Recent Developments in N=4 Yang-Mills Amplitudes

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Introduction

- Planar N=4 Yang-Mills scattering amplitudes have been computed to very high loop order.
- They have many remarkable properties, that have sparked interest from mathematicians working on combinatorics, algebraic geometry and number theory.
- At the same time, several methods that have been developed for N=4 Yang-Mills are directly applicable to, and have greatly aided, QCD computations.
- In this talk, I will review some recent developments in N=4 Yang-Mills amplitudes and describe some approaches that hope to explain their properties using various mathematical constructions.

Outline

- Introduction
- Planar N=4 Yang-Mills n-point amplitudes:
  -- status and tools
- n= 6, 7 amplitudes: cluster algebras
- New features for n>7 amplitudes from
  -- plabic graphs
  -- tensor diagrams
- Conclusions

Planar N=4 Yang-Mills
n-point scattering amplitudes

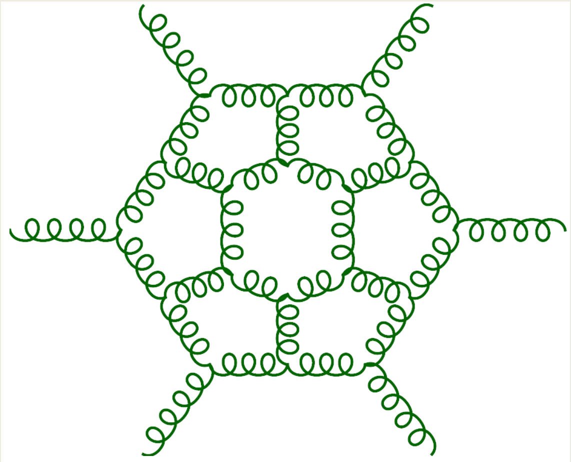
- n<6 all loops — Bern, Dixon, Smirnov '05
- n=6 through 7-loops
- n=7 through 4-loops

  Caron-Huot, Dixon, Drummond, Dulat, Foster, Gurdogan, von Hippel, Papathanasiou, review: 2005.06735
- n=8, 9 MHV, NMHV through 2-loops — He, Li, Zhang '20
- n>9 MHV through 2-loops — Caron-Huot '10

n=6 and n=7 Amplitudes Bootstrap

Write down the answer as linear combo of functions and determine the coefficients by solving a system of linear constraints.

Remaining number of parameters in the ansatz for (MHV, NMHV) n=6 amplitude after each constrain is applied at each loop order:



[Caron-Huot, Dixon, Drummond, Dulat, Foster, Gurdogan, von Hippel, Papathanasiou, review: 2005.06735]

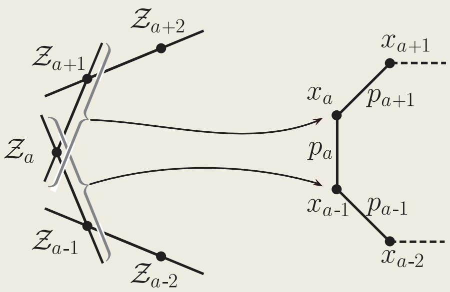
Constraint	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$	$L=6$
1. $\mathscr{H}_6$	6	27	105	372	1214	3692?
2. Symmetry	(2,4)	(7,16)	(22,56)	(66,190)	(197,602)	(567,1795?)
3. Final-entry	(1,1)	(4,3)	(11,6)	(30,16)	(85,39)	(236,102)
4. Collinear	(0,0)	(0,0)	$(0^*,0^*)$	$(0^*,2^*)$	$(1^{*3},5^{*3})$	$(6^{*2},17^{*2})$
5. LL MRK	(0,0)	(0,0)	(0,0)	(0,0)	$(0^*,0^*)$	$(1^{*2},2^{*2})$
6. NLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	$(0^*,0^*)$	$(1^*,0^{*2})$
7. NNLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	$(1,0^*)$
8. N^3LL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
9. Full MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
10. T^1 OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
11. T^2 OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)

Tools for N=4 Yang-Mills Amplitudes

- Momentum -> Momentum Twistors

$$\mathbf{Z_i^A} = (\mathbf{Z_i^1}, \mathbf{Z_i^2}, \mathbf{Z_i^3}, \mathbf{Z_i^4}) \in \mathbf{P^3}$$

$$\langle i\,jkl \rangle \equiv \langle Z_i Z_j Z_k Z_l \rangle = \det(Z_i Z_j Z_k Z_l)$$



Penrose, Hodges, Arkani-Hamed et al

- MHV and NHMV L-loop amplitudes can be expressed in terms of multiple polylogarithms of weight m=2L

$$dF_m = \sum_{\phi_{\alpha_1} \in \Phi} F_{m-1}^{\phi_{\alpha_1}} d\log \phi_{\alpha_1}$$

$$dF_{m-1}^{\phi_{\alpha_1}} = \sum_{\phi_{\alpha_2} \in \Phi} F_{m-2}^{\phi_{\alpha_2},\phi_{\alpha_1}} d\log \phi_{\alpha_2}$$

SYMBOL

$$\mathbf{S}[F_m] = \sum_{\phi_{\alpha_1},\phi_{\alpha_2},\ldots\phi_{\alpha_m} \in \Phi} F_0^{\phi_{\alpha_m},\phi_{\alpha_{m-1}},\ldots,\phi_{\alpha_2}\phi_{\alpha_1}} [\phi_{\alpha_m} \otimes \phi_{\alpha_{m-1}} \otimes \ldots \otimes \phi_{\alpha_2} \otimes \phi_{\alpha_1}]$$

Example

$$dLi_2(z) = -\log(1-z) d\log(z) \to \mathbf{S}[Li_2(z)] = -(1-z) \otimes z$$

Goncharov, Spradlin, Vergu, AV

FOCUS OF MY TALK:
SYMBOL ALPHABET

$$\phi_\alpha \in \Phi$$

-encodes singularities
-input in bootstrap

Example: 2-loop MHV Amplitudes

- n=6 symbol alphabet is given by 15 letters:

all Gr(4,6) Plucker coordinates <a a+1 b c>

$$R_6^{(2)} = \mathrm{Li}_4\left(-\frac{\langle 1234\rangle\langle 2356\rangle}{\langle 1236\rangle\langle 2345\rangle}\right) - \frac{1}{4}\mathrm{Li}_4\left(-\frac{\langle 1246\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1456\rangle}\right) + \cdots$$

GSVV

- n=7 symbol alphabet is given by 49 letters:

all Gr(4,7) Plucker coordinates <a a+1 b c> and

7 cyclic images <1(23)(45)(67)> and <1(27)(34)(56)>

$$R_7^{(2)} = \frac{1}{4}\mathrm{Li}_{2,2}\left(\frac{\langle 1267\rangle\langle 2345\rangle}{\langle 1237\rangle\langle 2456\rangle}, -\frac{\langle 2456\rangle\langle 1(23)(45)(67)\rangle}{\langle 1267\rangle\langle 1456\rangle\langle 2345\rangle}\right) - \frac{1}{2}\mathrm{Li}_{2,2}\left(\frac{\langle 1267\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1567\rangle}, \frac{\langle 1(27)(34)(56)\rangle}{\langle 1267\rangle\langle 1345\rangle}\right) + \cdots$$

$$\langle a(bc)(de)(fg)\rangle \equiv \langle abde\rangle\langle acfg\rangle - \langle abfg\rangle\langle acde\rangle$$

GPSV

So far I told you about
the results of amplitude calculations.
Is there an independent
mathematical description of
symbol letters?

Yes: Cluster Algebras.

We observed that symbol alphabets are
given by subsets of cluster coordinates of
Grassmannian Cluster Algebra

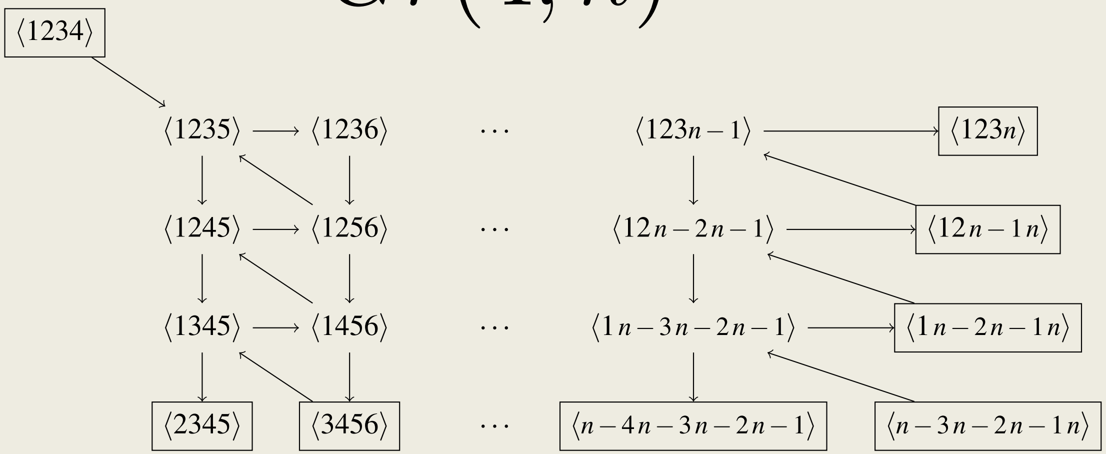
$$Gr(4, n)$$

Golden, Goncharov, Spradlin, Vergu, AV

Grassmannian Cluster Algebra

$$Gr(4,n)$$

Initial Quiver



Mutation Rule

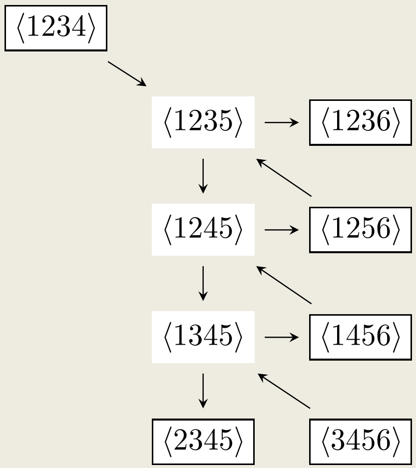
$$a_k \to a_k' = \frac{1}{a_k}\left(\prod_{\text{arrows } i\to k} a_i + \prod_{\text{arrows } k\to j} a_j\right)$$

Cluster Coordinates $\{a_k\}$

Fomin, Zelevinsky ‘02; Scott; Gekhtman, Shapiro, Vainshtein
Cluster Algebra Portal: http://www.math.lsa.umich.edu/~fomin/cluster.html

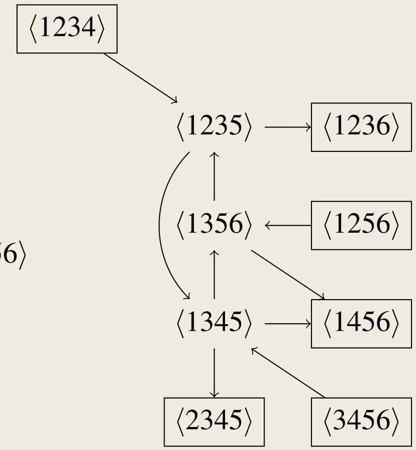
Cluster Coordinates: Gr(4,6) and Gr(4,7)

n=6



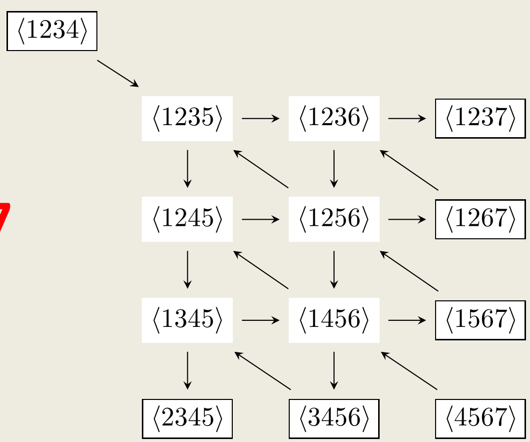
Mutate

⟨1245⟩ → ⟨1356⟩



14 quivers give
15 cluster coordinates
<a a+1 b c>

n=7



833 quivers give
49 cluster coordinates <a a+1 b c>,
7 cyclic images <1(23)(45)(67)>, <1(27)(34)(56)>

Matches symbol alphabets for n=6, 7 amplitudes!

Caron-Huot; Golden, Goncharov, Spradlin, Vergu, AV

New Features at n>7

- Gr(4,n) cluster algebra is infinite for n>7

Fomin, Zelevinsky

- Symbol letters involve square roots

n=8 Symbol Alphabet

He, Li, Zhang '19: amplitude calculation

180 RATIONAL LETTERS

- 68 Plücker coordinates of the form $\langle a\ a{+}1\ b\ c\rangle$,
- 8 cyclic images of $\langle 12\bar{4}\cap\bar{7}\rangle$,
- 40 cyclic images of $\langle 1(23)(45)(78)\rangle$, $\langle 1(23)(56)(78)\rangle$, $\langle 1(28)(34)(56)\rangle$, $\langle 1(28)(34)(67)\rangle$ $\langle 1(28)(45)(67)\rangle$,
- 48 dihedral images of $\langle 1(23)(45)(67)\rangle$, $\langle 1(23)(45)(68)\rangle$, $\langle 1(28)(34)(57)\rangle$,
- 8 cyclic images of $\langle \bar{2}\cap(245)\cap\bar{8}\cap(856)\rangle$,
- and 8 distinct dihedral images of $\langle \bar{2}\cap(245)\cap\bar{6}\cap(681)\rangle$.

$\bar{a}\equiv(a{-}1\ a\ a{+}1)$

$\langle ab(cde)\cap(fgh)\rangle=\langle acde\rangle\langle bfgh\rangle-\langle bcde\rangle\langle afgh\rangle$

$\langle \bar{x}\cap(abc)\cap\bar{y}\cap(def)\rangle\equiv\langle a,(bc)\cap\bar{x},d,(ef)\cap\bar{y}\rangle$

$\langle a,b,c,(de)\cap(fgh)\rangle\equiv\langle abcd\rangle\langle efgh\rangle-\langle abce\rangle\langle dfgh\rangle$

2 x 9 ALGEBRAIC LETTERS (SQUARE ROOTS)

$$\Delta_{1357}=(\langle 1256\rangle\langle 3478\rangle-\langle 1278\rangle\langle 3456\rangle-\langle 1234\rangle\langle 5678\rangle)^2-4\langle 1234\rangle\langle 3456\rangle\langle 5678\rangle\langle 1278\rangle$$

and 1 cyclic

n=9 Symbol Alphabet

He, Li, Zhang '20: amplitude calculation

59 x 9 RATIONAL LETTERS

- 13 cyclic classes of $\langle 12kl\rangle$ for $3 \leq k < l \leq 8$ but $(k,l) \neq (6,7),(7,8)$;
- 7 cyclic classes of $\langle 12(ijk)\cap(lmn)\rangle$ for $3 \leq i < j < k < l < m < n \leq 9$;
- 8 cyclic classes of $\langle\bar{2}\cap(245)\cap\bar{6}\cap(691)\rangle$, $\langle\bar{2}\cap(346)\cap\bar{6}\cap(892)\rangle$, $\langle\bar{2}\cap(346)\cap\bar{2}\cap(782)\rangle$, $\langle\bar{2}\cap(245)\cap\bar{7}\cap(791)\rangle$, $\langle\bar{2}\cap(245)\cap(568)\cap\bar{8}\rangle$, $\langle\bar{2}\cap(245)\cap(569)\cap\bar{9}\rangle$, $\langle\bar{2}\cap(245)\cap(679)\cap\bar{9}\rangle$, $\langle\bar{2}\cap(256)\cap(679)\cap\bar{9}\rangle$;
- 10 cyclic classes of $\langle 1(i\,i{+}1)(j\,j{+}1)(k\,k{+}1)\rangle$ for $2 \leq i,\ i+1 < j,\ j+1 < k \leq 8$;
- 6 cyclic classes $\langle 1(2i)(j\,j{+}1)(k9)\rangle$ for $3 \leq i < j,\ j+1 < k \leq 8$, but $(i,k) \neq (3,8),(4,7)$;
- 14 cyclic classes of $\langle 1(29)(ij)(k\,k{+}1)\rangle$ for $3 < i < j \leq 8$, $3 \leq k \leq i-2$ or $j+1 \leq k \leq 7$;
- 1 cyclic class of $\langle 1,(56)\cap\bar{3},(78)\cap\bar{3},9\rangle$.

11 x 9 ALGEBRAIC LETTERS (SQUARE ROOTS)

$$\Delta_{1357} = (\langle 1256\rangle\langle 3478\rangle - \langle 1278\rangle\langle 3456\rangle - \langle 1234\rangle\langle 5678\rangle)^2 - 4\langle 1234\rangle\langle 3456\rangle\langle 5678\rangle\langle 1278\rangle$$

and 8 cyclic

New Features at n>7

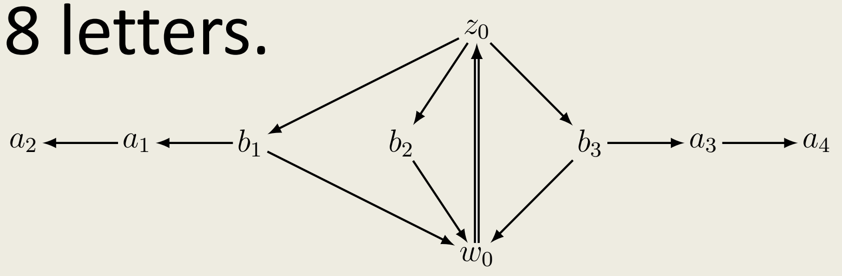
- Gr(4,n) cluster algebra is infinite for n>7
- Symbol letters involve square roots

Is there a mathematical description?

1. Tropical Geometry Drummond, Foster, Gurdogan, Kalousios ’19 Henke, Papathanasiou ’19 ‘21
2. Dual Polytopes Arkani-Hamed, Lam, Spradlin ’19
3. Plabic Graphs Mago, Schreiber, Spradlin, Yelleshpur, AV ’20 ‘21
4. Tensor Diagrams Ren, Spradlin, AV ’21
5. Scattering Diagrams Herderschee ‘21

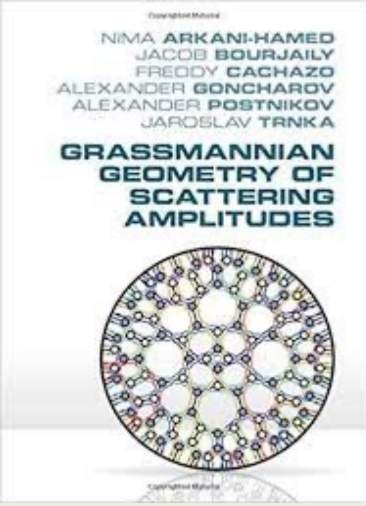
1. Tropical Geometry

- Speyer-Williams'03 associated a fan to the positive Grassmanian by solving tropicalized Plucker relations (multiplication->addition, addition->minimum).
- Building on this idea Drummond, Foster, Gurdogan, Kalousios'19 looked at a "smaller" version of Gr(4,8) fan by looking at particular Plucker coordinates.
- This fan has 272 rays that are g-vectors for cluster coordinates that include 180 rational n=8 letters.
- There are 2 exceptional rays from which they reproduced 18 algebraic n=8 letters.



2. Dual Polytopes

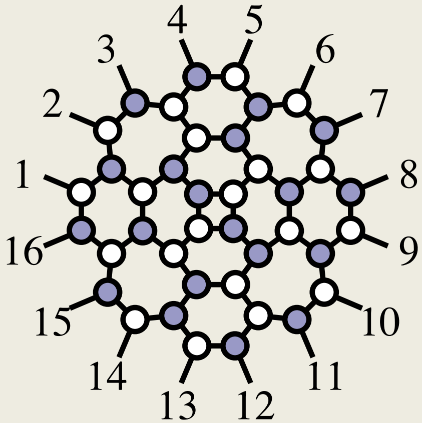
- Arkani-Hamed, Lam and Spradlin’19 looked at polytopes dual to these fans.
- To compute variables associated to the exceptional rays they used the method of Chang, Duan, Fraser, Li’19 and found evidence for the expected type of square roots.



3. Plabic Graphs

The building blocks of N=4 SYM amplitudes are Yangian invariants which are given by integrals

$$\mathcal{Y}_{n,k}(\mathcal{Z}) = \frac{1}{\text{vol}[\text{GL}(k)]}\int \frac{d^{k\times n}C_{\alpha a}}{(1\cdots k)(2\cdots k{+}1)\cdots(n\cdots k{-}1)}\prod_{\alpha=1}^{k}\delta^{4|4}(C_{\alpha a}\mathcal{Z}_a)$$

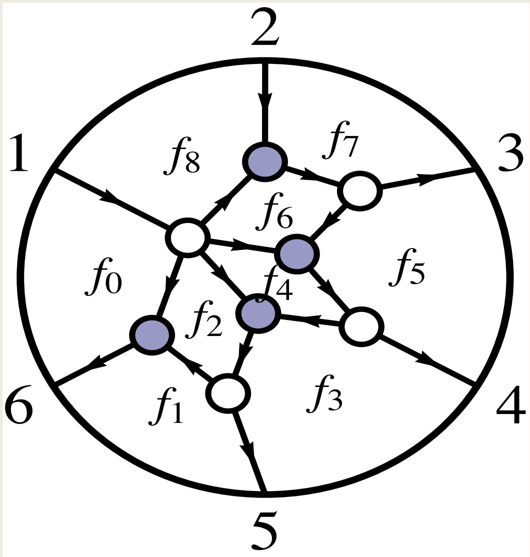


The matrix C parameterizes a cell of the positive Grassmannian; such cells are in correspondence with (equivalence classes) of plabic graphs.

Our Strategy: start with plabic graph, solve C Z=0, compare with known symbol letters.

Mago, Schreiber, Spradlin, AV'20 He, Li'20

Example: n=6, k=2



Solution to C Z=0

$$f_0 = -\frac{\langle 1234\rangle}{\langle 2346\rangle}\,, \quad f_1 = -\frac{\langle 2346\rangle}{\langle 2345\rangle}\,, \quad f_2 = \frac{\langle 2345\rangle\langle 1236\rangle}{\langle 1234\rangle\langle 2356\rangle}\,,$$

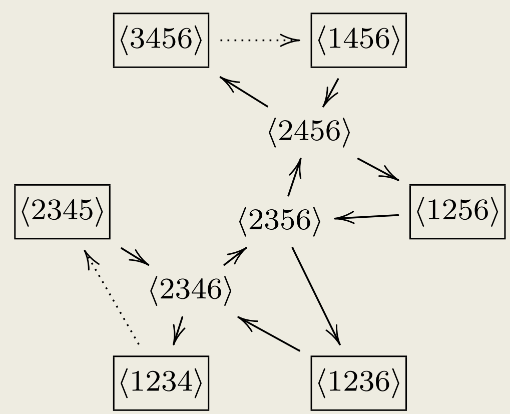
$$f_3 = -\frac{\langle 2356\rangle}{\langle 2346\rangle}\,, \quad f_4 = \frac{\langle 2346\rangle\langle 1256\rangle}{\langle 2456\rangle\langle 1236\rangle}\,, \quad f_5 = -\frac{\langle 2456\rangle}{\langle 2356\rangle}\,,$$

$$f_6 = \frac{\langle 2356\rangle\langle 1456\rangle}{\langle 3456\rangle\langle 1256\rangle}\,, \quad f_7 = -\frac{\langle 3456\rangle}{\langle 2456\rangle}\,, \quad f_8 = -\frac{\langle 2456\rangle}{\langle 1456\rangle}\,,$$

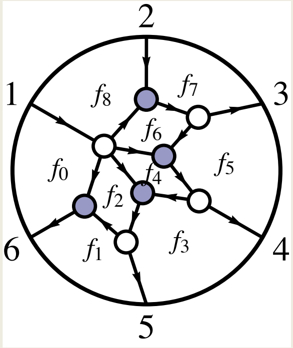
$$C = \begin{pmatrix} 1 & 0 & c_{13} & c_{14} & c_{15} & c_{16} \\ 0 & 1 & c_{23} & c_{24} & c_{25} & c_{26} \end{pmatrix}$$

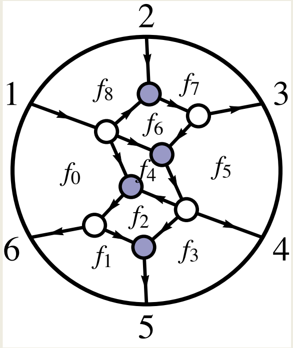
$$c_{13} = -f_0 f_1 f_2 f_3 f_4 f_5 f_6\,, \qquad c_{23} = f_0 f_1 f_2 f_3 f_4 f_5 f_6 f_8\,,$$
$$c_{14} = -f_0 f_1 f_2 f_3 f_4 (1 + f_6)\,, \qquad c_{24} = f_0 f_1 f_2 f_3 f_4 f_6 f_8\,,$$
$$c_{15} = -f_0 f_1 f_2 (1 + f_4 + f_4 f_6)\,, \qquad c_{25} = f_0 f_1 f_2 f_4 f_6 f_8\,,$$
$$c_{16} = -f_0 (1 + f_2 + f_2 f_4 + f_2 f_4 f_6)\,, \qquad c_{26} = f_0 f_2 f_4 f_6 f_8\,.$$

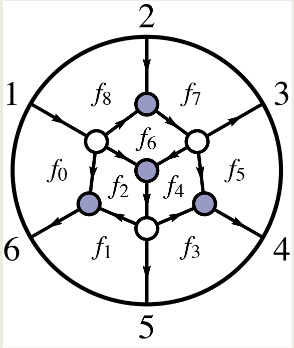
Letters corresponding to this graph can be summarized by quiver:



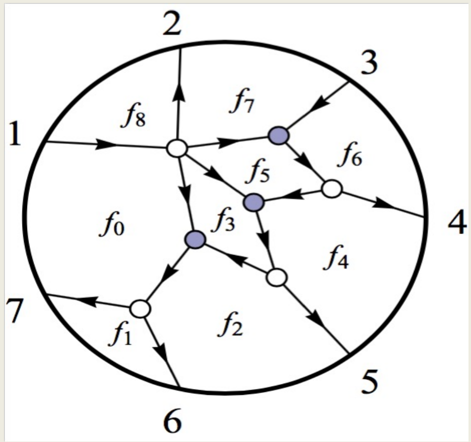
n=6 and n=7

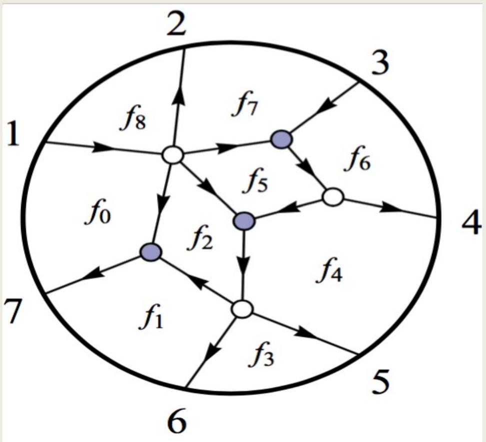






We exactly reproduce n=6 symbol alphabet

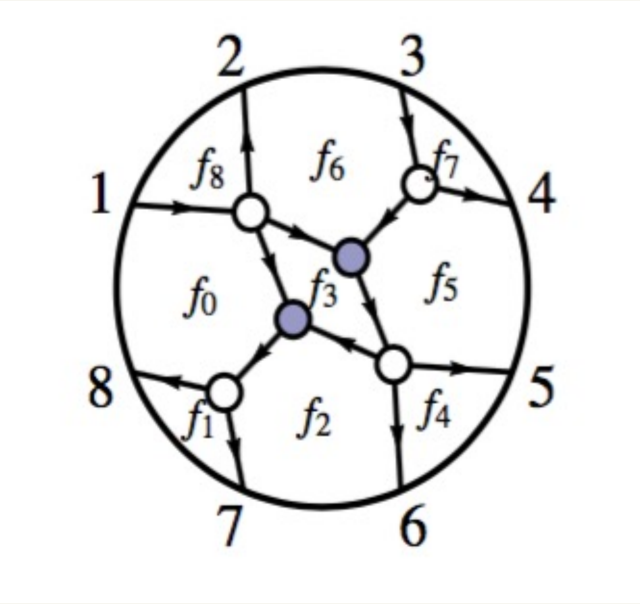




We exactly reproduce n=7 symbol alphabet

Algebraic letters: n=8

This graph gives 8 algebraic letters:



$$f_0 = \sqrt{\frac{\langle 7(12)(34)(56)\rangle \langle 1234\rangle}{a_5 \langle 2(34)(56)(78)\rangle \langle 3478\rangle}}, \qquad f_5 = \sqrt{\frac{a_1 a_6 a_9 \langle 3(12)(56)(78)\rangle \langle 5678\rangle}{a_4 a_7 \langle 6(12)(34)(78)\rangle \langle 3478\rangle}},$$

$$f_1 = -\sqrt{\frac{a_7 \langle 8(12)(34)(56)\rangle}{\langle 7(12)(34)(56)\rangle}}, \qquad f_6 = -\sqrt{\frac{a_3 \langle 1(34)(56)(78)\rangle \langle 3478\rangle}{a_2 \langle 4(12)(56)(78)\rangle \langle 1278\rangle}},$$

$$f_2 = -\sqrt{\frac{a_4 \langle 5(12)(34)(78)\rangle \langle 3478\rangle}{a_8 \langle 8(12)(34)(56)\rangle \langle 3456\rangle}}, \qquad f_7 = -\sqrt{\frac{a_2 \langle 4(12)(56)(78)\rangle}{a_1 \langle 3(12)(56)(78)\rangle}},$$

$$f_3 = \sqrt{\frac{a_8 \langle 1278\rangle \langle 3456\rangle}{a_9 \langle 1234\rangle \langle 5678\rangle}}, \qquad f_8 = -\sqrt{\frac{a_5 \langle 2(34)(56)(78)\rangle}{a_3 \langle 1(34)(56)(78)\rangle}},$$

$$f_4 = -\sqrt{\frac{\langle 6(12)(34)(78)\rangle}{a_6 \langle 5(12)(34)(78)\rangle}},$$

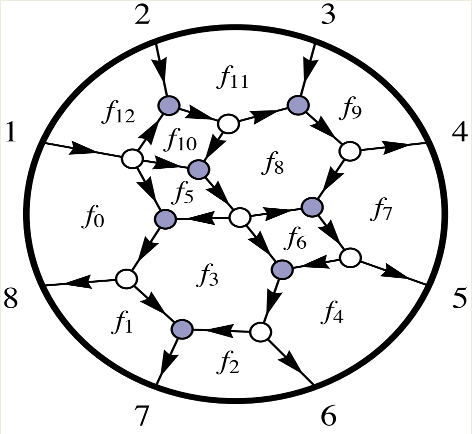
$$\sqrt{\Delta_{1357}}$$

To obtain the 9th: square move on f3.

Cycling by one:
we reproduce all n=8 algebraic letters.

Rational Letters

- It is not possible to obtain all rational symbol letters from just plabic graphs.
- We have to consider non-plabic C-matrices.



Mutation of face f_8 gives non-plabic C'

- In some cases, solutions involve non-cluster coordinates.
- We showed that restricting to the top cell (k=n-4) of the Grassmannian but allowing arbitrary non-plabic C-matrices, we will always produce cluster variables.

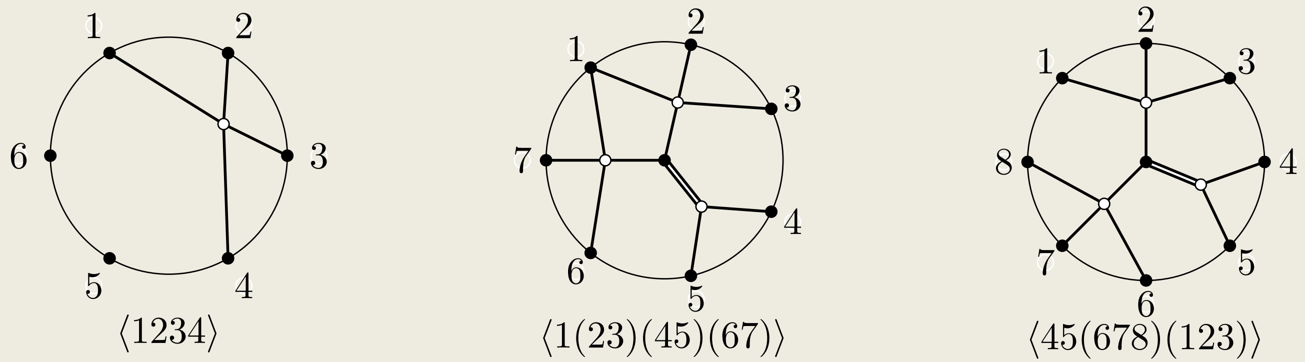
Symbol Alphabet from Plabic Graphs

- We identified set of graphs that reproduced all known n=8 and n=9 symbol alphabets.
- We do not have a theory to explain the pattern of which cells are associated to which symbol letter observed in amplitudes.
- We provided some “phenomenological” data in hope that future work will shed more light on this interesting problem.

Mago, Schreiber, Spradlin, Yelleshpur Srikant AV’20 ‘21

4. Tensor Diagrams

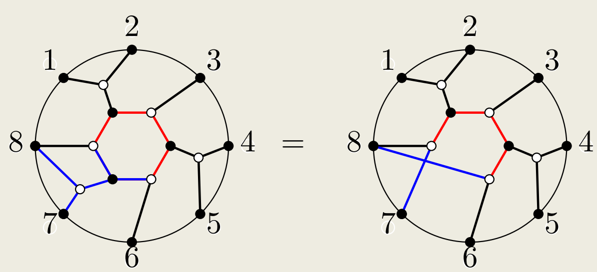
Cluster variables can be represented by tensor diagrams Fomin Pilyavsky'16

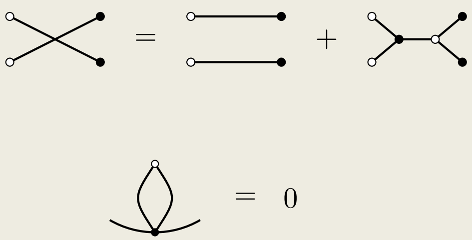


- All boundary vertices are colored black
- Each internal vertex either black or white (valence k for G(k,n))
- Each edge connects black vertex to white vertex
- To each diagram one associates a tensor invariant

Rational Letters from Tensor Diagrams

- An planar tensor diagram (web) is aborizable iff it can be turned into a tree diagram using skein relations.

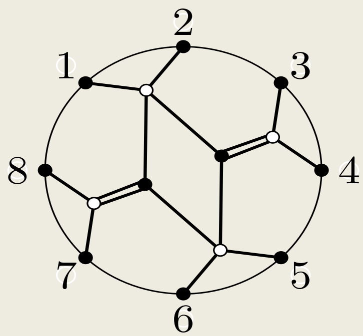




- Fomin-Pilyavsky ‘16 conjecture: for an arborizable web the tensor invariant is a cluster variable. [Proven by Fraser ‘17 for Gr(3,9) and Gr(4,8).]

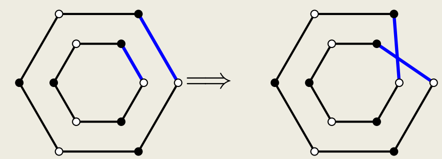
Algebraic Letters from Tensor Diagrams

- We proposed to look at webs that can be reduced to one inner loop, and assign to it a "web series"



$$\mathcal{W} = 1 + \sum_{m=1}^{\infty} t^m W_m$$

the coefficients can be derived graphically by twisting the inner loop



- Then we showed that the series takes the form:

$$\frac{1 - B\,t^2}{1 - A\,t + B\,t^2}$$

$$A = \langle 1\,2\,5\,6\rangle\langle 3\,4\,7\,8\rangle - \langle 1\,2\,7\,8\rangle\langle 3\,4\,5\,6\rangle - \langle 1\,2\,3\,4\rangle\langle 5\,6\,7\,8\rangle$$
$$B = \langle 1\,2\,3\,4\rangle\langle 3\,4\,5\,6\rangle\langle 5\,6\,7\,8\rangle\langle 1\,2\,7\,8\rangle\,.$$

- We observe square roots in the poles: $A \pm \sqrt{A^2 - 4B}$
- We reproduce square roots up to n=9.

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Conclusions

- Symbol Alphabet of N=4 Yang-Mills amplitudes is described by Gr(4,n) cluster algebras for n=6, 7.
- Starting with n=8 one needs a mechanism producing finite subsets in Gr(4,n) and square roots.
- We studied candidate mechanisms coming from plabic graphs and tensor diagrams.
- Future: more systematics, more examples, cluster adjacency, cluster functions, non-N=4 SYM…..