

Flux-Tube S-matrix and Spacetime S-matrix Pedro Vieira, with B.Basso and A.Sever [1303.1396,1306.2058]

Null Polygonal Wilson Loops in Conformal Gauge Theories

Wilson Loops are important observables in Gauge theories.

(btw, smooth curves can be approximated by null polygons with many edges) [Alday, Gaiotto, Maldacena, Sever, PV]

=

* In the Ising model of QFTs, planar N=4 SYM, WL = Scattering Amplitudes

[Alday, Maldacena; Drummond, Korchemsky, Sokatchev; Brandhuber, Heslop, Travaglini; Drummond, Henn, Korchemsky, Sokatchev; Berkovits, Maldacena]

 $= 1 + \frac{1}{2} + \cdots + \frac{1}{2} = 9$

Solving Scattering Amplitudes in planar N=4 SYM is tantamount to summing over all flux tube excitations at any finite 't Hooft coupling. other elementary building blocks, which are unions of two consecutive squares, are unions of two consecutive squares, are α **Example 19 In the** *permetual* in the state is the *i*² deniations in the state of any the state of any independent of $\frac{1}{2}$. The state of any independent of $\frac{1}{2}$. The state of any independent of $\frac{1}{2}$. the global geometry and fully determined by the flux-tube excitations at any Initie the coupling.

and angular momentum *{Ei, pi, mi}* of the flux-tube state *ⁱ* defined on the *i*-th square. The

OPE for Correlation Functions OPE for Wilson Loops

Perturbation theory and the OPE expansion are different. tach expansion provides invaluable data for the Other. Each expansion provides invaluable data for the other.

theory: in this case all the OPE data become coupling dependent and the decomposition dependent and the decompo
The OPE data become coupling dependent and the decomposition of the decomposition of the decomposition of the

A bit slower, building the tessellation:

- Number of edges = n
- Number of squares $= n-3$ *
- Number of middle squares $= n-5$ *
- Number of pentagons $= n-4$

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Propagation in the n-5 middle squares

 $e^{-E_i \tau_i + i p_i \sigma_i + i m_i \phi_i}$

 $\overline{1}$

O

Propagation in the λ pentagon transition P between n-5 middle squares each pair of consecutive squares A **pentagon transition** P between each pair of consecutive squares

 $P(0|\psi_1) P(\psi_1|\psi_2) \ldots P(\psi_{n-6}|\psi_{n-5}) P(\psi_{n-5}|0)$

into the sequence

 ψ_i

 $\int_0^{\frac{n-5}{n}}$

i=1

pentagons. The geometrical data of the loop, or equivalently the set of (4D) cross ratios

 $\lceil n-5 \rceil$

 $\mathcal{W} = \sum_{i} \prod_{i} e^{-E_i \tau_i + ip_i \sigma_i + im_i \phi_i} P(0)$

 $\overline{\psi_i}$ $\overline{\psi_i}$ $\overline{\psi_i}$ $\overline{\psi_i}$

- $\frac{1}{2}$, $\frac{1}{2}$, and angular momentum *{Ei, pi, mi}* of the flux-tube state *ⁱ* defined on the *i*-th square. The To get Scattering Amplitudes in planar N=4 SYM non-perturbatively we need:
- $\mathbb T$ $\mathbb T$ $\mathbb T$ $\mathbb T$ $\mathbb T$ $\mathbb T$ The exact spectrum of the flux tube excitations. $\sqrt{43}$ [Basso 2010]
	- $330,36$ ver, ivez of 3° (basso, sever, ive an published, working the progress [Basso,Sever,PV 2013] + [Basso,Sever,PV unpublished/work in progress] *O* All pentagon transitions between any two flux tube eigenstates at any coupling.

trivial fashion in the first square bracket. They are not geometrical. The geometry enters that The spectrum and the pentagon transitions are dynamical observables of the color flux tube.

Questions for the rest of the talk

- What is a nice coordinatization of null polygons from the OPE point of view? I.e, what exactly is τ_i, σ_i, ϕ_i ?
- The null polygonal WL is UV divergent because of the cusps. What exactly are we computing?
- What are the flux tube eigenstates ψ_i ? How to sum over them?
- How to compute the pentagon transitions at finite coupling?
	- What happens at weak coupling?
	- What happens at strong coupling?

II cusp Henn, Korchemsky, Sokatchev] cancels out as well. This is a finite conformal invariant object. \overline{a} All cusp divergences drop out of this object. The conformal anomaly of *[Drummond,*
Here Kambangly Schotched Gancols out as well. This is a finite conformal invariant object

2⇡

! *^F*

Hence it can only depend on the cross ratios made out of the positions of the cusps of the original polygon. For a *null* polygon with n edges there are 3n-15 such crossratios. (for the heptagon in the figure there are 6 cross-ratios.)

Squares and pentagons have no cross-ratios hence they are fixed by conformal symmetry and are given by the BDS ansatz. (This is the analogue of the statement that 2 and 3 $P₁$ point functions are fixed by conformal symmetry in a CFT.)

 $\mathbf{F}=\mathbf{F}$ furthermore, in the middle transition, it can change its nature. This is because the pentagon it can change its nature. This is because the pentagon in the pentagon is because the pentagon in the pentagon i

associated the *UCC* and *U*(1)-preserved respectively to the *UCC* 100-violation in CONSIDERING and *I*(1)-violation in the *Fallo*. Hence we loose no information in considering this finite ratio.

! *^F*¯ (*F*). In sum, we have

and *F*

F¯ ī

- To measure some charge of the states flowing from A to B we act with the symmetry * generators (corresponding to that charge) on A (or on B). See e.g. the usual OPE for 4pt correlation functions where we act with dilatations on two points to measure what flows from two operators to the other two. (Of course acting on both A and B means doing nothing by definition.)
- Similarly, each middle square in our tessellation has 3 symmetries corresponding to a time translation, a space translation and a rotation of the orthogonal directions. We act with those symmetries on the cusps below each that square. In this way we measure the energy, momentum and angular momentum flowing in each middle square.

Equivalently:

$$
u_i = u_{i+3} \equiv \frac{x_{i-1,i+1}^2 x_{i-2,i+2}^2}{x_{i-1,i+2}^2 x_{i+1,i-2}^2}
$$

FIG. 2. For any middle square in the framework of the framework we associate the framework we associate the fr
In the framework we associate the framework we associate the framework we associate the framework we associate

$$
\frac{1}{u_2} = 1 + e^{2\tau}
$$

\n
$$
\frac{1}{u_3} = 1 + (e^{-\tau} + e^{\sigma + i\phi})(e^{-\tau} + e^{\sigma - i\phi})
$$

\n
$$
\frac{u_1}{u_2 u_3} = e^{2\sigma + 2\tau}
$$

There are n-5 middle squares so the 3n-15 parameters τ_i, σ_i, ϕ_i coordinatize all conformally inequivalent polygons.

FIG. 3. We construct a conformal invariant finite ratio by

 τ

 σ

 ϕ

When \ast So far we considered mostly kinema S $\frac{1}{2}$ Our next example is the MHV heptagon for which we have So far we considered mostly kinematics and thus very general (it applies to any 4D conformal theory). We found that

the WL (34) becomes

recall that P does not depend on geometry since all pentagons are conformal equivalent. P is a flux tube observable.

F¯

! *^F*

F¯

***** Now we move to the most interesting part, the dynamics. What are the flux tube eigenstates 2. Prow to sum over them? How to compute the pentagon transitions² at finite coupling? y_i involving a single gluonic excitation *F* or *F*¯. Such gluonic excitation can be produced at Furthermore, in the middle transition of the middle transition in the pental control of the pental with the pentagon of the pental with the pe to compute the pentagon transitions ²/₂ finite coupling? have two possible gluonic transitions

We will now specialize to planar N=4 SYM.

associated respectively to the *U*(1)-preserving and *U*(1)-violating processes: *F*

- There are several equivalent descriptions of the excited flux tube: 米
	- As null Wilson lines with insertions
	- As large spin operators *F* = *F^z* on top of the flux tube by means of a linear combination of local operators

 $O = \text{tr} (Z D D D D ... D D D D F D D D D ... D D D D F D D D D ... D D D D Z)$

As an excited GKP [Gubser,Klebanov,Polyakov] folded string.

These states have a fixed number of excitations with given momenta and we know their spectrum exactly from Integrability [Basso 2010] (we also know how these excitations scatter amongst themselves [Ba<mark>sso,Rej;Fioravanti,Piscaglia,Rossi;Basso,Sever,PV]</mark>). Hence to this mapping to this mapping to the integrable spin chain, the complete spectrum of \mathcal{L} riese states have a fixed humber of excitations with given monenta and we know ien spectrum exactly from integrability [Basso 2010] (we also know how these excitations scatter R in the series entergy consider R . The energy consideration is an any consideration of the energy consideration of R

$$
\sum_{\psi} = \sum_{\mathbf{a}} \int du_1 \dots du_N \mu(u_1) \dots \mu(u_N)
$$

$$
E = \epsilon(u_1) + \cdots + \epsilon(u_N), \qquad p = p(u_1) + \cdots + p(u_N)
$$

The vector a indicates which kind of excitations we are considering. For example, the state above has two gluonic excitations so that $a = {F, F}$.

For example, (20) is dual to a folded string in *AdS*⁵ with two bumps that are dual to

- Agaington and the challenge is now to compute the pentagon transitions between a state w The chancityc is now to compute the peritagon transitions between a sta.
For particles and another state with M excitations the both above and allowing beat at the the top. It can also be produced before. The challenge is now to compute the pentagon transitions between a state with N particles and another state with M excitations.
- **If** $*$ The simplest transitions where a single excitation propagates from one square to the next. Multi-particles can be more or less built out of those, see below. This is a manifestation of Integrability for the pentagon transitions.
- Single particle transitions are associated to the lightest states so they are also the most important ones. E.g., they determine the dominant behavior of the WL in the
near collinear limit of large tau's and *F* near collinear limit of large tau's.

+ 2 cos(1)*f*(⌧1*,* 1) vacuum ! *^F* (*F*¯) ! vacuum ! vacuum

We consider for illustration the case where the bottom and the top excitations are gluons. We want to compute the single particle pentagon transition at any coupling *Pute the single particle perhage*

^PF¯*F*¯(*u|v*) and *^P*¯(*u|v*) ⌘ *^P^F ^F*¯(*u|v*) = *^PF F*¯ (*u|v*). We now

 \parallel

is sent to the neighboring edge on its right. This is consistent

 \mathbb{R}^n

 $\|$

 $| \cdot |$

becomes a *P*¯ transition. This justifies some occurrences of *P*¯

versus *P* in the main text, see e.g. (8), and will be motivated

^P¯(*u*²*|v*)*/P*(*u*3*|v*) = *^S*(*v, u*). This equation has a neat

interpretation: we can bring a particle from the bottom

 \vert top of the \vert

^u ! *^u*² or through the right through *^u* ! *^u*3. Both

give us two *F*'s on the top but depending on which option

 \parallel

the top excitation *v*. To compare both options we have

to permute the two excitations thereby acquiring an *S*-

 \mathbf{E}

 \mathbb{R}^n

 \mathbf{r}

tion of it. At the same time, these kind of manipulations

in [9].

puted exactly using integrability integrability μ is the grabitation of the grabitation μ is the grabitation of the grabitation μ

Fig. 5. Fig. 5. Fig. of the sign of both momenta is equivalent to a set of both momenta is equivalent to They take the form of functional equations. We found one solution to We dub the second axiom as the *fundamental relation*: We postulate three Bootstrap like axioms that this object should satisfy. these equations which, we conjecture, gives the pentagon transition at any finite coupling.

I.
$$
P(u|v) = P(-v|-u)
$$

\nII. $P(u|v) = S(u, v)P(v|u)$
\nIII. $P(u^{-\gamma}|v) = P(v|u)$

* Axiom 1, $P(-u|-v) = P(v|u)$, is the obvious reflection symmetry of the pentagon.

- * Axiom 1, $P(-u|-v) = P(v|u)$, is the obvious reflection symmetry of the pentagon.
- Axiom 3 comes from the mirror symmetry of the flux tube. There exists a non-米 perturbative path in the rapidity u which implements the Wick rotation:

- Axiom 1, $P(-u|-v) = P(v|u)$, is the obvious reflection symmetry of the pentagon. $* A$
	- Axiom 3 comes from the mirror symmetry of the flux tube. There exists a nonperturbative path in the rapidity u which implements the Wick rotation: **perturbative path in the rapidity u which the value of** \mathbf{r}

$$
E(u^{\gamma}) = ip(u)
$$

\n
$$
p(u^{\gamma}) = iE(u)
$$

\nHence we expect
\n
$$
\underbrace{\begin{pmatrix} \overline{v} \\ \overline{v} \\ \overline{v} \end{pmatrix}}_{u_{\gamma}} = \underbrace{\begin{pmatrix} \overline{v} \\ \overline{v} \\ \overline{v} \\ \overline{v} \end{pmatrix}}_{u_{\gamma}}
$$
 or
$$
P(u^{-\gamma}|v) = P(v|u)
$$

Axiom 2, $P(u|v) = S(u,v)P(v|u)$, together with the other two, implies Watson's $P(0|u,v) = S(v,u)P(0|v,u)$ where $P(0|u,v)$ is given by \mathbf{v} \mathbf{v} \mathbf{v} factor is a consequence of the see it, we can access that we can access that $P(0)$ **both we are performed on the bottom excited on the bottom excited with the set of the inverse invers**

0i = *P*(0*|u, v*)*ij ,* (69)

h*i*(*u*)*^j* (*v*)*|P |*

b

This is a nice self-consistency check and is the main motivation for axiom 2. This is a pies self consistency sheak and is the main mativation for exism? this is a nice self-consistency check and is the main motivation for axiom 2. \sim

Axiom I is especially important since it relates the pentagon transition *P*(*u, v*) to the S-

* We can solve the bootstrap equations. For example, a solution for the scalar excitations is

$$
P(u|v)^2 = \frac{S(u,v)}{g^2(u-v)(u-v+i)S(u^{\gamma},v)}
$$

when taking the square-root of (79). This one is easily fixed by comparison with data, at α

29

is exactly equal to 1*/g*² (within the normalization assumed in this paper).

It provides a precise connection between the space-time and the flux tube Smatrices. They hold at any coupling.

Integrable of the choice of the branch o

The flux tube S-matrices can be computed at any coupling using Integrability
The state fact fiscal propertial processing and the two towers and the two to linear integral equal I he flux tube S-matrices can be computed at any coupling using integrability
[Basso,Rej;Fioravanti,Piscaglia,Rossi;Basso,Sever,PV] The main ingredients are solutions to linear integral equations akin to the so-called BES equation for the flux-tube vacuum. the very same kind of axioms as those for the twist-one particles considered in this paper. This paper is paper

transitions for fermions a bit harder and our conjecture for them is based to a large extent on

the experience acquired here with the gluonic and scalar transitions. We have the gluonic and scalar transition
The gluonic and scalar transitions in the gluonic and scalar transitions. We have that a more that a more that

cordinate to our previous and the measure of the measure of the measure of the measure $\frac{1}{2}$ is reading the measure $\frac{1}{2}$ Multi-particles are built in terms of the single particle transitions: coupling and they will be studied in greater details in greater details in a forthcoming publication \mathcal{S} .

$$
P({u_i}|{v_j}) = \frac{\prod_{i,j} P(u_i|v_j)}{\prod_{i>j} P(u_i|u_j) \prod_{i
$$

number or particles (at most eignt). For example, for two scalars We have an algorithm for getting the SU(4) matrix part but so far we only checked it up to a small we have an algorithm for getting the SO(4) matrix part but so far
2 number of particles (at most eight). For example, for two scalars

$$
P(\mathbf{u}|\mathbf{v})_{i_1i_2}^{j_1j_2} = P_{\text{dyn}}(\mathbf{u}|\mathbf{v}) \left[\pi_1(\mathbf{u}|\mathbf{v}) \delta_{i_1}^{j_1} \delta_{i_2}^{j_2} + \pi_2(\mathbf{u}|\mathbf{v}) \delta_{i_1}^{j_2} \delta_{i_2}^{j_1} + \pi_3(\mathbf{u}|\mathbf{v}) \delta_{i_1i_2} \delta_{i_1i_2}^{j_1j_2} \right]
$$

where the matrix functions π_j are simple, rational functions of the rapidities, independent of the coupling. where the matrix functions π_j are simple, rational functions of the rapidities, independent of the coupling.

6 and indicate which pair of scalars we insert at the bottom while the outgoing indices

, (145)

with waak and strong coupling results with weak and strong coupling results. with weak and strong coupling results.

and *F*

At strong coupling we can derive the Y-system in [Alday,Gaiotto,Maldacena; Alday,Maldacena,Sever,Vieira] This involves re-summing infinitely many can absorpt the constitution of the continuity of the excitence of the excitence of the contract of the contra have two possible gluonic transitions multi-particles and boundstates.

 \mathbf{x}_{i+1} vacuum \mathbf{x}_{i+1}

+ 2 cos(1)*f*(⌧1*,* 1) vacuum ! *^F* (*F*¯) ! vacuum ! vacuum

 $\overline{1}$

F¯ ! *^F*

F¯ Ļ.

This is quite exciting since the strong coupling and the strong of the strong coupling associated respectively to the *U*(1)-preserving and *U*(1)-violating processes: *F* result was begging for such an interpretation.

- At weak coupling we checked the single particle transitions against all available data in * the literature for Wilson loops up to 3 loops.
	- MHV Hexagon at 1 Loop [Bern,Dixon,Smirnov], 2 Loops [Del Duca,Duhr,Smirnov; * Goncharov,Spradlin,Vergu,Volovich], 3 Loops [Dixon,Drummond,Henn], 4 Loops [Dixon,Duhr,Pennington, to appear]
	- MHV Heptagon at 1 Loop [Bern,Dixon,Smirnov], 2 Loops [Caron-Huot] *
	- NMHV Hexagon at 1 Loop [Bern,Dixon,Dunbar,Kosower; * Drummond,Henn,Korchemsky,Sokatchev], 2 Loops [Dixon,Drummond,Henn]
	- NMHV Heptagon at 1 Loop [Bourjaily,Caron-Huot,Trnka], 2 Loops [Caron-Huot,He] * (MHV amplitudes = Bosonic WL, non-MHV amplitudes = Superloop [Skinner,Mason;Caron-Huot]; OPE still applies [Sever, PV,Wang])
	- We also generated infinitely many higher loop predictions *
- Not everything is done:
	- We have conjectures for transitions * with **fermions**. But they are not as well motivated since mirror (and crossing) transformation for fermions is not well understood/does not seem to exist.

 u^{γ} *u* = $p(u^{\gamma}) = iE(u)$ $E(u^{\gamma}) = ip(u)$

- We seem to be able to compute **matrix part** case by case but a general expression (which would render the re-summation easy) is still lacking.
- Once we have all the transitions, or at least many transitions, would be nice to think * what is the best way to **plot** the amplitude.

re-sum into some beautiful object from the Integrability point of view? is sain this some peaamal expect home the this graphly penne of *Home* re-sum into some beautiful object from the Integrability point of view?

At strang coupling it desel: The Veng Veng functional and its sesseigted The middle transition of the mature. The many range in the middle it can consider the pentagon of the pentagon TBA equations.* What about finite coupling? We don't know yet but in any case, we should be able to At strong coupling it does!: The Yang-Yang functional and its associated case, we should be able to plot the finite coupling amplitude nevertheless.

Could be interesting to see how WL data from other conformal gauge associated respectively to the *U*(1)-preserving and *U*(1)-violating processes: *F* theories looks like when OPE decomposed.

! *^F* * for the experts: the strong coupling result for the Wilson loop contains several contributions but
once we compute the finite ratio W everything cancels out except for the nicest of all (from the Integrability point of view) which is the Yang-Yang functional! once we compute the finite ratio W everything cancels out except for the nicest of all (from the

+ 2 cos(1)*f*(⌧1*,* 1) vacuum ! *^F* (*F*¯) ! vacuum ! vacuum

감사합니다 Thank you

Multi-particles the very same kind of axioms as those for the twist-one particles considered in this paper. **Examine Family comparts family comprise the two towers** \mathbb{R}^n **denotes**

where

while

P(*u*1*, u*² 3*i|v*¹ 2*i, v*2)

*i*1*i*²

$$
P({u_i}|{v_j}) = \frac{\prod_{i,j} P(u_i|v_j)}{\prod_{i>j} P(u_i|u_j) \prod_{i
$$

fundamental twist-one constituents. It follows then that the bound-state transitions satisfy

The matrix part encodes the SU(4) symmetry of the excitations. For gluons the matrix part is 1. For scalars and fermions it is non-trivial. For example, for two scalars we have in [3] with gauge fields. More generally we believe that the that their solution can always be written as the
International we believe that the solution can always be written as the solution can always be written as the **a** product of two contributions: the *dynamical participal particip* respective as social and tensor under the R-symmetry and tensor under the R-symmetry of Second Processes and tensor under the R-symmetry of Second Processes and tensor under the R-symmetry of Second Processes and the R-sym 20 m 2 transition and the fraction and that 20 m

$$
P(\mathbf{u}|\mathbf{v})_{i_1i_2}^{j_1j_2} = P_{\text{dyn}}(\mathbf{u}|\mathbf{v}) \left[\pi_1(\mathbf{u}|\mathbf{v}) \delta_{i_1}^{j_1} \delta_{i_2}^{j_2} + \pi_2(\mathbf{u}|\mathbf{v}) \delta_{i_1}^{j_2} \delta_{i_2}^{j_1} + \pi_3(\mathbf{u}|\mathbf{v}) \delta_{i_1i_2} \delta_{i_1i_2}^{j_1j_2} \right]
$$
\nwhere

 $\mathcal{A}(\mathcal{A})$ and indicate which pair of scalars we insert at the outgoing indices we indices $\mathcal{A}(\mathcal{A})$

, (145)

*^j*1*i*¹ . Further constraints come from the Wat-

$$
\pi_1(\mathbf{u}|\mathbf{v}) + \pi_2(\mathbf{u}|\mathbf{v}) = 1, \qquad \pi_2(\mathbf{u}|\mathbf{v}) = \frac{(u_1 - v_1)(u_2 - v_2 + i)}{(u_1 - u_2 - i)(v_1 - v_2 + i)}
$$

$$
\pi_2(\mathbf{u}|\mathbf{v}) + \pi_3(\mathbf{u}|\mathbf{v}) = \frac{(u_1 - v_1)(u_2 - v_2 + i)(u_1 - v_1 - i)(u_2 - v_2 + 2i)}{(u_1 - u_2 - i)(u_1 - u_2 - 2i)(v_1 - v_2 + i)(v_1 - v_2 + 2i)}
$$

= *P*(*v*1*, u*1*|v*2*, u*2)

same object up to a relation of R-symmetry indices and rapidities. More precisely, since \mathcal{L} and also for a similar pattern for a similar pattern for a similar pattern for a larger number of scalars and its for scalars we have an algorithm for getting the matrix part but so far we only enceived it
up to a small number of particles, at most 8) *j*1*j*² $\frac{1}{2}$ (we believe we have an algorithm for getting the matrix part but so far we only checked it part (into a product of single particle transitions) is *universal* and should be a consequence of (We believe we have an algorithm for getting the matrix part but so far we only checked it up to a small number of particles, at most 8)

I * For the experts, in terms of the momentum twistors appearing in the previous talks we have **the momentum twistors as a set of the momentum terms of the momentum terms** of the momentum twistors as $\frac{1}{2}$

General n-gons

$$
e^{2\tau_{2j}} \equiv \frac{\langle -j, j+1, j+2, j+3 \rangle \langle -j-1, -j, -j+1, j+2 \rangle}{\langle -j-1, -j, j+2, j+3 \rangle \langle -j, -j+1, j+1, j+2 \rangle},
$$

\n
$$
e^{\sigma_{2j}+\tau_{2j}-i\phi_{2j}} \equiv \frac{\langle -j-1, -j, j+1, j+2 \rangle \langle j, j+1, j+2, j+3 \rangle}{\langle -j-1, j+1, j+2, j+3 \rangle \langle -j, j, j+1, j+2 \rangle},
$$

\n
$$
e^{\sigma_{2j}+\tau_{2j}+i\phi_{2j}} \equiv \frac{\langle -j-2, -j-1, -j, -j+1 \rangle \langle -j-1, -j, j+1, j+2 \rangle}{\langle -j-2, -j-1, -j, j+2 \rangle \langle -j-1, -j, -j+1, j+1 \rangle},
$$

\n
$$
e^{2\tau_{2j+1}} \equiv \frac{\langle -j-1, j+1, j+2, j+3 \rangle \langle -j-2, -j-1, -j, j+2 \rangle}{\langle -j-2, -j-1, j+2, j+3 \rangle \langle -j-1, -j, j, j+2 \rangle},
$$

\n
$$
e^{\sigma_{2j+1}+\tau_{2j+1}-i\phi_{2j+1}} \equiv \frac{\langle -j-2, -j-1, -j, -j+1 \rangle \langle -j-1, -j, j+2, j+3 \rangle}{\langle -j-2, -j-1, -j, j+3 \rangle \langle -j-1, -j, -j+1, j+2 \rangle},
$$

\n
$$
e^{\sigma_{2j+1}+\tau_{2j+1}+i\phi_{2j+1}} \equiv \frac{\langle j+1, j+2, j+3, j+4 \rangle \langle -j-1, -j, j+2, j+3 \rangle}{\langle -j-1, j+2, j+3, j+4 \rangle \langle -j, j+1, j+2, j+3 \rangle},
$$

where in the convention of (2) the convention of (2) is the very bottom one, the edge 2 is the edge 2 is the next one, the next one, the next one, the edge 2 is the next one, the next one, the edge 2 is the next one, th

its right, Here, h*i, j, k, l*i is a shorthand for h*Zi, Z^j , Zk, Zl*i and our definitions here are

equivalent to the ones in figure 2 of \mathbb{R}^2 . Note that we have divisions for even and we have divisions for

We have a matrix: *<i>u*
 ِ^{*s*}</sup> cosh (⇡*u*)

For the measure we have well as the measure we have the measure we have the measure we have the measure we have
The measurement of the measurement

⌘ = 1 for gluons and describe *fⁱ* for both cases at once. We have

given by

cosh (⇡*u*)

dt

1

For the measure we have

matrix:
$$
K_{ij} = 2j(-1)^{j(i+1)} \int_{0}^{\infty} \frac{dt}{t} \frac{J_i(2gt)J_j(2gt)}{e^t - 1}
$$

e ϵ ectively truncate the range of the indices as explained below. The matrix elements as explained below. The matrix elements are ϵ

e^t 1

t

where *Jⁱ* is the *i*-th Bessel function. This same matrix *K* is useful for computing the *f*-

vectors *,* ˜, on the other hand, depend on which case we are interested in, but, actually, in

a very minimal way. In fact, we can introduce a parameter \mathbb{R}^n such that \mathbb{R}^n

functions for the scalar as well as for the gluon excitations. It corresponds to the kernel

¹ = 1 *^K* ⁺ *^K*² *^K*³ ⁺ *...* (182)

*f*1(*u, v*)=2˜(*u*) *·* Q *·* M *·* (*v*)*, f*2(*u, v*)=2˜(*v*) *·* Q *·* M *·* (*v*) (183)

over *i, j* = 1*,* 2*,...,*⇤ 1 and get accurate results to order *g*⇤. The inverse M is also trivial

to compute exactly in perturbation theory since we can truncate (182). For example, to get

(178)

¹⁵ 2 0

A (185)

A (185)

 $\int_{0}^{\infty} dt J_j(2gt) (J_0(2gt) - \cos(ut) \left[e^{t/2} \right]^{(-1)^{n \times j}})$ **for the sections similar to each other.** One of them is $\kappa_j(u) = \frac{1}{\pi}$ $\frac{1}{t}$ $e^t - 1$ of the Beisert-Eden-Staudacher equation \mathbb{R}^n , when written in the manner of $[69]$ $\kappa_j(u)$ = $\int_{}^{\infty}$ 0 *dt t* $J_j(2gt)(J_0(2gt) - \cos(ut) [e^{t/2}]^{(-1)^{\eta \times j}})$ e^t-1 And 2 vectors similar to each other. One of them is **Keta we And 2 vectors similar to each other. One of them is** ¹ = 1 *^K* ⁺ *^K*² *^K*³ ⁺ *...* (182) $\hat{f}(u) = -\int_0^\infty$ $\frac{dt}{dt} J_j(2gt) (J_0(2gt) - \cos(ut) [e^{t/2}]^{(-1)^{n \times j}}$ • And 2 vectors similar to each other. One of them is $\kappa_j(u) = -\int_0^u \frac{dt}{t} \frac{J_j(2gt)(J_0(2gt) - \cos(ut)) e^{t/2}}{e^t - 1} e^{(t/2 - t/2)}$ together with a trivial diagonal matrix Q with entries Q*ij* = *ij* (1)*ⁱ*+1*i*. The functions *fⁱ* dt *,* (9_{at}) where we used that *f*1(*u, u*) *f*2(*u, u*) = 0, see (172).

vectors *,* ˜, on the other hand, depend on which case we are interested in, but, actually, in

@

- ***** Finally we have a matrix of integers $\mathbb{Q}_{ij} = \delta_{ij}(-1)^{i+1}i$ and $\mathbb{M} \equiv (1+K)^{-1} = 1 K + K^2 K^3 + ...$ $-\kappa$ ² ***** Finally we have a matrix of integers $\mathbb{Q}_{ij} = \delta_{ij}(-1)^{i+1}i$ and $\mathbb{M} \equiv (1 + K)^{-1} = 1 - K + K^2 - K^3 + ...$ are then given by are then given by **where is a finally we have a matrix of integers**
- ***** Then we construct four similar functions $f_{1,2,3,4}$. For example $f_4(u,v) = 2 \kappa(v) \cdot \mathbb{Q} \cdot \mathbb{M} \cdot \kappa(v)$ together with a trivial diagonal matrix Q with entries Q*ij* = *ij* (1)*ⁱ*+1*i*. The functions *fⁱ* Formulae for Gluon
- Hold thind high calont in the point going on transitions, reach The gluon S-matrix which is the non-trivial ingredient in the pentagon transitions, reads Similarly, we can write non-trivial ingredient in the penta gon trans

$$
S(u,v) = \frac{\Gamma(\frac{3}{2} - iu)\Gamma(\frac{3}{2} + iv)\Gamma(iu - iv)}{\Gamma(\frac{3}{2} + iu)\Gamma(\frac{3}{2} - iv)\Gamma(iv - iu)}F(u,v)
$$

⌘ = 1 for gluons and describe *fⁱ* for both cases at once. We have

^j (*u*) ⌘ ^Z

to compute exactly in perturbation theory since we can truncate (182). For example, to get where **where** where where $\frac{1}{2}$

0

where
\n
$$
\log F = 2i \int_{0}^{\infty} \frac{dt}{t} (J_0(2gt) - 1) \frac{e^{-t/2} (\sin (ut) - \sin (vt))}{e^t - 1} - 2i f_1 + 2i f_2
$$

xternal text The mirror S-matrix uses the other two functions $f_{1,2,3,4}$ **de** 1 The mirror S-matrix uses the other two functions $f_{1,2,3,4}$

alous dimension is nothing but (4g² times) $[\mathbb{Q} \cdot \mathbb{M}]_{1,1}$ *results uses the other two functions* $t_{1,2,3,4}$
A The famous cusp apomalous dimension is nothing but (4 a^2 times) $[0]$. MI Q *·* M = $\left[\mathbb{Q} \cdot \mathbb{M} \right]_{1,1}$ 4*g*³⇣(3) ²*g*4⇡⁴ The famous cusp anomalous dimension is nothing but (4g² times) $\left[\mathbb{Q}\cdot\mathbb{M}\right]_{1,1}$

$$
F_0^{(0)} = 1
$$

\n
$$
F_1^{(1)} = -2H_0 - 4H_1
$$

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$$
F_0^{(1)} = -2H_0 - 4H_1
$$

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$$
F_0^{(2)} = 8H_1B_0 + B_0^2 + 8H_1^2 + \frac{\pi^2}{6}
$$

\n
$$
F_2^{(2)} = 8H_1B_0 + B_0^2 + 8H_1^2 + \frac{\pi^2}{6}
$$

\n
$$
F_1^{(2)} = 4B_0H_{0,1} + 12H_1^2B_0 + 2H_1B_0^2 - \frac{4}{3}\pi^2B_0 + 8H_1^3 + 2\pi^2H_1 - 4\zeta(3)
$$

\n
$$
F_0^{(2)} = 4H_1B_0H_{0,1} + \frac{1}{2}B_0^2H_{0,1} - 4H_0H_{0,0,1} + 2\zeta(3)B_0 + 4H_1^2B_0 + H_1^2B_0^2 + \frac{4}{3}\pi^2H_1B_0
$$

\n
$$
+ \frac{1}{12}\pi^2B_0^2 + \frac{1}{6}\pi^2B_0 + 14H_{0,0,0,1} + 2\zeta(3)B_0 + 4H_1^2B_0 + H_1^2B_0^2 + \frac{4}{3}\pi^2H_1B_0
$$

\n
$$
+ \frac{1}{12}\pi^2B_0^2 + \frac{1}{6}\pi^2B_0H_1 + 4\pi^2H_{0,0,1} - 4H_1\zeta(3) - 2H_1^4 + \pi^2H_1^2 + \frac{\pi^4}{60}
$$

\n
$$
-8B_0H_{0,1}H_1^2 + 8B_0H_1^2 + 2\pi^2H_1 - \frac{2}{9}H_0^2H_1^2 + \frac{38}{9}\pi^3H_0H_1^3 + 44B_0H_0, H_1^3
$$

\n
$$
= 4B_0H_{0,1}H_1^2 + 8B_0H_{0,0,1}H_1^2 + 8B_0H_{0,1}H
$$

We have (*^H* ⁼ *^H*(*x*) and *^H*¯ ⁼ *^H*(+*x*))

We can solve the bootstrap equations. A solution for the scalar excitations is Combining everything together our proposal for the transition of a single scalar over the

$$
P(u|v)^{2} = \frac{S(u,v)}{g^{2}(u-v)(u-v+i)S(u^{\gamma},v)}
$$

is exactly equal to 1*/g*² (within the normalization assumed in this paper).

for the scalar, see (27).

while for gluonic excitations we have (f is a simple function of the so called Zhukowsky variables) W_{α} while for grading explicitions we have it is a simple function of the soletime zhanchon, while for gluonic excitations we have (f is a simple function

$$
P(u,v)^2 = \frac{f(u,v)}{g^2(u-v)(u-v-i)} \frac{S(u,v)}{S(u^{\gamma},v)}
$$

These formulae establish a precise connection between the space-time and the flux tube S-matrices. They hold at any coupling. T to complete our construction we also need to get the measure. According to get the measure. Ac-These formulae establish a precise connection between the space-time and polytophere of *P*(*n*₂) and the *y* riold at any coupling. tube S-matrices. They hold at any coupling.

The flux tube S-matrices can be computed at any coupling using Integrability [Basso, Rej; Fioravanti, Piscaglia, Rossi; Basso, Sever, PV] The main ingredients are the solutions to the so called BES [Beisert, Eden, Staudacher] equation describing the flux tube vacuum. *i*

15This follows in the property of the standing of the single particle transitions: Multi-particles are built in terms of the single particle transitions:

$$
P(\{u_i\}|\{v_j\}) = \frac{\prod_{i,j} P(u_i|v_j)}{\prod_{i>j} P(u_i|u_j) \prod_{i
$$

4

up to a small number of particles, at most 8) *u>*
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 iva in the matrix part but

up to a small number of particles, at most 8) the part but so far we only checked it (We believe we have an algorithm for getting the matrix part but so far we only checked it

4