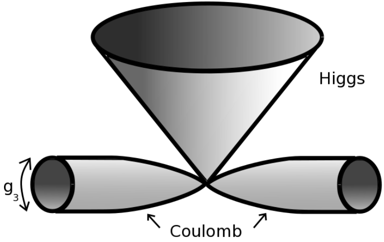
- Next consider an interacting gauge theory: SQED with chirals Q and $\tilde{Q}$ of charge 1 and -1. We will consider two relevant parameters:

$$\text{gauge coupling } g_3^2, \qquad \text{Fayet-Iliopolous (FI) parameter } \zeta$$

- The potential for the scalar fields is:

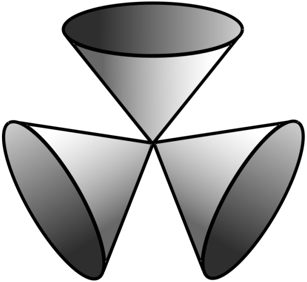
$$V = \frac{g_3^2}{8}(|Q|^2 - |\tilde{Q}|^2 - \frac{\zeta}{2\pi})^2 + \sigma^2(|Q|^2 + |\tilde{Q}|^2)$$

- Then the moduli space of vacua consists of two branches:
  - A Higgs branch, where $\sigma = 0$, but $Q, \tilde{Q}$ are non-zero,
  - When $\zeta = 0$, a Coulomb branch, where $Q = \tilde{Q} = 0$, but σ is non-zero.



Example: $U(1)$ $N_f = 1$ (cont'd)

- While the Coulomb branch is asymptotically a cylinder of radius g_3, the Higgs branch is independent of g_3.
- In the IR limit, $g_3 \to \infty$, these three branches appear symmetrically.



- We find an IR dual description as an WZ model with superpotential $W = XYZ$.

Reducing $U(1)$ $N_f = 1$

- If we place this theory (with $\zeta = 0$) on $\mathbb{R}^2 \times S^1_r$, we find at energies below $1/r$ an effective $2d$ description.
- Asymptotically on the Coulomb branch this is a sigma model with radius $\sqrt{g_3^2 r} = \sqrt{\gamma}$.
- For finite γ, this theory is *not* equivalent to the $2d$ XYZ theory.
- We can try to engineer a $2d$ UV description. A natural guess is as $2d$ SQED with one flavor. However note:

$$g_2^2 = \frac{1}{r} g_3^2$$

  So to obtain a finite g_2, we must take $\gamma \to 0$.
- In particular, $2d$ SQED is *not* equivalent to *XYZ*! (*e.g.*, note they have different numbers of branches in the IR.)

Fix 1- Focus on Higgs branches

- Since the problem was with the Coulomb branch, we might look for deformations which lift it.
- The FI parameter ζ does precisely this. Let us then instead take:

$$t = \zeta r$$

non-zero and finite.

- Then we claim one finds the $2d$ $U(1)$ theory with non-zero FI parameter t.
- On the XYZ side, this gives a large mass to two of the chirals, X and Y, and they can be integrated out. We find a single free chiral Z.
- This leads to the $2d$ duality:

$$U(1) \text{ with chirals of charge 1 and } -1, \quad \leftrightarrow \quad \text{a free chiral}$$

- This appears to be a valid $2d$ duality (up to deformations of the asymptotic Kahler potential).
- For example, we can check the matching of supersymmetric partition functions in $2d$, namely, the S^2 partition function and elliptic genus.

An aside - supersymmetric partition functions

- A useful tool for studying dualities are supersymmetric partition functions. These are computed by localization, and are a function of background fields and geometric parameters:

$$\mathcal{Z}_{\mathcal{M}_d}(\text{background fields } \mu_a, \text{geometric parameters } \beta_i) = \sum\!\!\!\!\!\!\int \text{ BPS configurations}$$

- For studying reduction, we consider the 3*d* supersymmetric index, or $S^2 \times S^1_\tau$ partition function. It satisfies:

$$\mathcal{Z}_{S^2 \times S^1_\tau}(\mu_a; \tau) \quad \underset{\tau \to 0}{\longrightarrow} \quad \tau^{-c_{2d}+\cdots} \mathcal{Z}_{S^2}(\mu_a) + \ldots$$

- Then starting from a 3*d* duality, the 3*d* indices of the theories match, and so the S^2 partition functions of the 2*d* reductions must match.
- However, there may be some subtleties. *E.g.*, if a direct sum of multiple theories arises in 2*d*, only the one with maximal c_{2d} will be seen.
- The elliptic genus (or T^2 partition function), however, is defined with a twist by the left-moving R-symmetry, and so does not have a 3*d* uplift. Thus matching of the elliptic genus is a strong independent check of 2*d* dualities.

Fix 2- Duality of massive theories

- Another approach is to look at the theory with generic mass parameters turned on, which lifts the moduli space to discrete vacua.
- Here a useful object is the effective twisted superpotential $\tilde{W}(\Sigma; r)$ of the $3d$ theory compactified on S^1_r. [Aganagic, Hori, Karch, Tong]
- Taking the $r \to 0$ limit, and defining $X = r\Sigma$, one finds the gauge theory is described at low energies by the twisted superpotential:

  $$\tilde{W}(X) = \zeta X + m \log \sinh \frac{X}{2}$$

  giving a massive Landau-Ginzburg model for the twisted chiral field X.
- One can check that this matches the mass-deformed XYZ theory in $2d$. *E.g.*, one can compute their S^2 partition functions and check:

  $$Z_{S^2}(\zeta, m)[\text{LG model}] = Z_{S^2}(\zeta, m)[XYZ]$$

- This also follows by carefully taking the $\tau \to 0$ limit of the identity of the $3d$ index identities.
- However, the duality of the massive theories need not imply a duality at zero mass.

$U(N_c)$ theory with N_f flavors

- Next let us consider a more complicated example, $U(N_c)$ gauge theory with N_f pairs of (anti-)fundamental chirals.
- This theory has the following IR-dual description [Aharony]:

$$U(N_f - N_c) \ + \ N_f \text{ flavors} \ + \ N_f^2 \ \text{ mesons and singlets } \tilde{V}_\pm$$

$$\text{with superpotential } \ W = Mq\tilde{q} + V_+\tilde{V}_- + V_-\tilde{V}_+$$

- As before, reducing the undeformed theory leads to problems with the Coulomb branch, so we set $t = \zeta r$ finite.
- This gives a mass to the fields $\tilde{V}_\pm$, and we arrive at the following $2d$ duality, found previously by [Benini, Park, Zhao]:

$$U(N_c) \ + \ N_f \text{ flavors} \quad \leftrightarrow \quad U(N_f - N_c) + \ N_f \text{ flavors} \ + \ \text{mesons}$$

- The identity of S^2 partition functions follows from that of the $S^2 \times S^1$ identity of the $3d$ dual theories. The identity of the elliptic genus also holds, as an independent $2d$ check.

$U(N_c)$ theory with N_f flavors and CS level k

- Next we consider the same theory with the addition of a SUSY Chern-Simons term for the gauge field, defined for $k \in \mathbb{Z}$:

$$S_{CS} = \frac{k}{4\pi} \int d^3x Tr\left(\epsilon^{\mu\nu\rho}(A_\mu \partial_\nu A_\rho + \frac{2i}{3} A_\mu A_\nu A_\rho) + 2\sigma D + \lambda^\dagger \lambda\right)$$

- Then is theory has the following IR-dual description [Giveon, Kutasov]:

$$U(|k| + N_f - N_c)_{-k} + \; N_f \text{ flavors } + \; N_f^2 \text{ mesons, with } \; W = Mq\tilde{q}$$

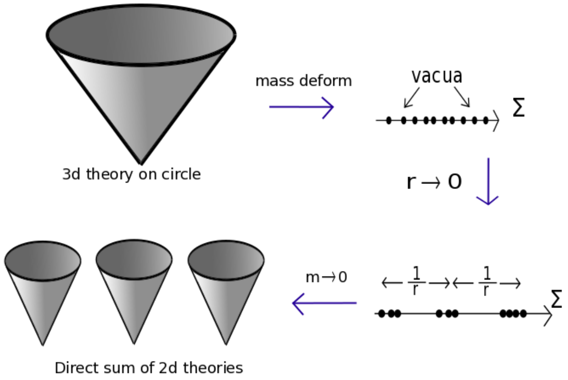
- The effect of the Chern-Simons term in the $3d$ theory on a circle naively vanishes as we take $r \to 0$. Proceeding as before, we seem to find a $2d$ duality:

$$U(N_c) \; + \; N_f \text{ flavors } \quad \overset{?}{\leftrightarrow} \quad U(|k| + N_f - N_c) + \; N_f \text{ flavors } + \text{ mesons}$$

for any $k \in \mathbb{Z}$, which is clearly wrong.

$U(N_c)$ theory with N_f flavors and CS level k (cont'd)

- To see what's going on, we mass deform the theory and study the behavior of the vacua as $r \to 0$ using $\tilde{W}(\Sigma, r)$.



- We find the low energy limit of $U(N_c)_k + N_f$ flavors on S^1_r is a direct sum:

$$\Big(U(N_c) \; + \; N_f \;\; \text{flavors}\Big) \oplus \Big(U(N_c-1) \; + \; N_f \;\; \text{flavors}\Big) \oplus \ldots \oplus \Big(U(N_c-k) \; + \; N_f \;\; \text{flavors}\Big)$$

- The dual decription is similarly given by a direct sum, which is term-wise dual:

$$\Big(U(|k|+N_f-N_c) \; + \; N_f \;\; \text{flavors}\Big) \oplus \ldots \oplus \Big(U(N_f-N_c) \; + \; N_f \;\; \text{flavors}\Big)$$

Some other examples

- 3d duality: Abelian mirror symmetry
  $\Rightarrow$ 2d duality: Hori-Vafa/Hori-Kapustin duality (as shown by [Aganagic et al])

- 3d duality: $SU(N_c)_k$ (N_f, N_a) (anti-)fundamental chirals dual to $SU(N_f - N_c)_{-k}$ with (N_f, N_a) for $N_f > N_a + 1,\ k < \frac{N_f - N_a}{2}$. [Aharony,Fleischer]
  $\Rightarrow$ 2d duality: $SU(N_c)$ (N_f, N_a) dual to $SU(N_f - N_c)$ (N_f, N_a) for $N_f > N_a + 1$ $\rightarrow$ generalization of [Hori,Tong].

- 3d duality: $Sp(2N_c)_k$ $2N_f$ flavors dual to $Sp(2(N_f + k - N_c - 1))_{-k}$ + $2N_f$ flavors
  $\Rightarrow$ 2d duality: For $2k$ odd, gives $Sp(2N_c)$ $2N_f$ flavors dual to $Sp(2(N_f - N_c - \frac{1}{2}))$ + $2N_f$ flavors [Hori]. For $2k$ even, there is an unlifted Coulomb branch, and we do not find duality of 2d gauge theories.

- S^2 partition functions match in these examples, as required by reducing the 3d index identites. Elliptic genera also match.

Summary

- We have seen various physical subtleties arise when studying the compactification of IR dualities of 3*d* theories.
- Because of the dependence of the 2*d* theory on the gauge coupling, naive reduction of dualities does not work, but sometimes fixes are available.
- The effective twisted superpotential and supersymmetric partition functions give useful tools for studying the reduction.
- We have recovered known dualities in 2*d*, and found new ones.

Open questions

- Better understanding the reduction of theories with Coulomb branches.
- Reducing other dualities from 3*d* to 2*d*, eg, nonabelian mirror symmetry, dualities derived from class *S* in 4*d*.
- Study reductions between other dimensions - eg, $5 \to 4, 4 \to 2$ (choice of Riemann surface, fluxes), etc.