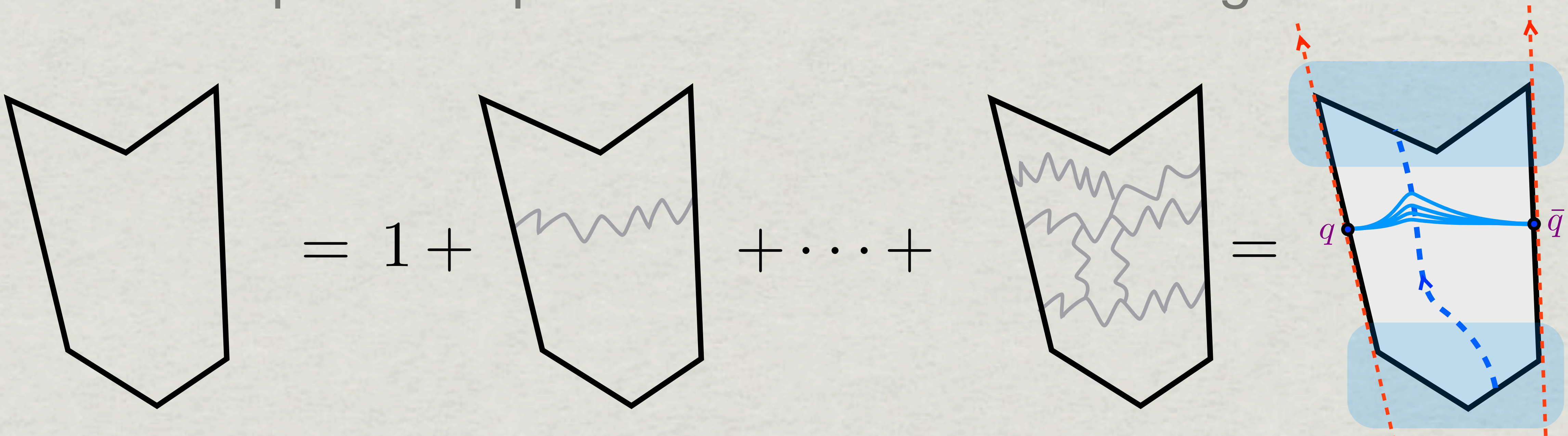
Flux-Tube S-matrix and Spacetime S-matrix

Pedro Vieira, with B.Basso and A.Sever

[1303.1396,1306.2058]

Null Polygonal Wilson Loops in Conformal Gauge Theories

- Wilson Loops are important observables in Gauge theories.

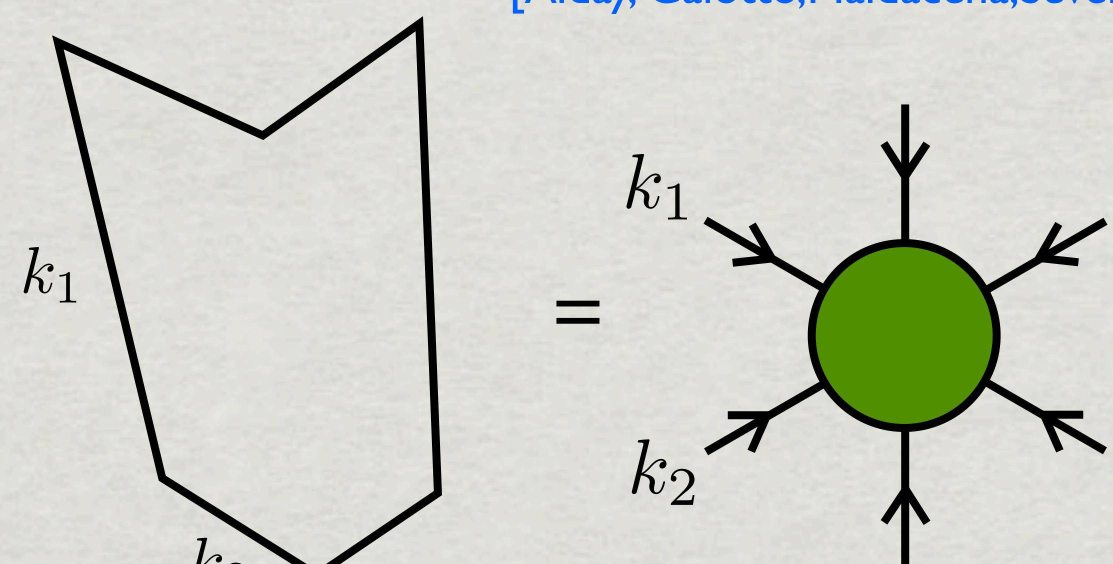


(btw, smooth curves can be approximated by null polygons with many edges)

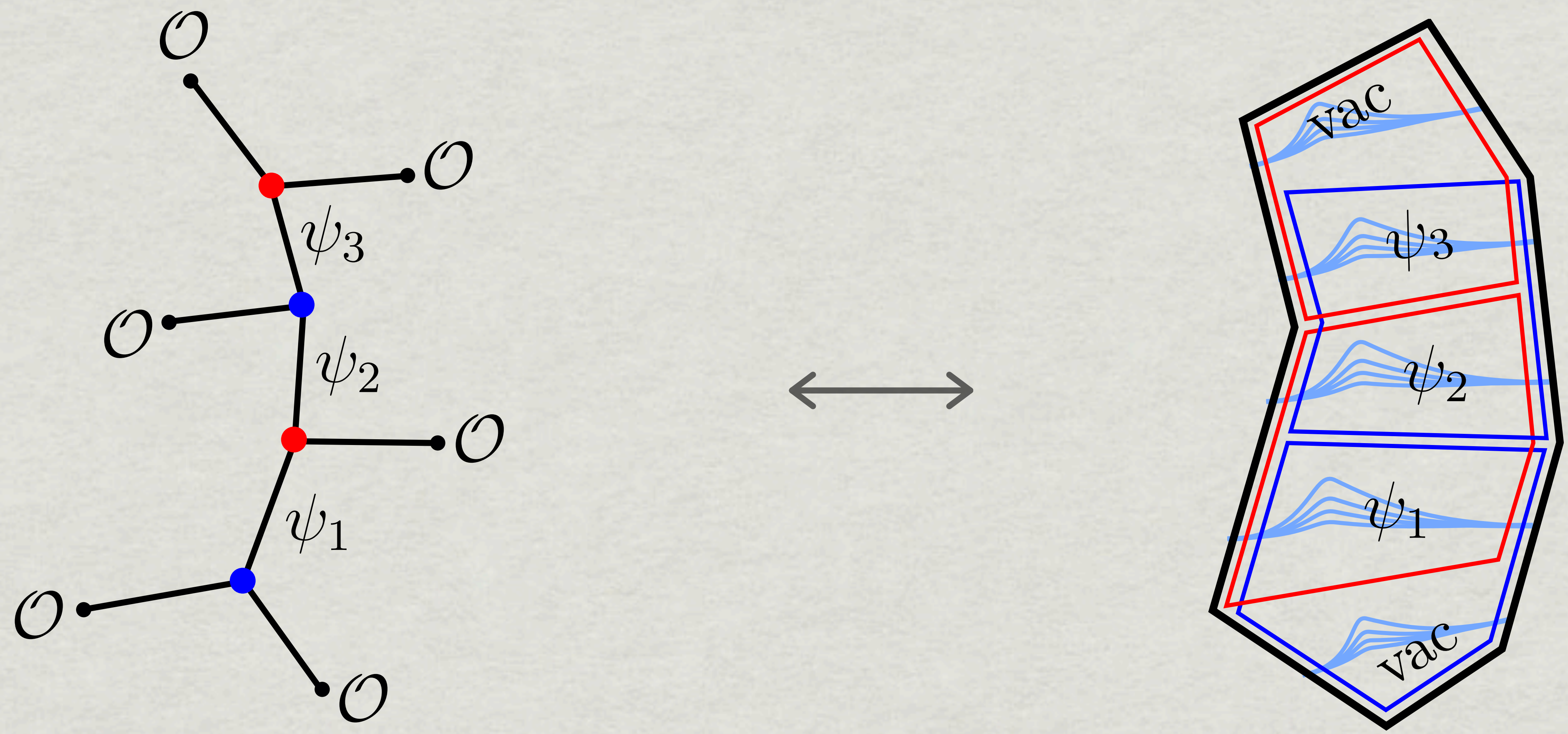
[Alday, Gaiotto,Maldacena,Sever,PV]

- In the Ising model of QFTs, planar N=4 SYM, WL = Scattering Amplitudes

[Alday, Maldacena; Drummond, Korchemsky, Sokatchev; Brandhuber, Heslop, Travaglini; Drummond, Henn, Korchemsky, Sokatchev; Berkovits, Maldacena]



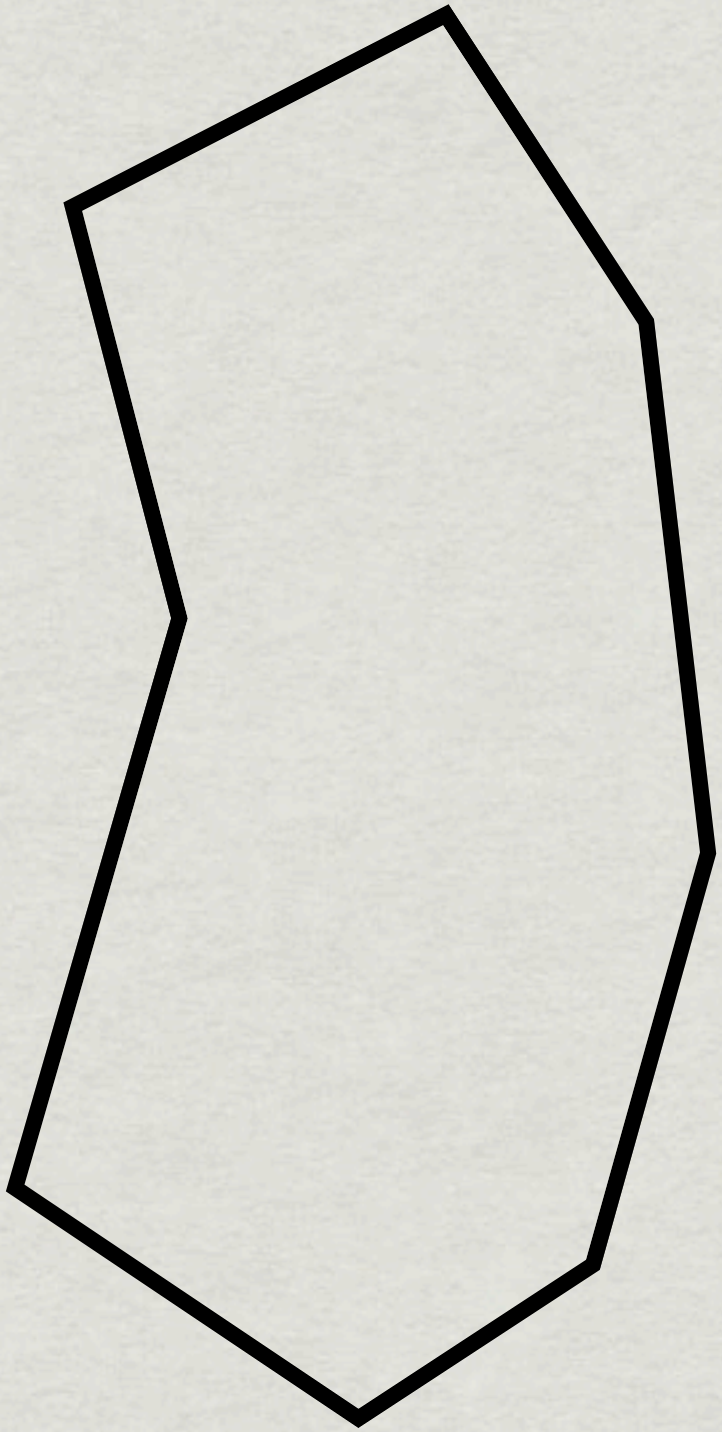
- Solving Scattering Amplitudes in planar N=4 SYM is tantamount to summing over all flux tube excitations at any finite ‘t Hooft coupling.



OPE for Correlation Functions ⟷ OPE for Wilson Loops

Perturbation theory and the OPE expansion are different.
Each expansion provides invaluable data for the other.

- A bit slower, building the tessellation:



- Number of edges = n
- Number of squares = n-3
- Number of middle squares = n-5
- Number of pentagons = n-4

A given tesselation will naturally come with a radius of convergence. Same as for the OPE of local operators.

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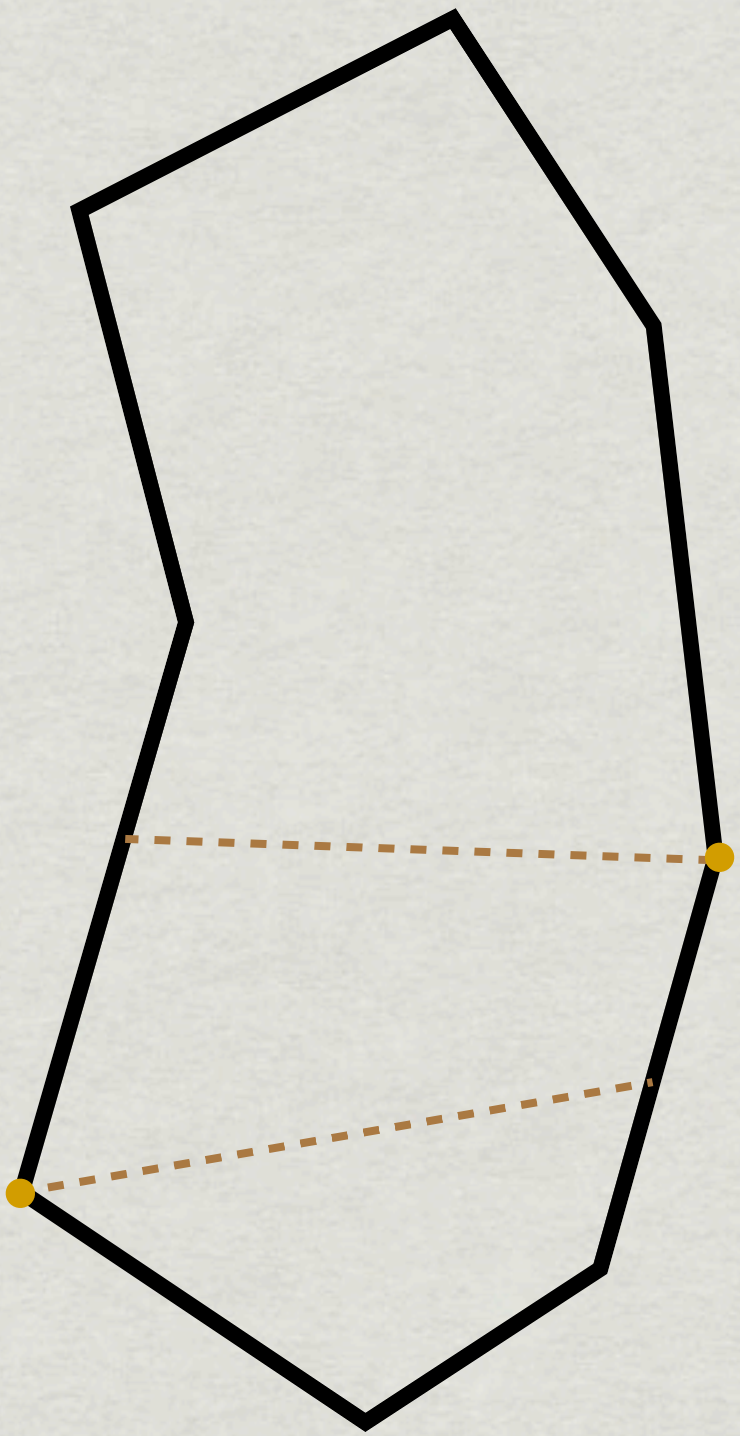

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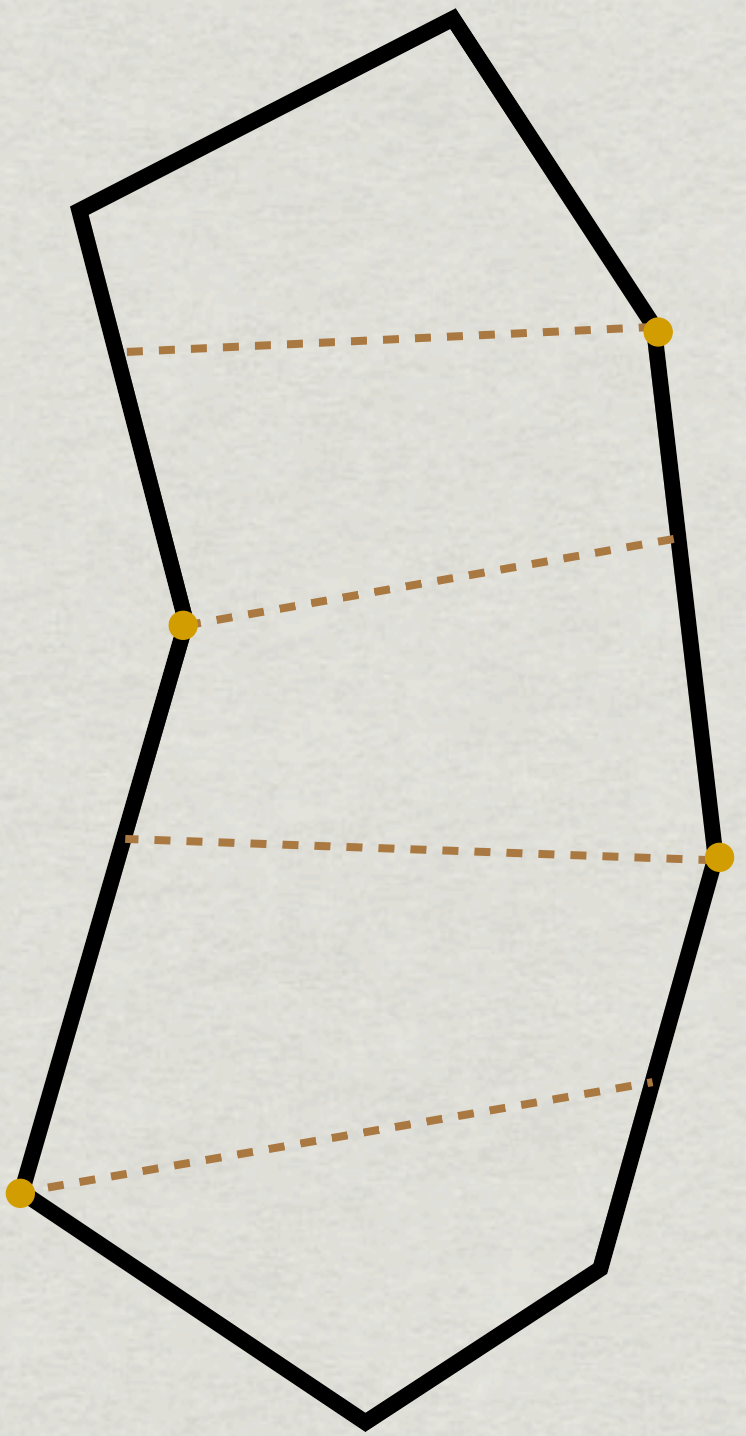

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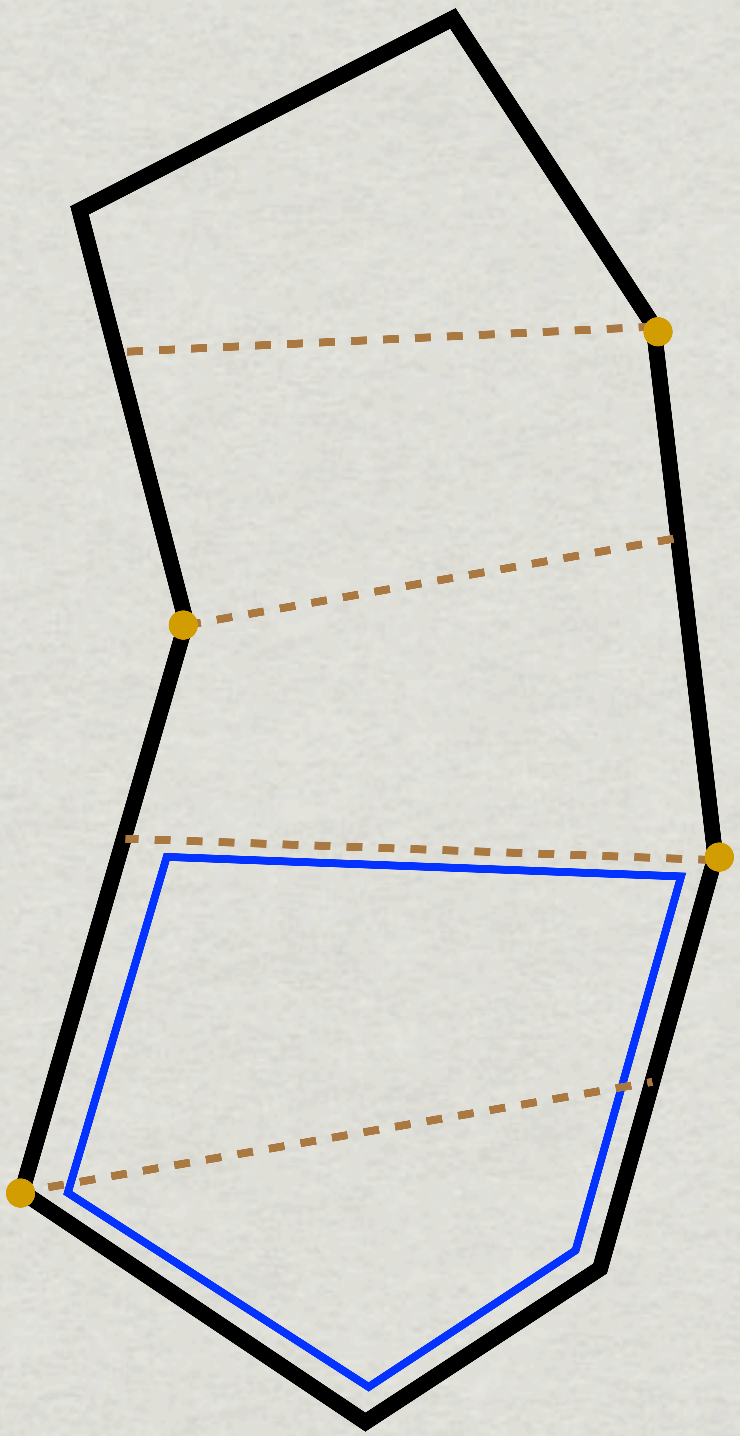

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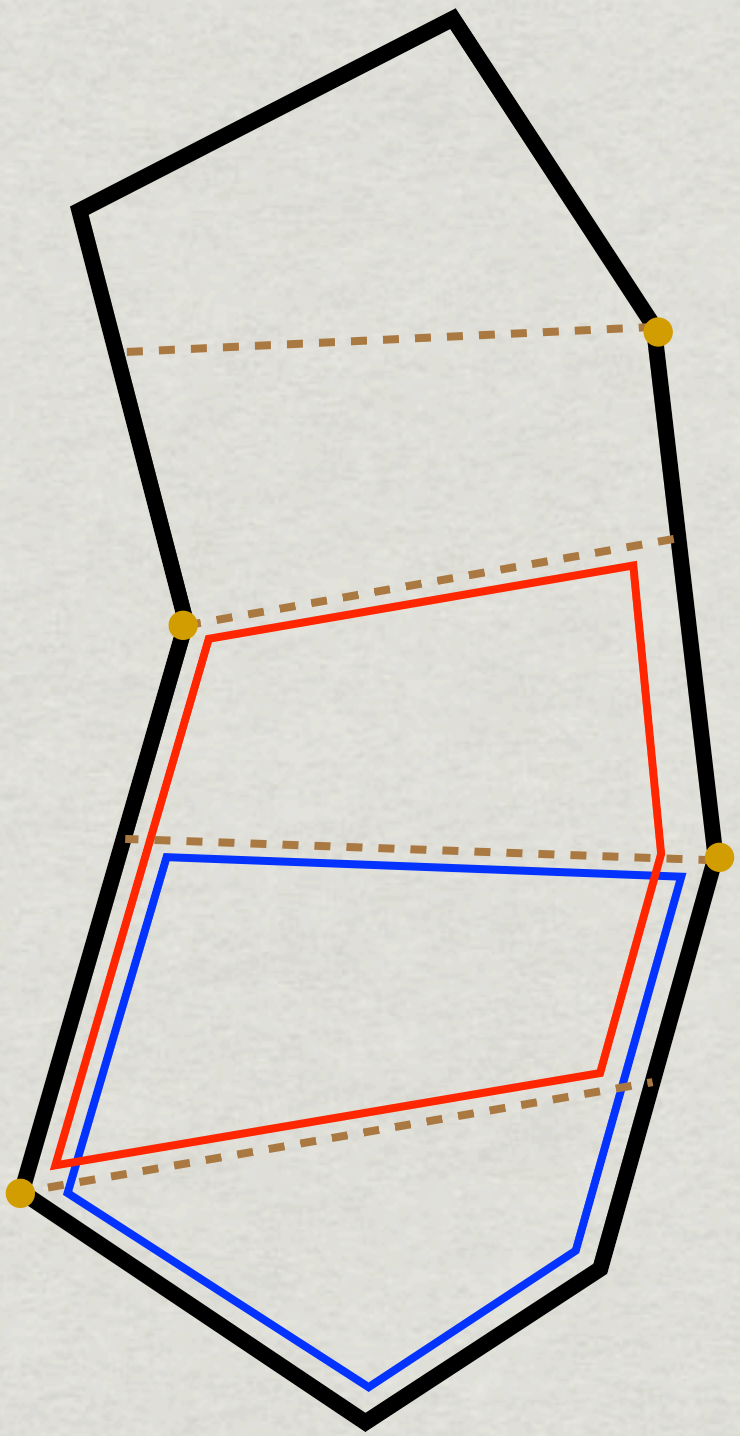

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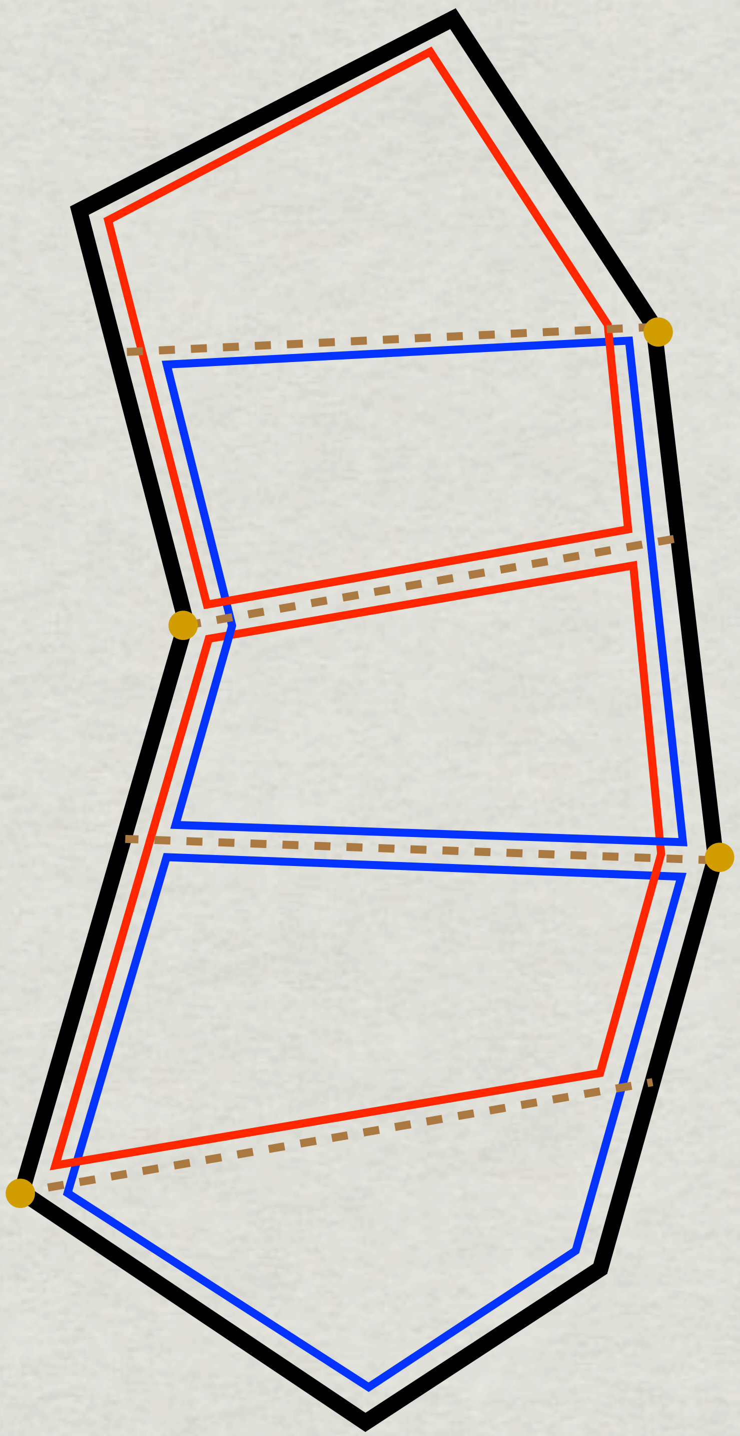

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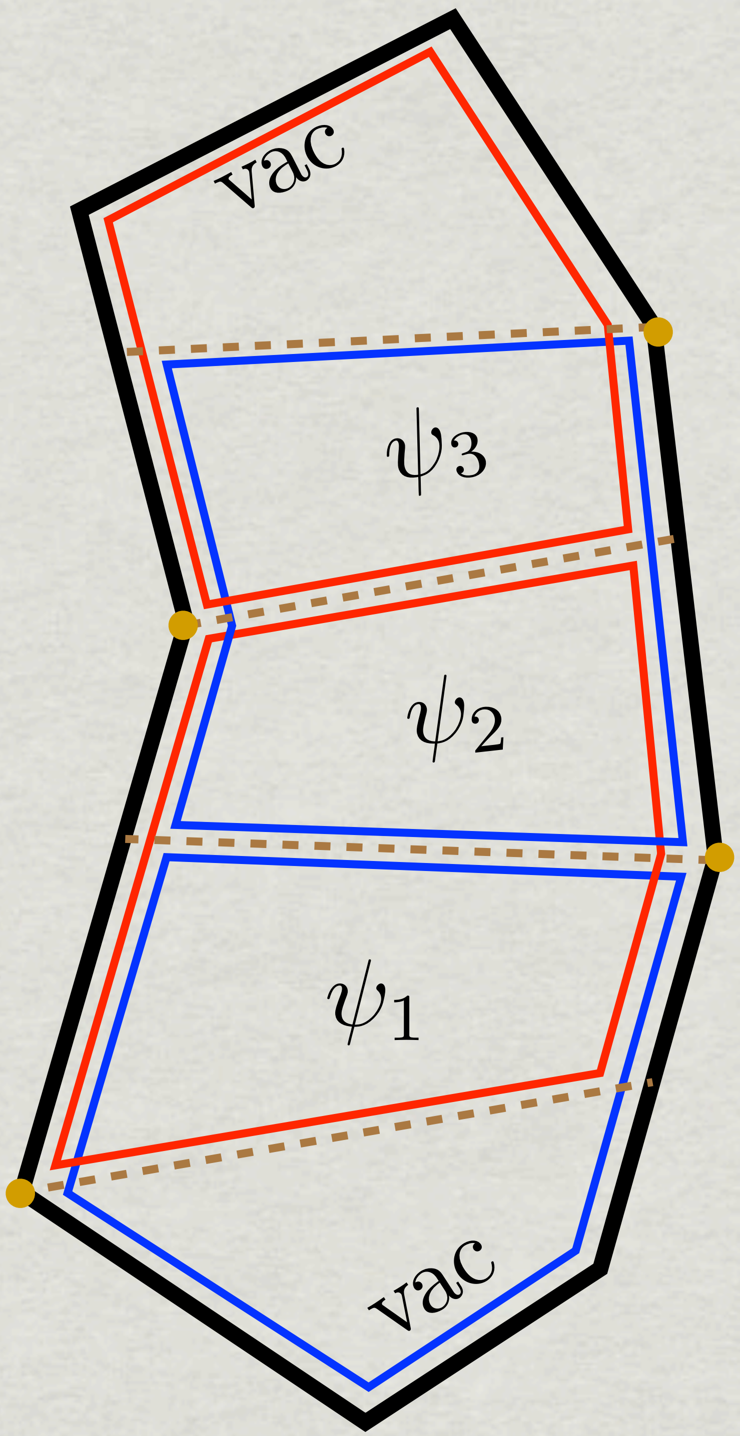

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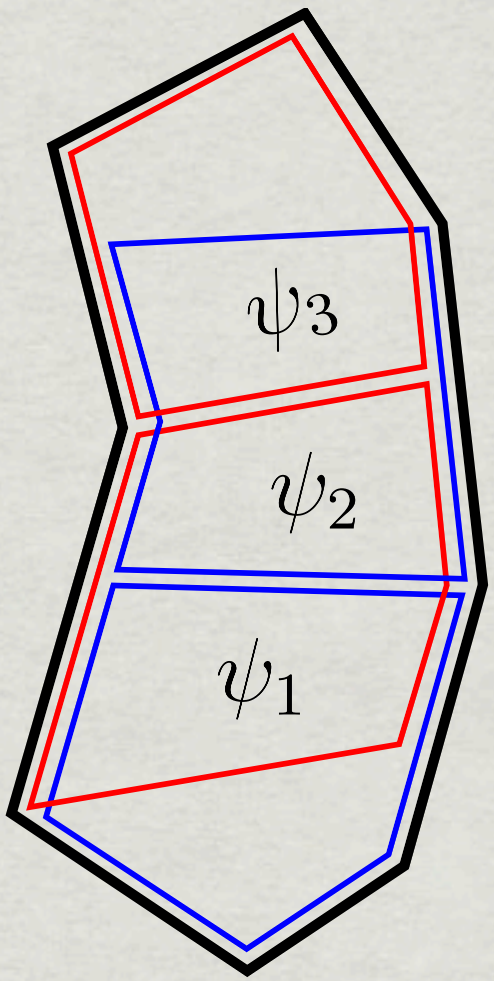
A given tesselation will naturally come with a radius of convergence. Same as for the OPE of local operators.

$$\mathcal{W} = \sum_{\psi_i} \left[\prod_{i=1}^{n-5} e^{-E_i\tau_i + ip_i\sigma_i + im_i\phi_i} \right] P(0|\psi_1)\ P(\psi_1|\psi_2)\ \ldots\ P(\psi_{n-6}|\psi_{n-5})\ P(\psi_{n-5}|0)$$

Propagation in the n-5 middle squares

A **pentagon transition** P between each pair of consecutive squares



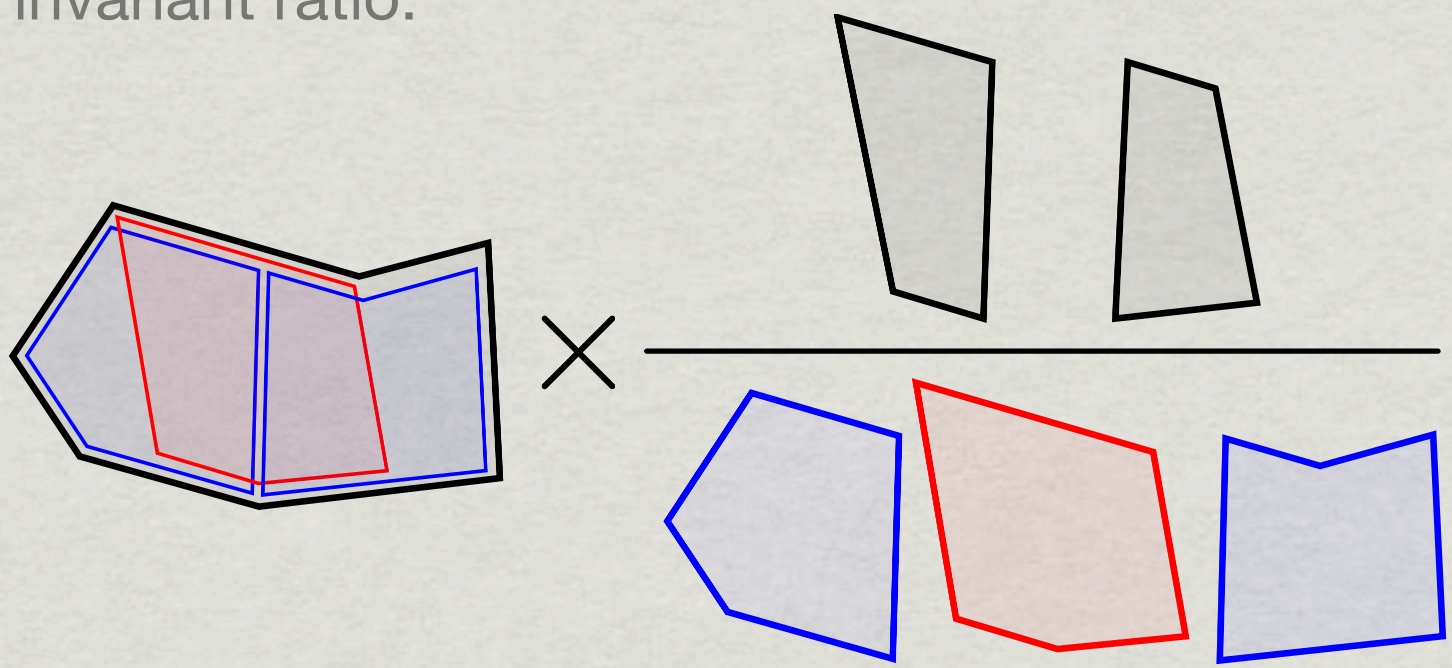
- To get Scattering Amplitudes in planar N=4 SYM non-perturbatively we need:
  - The exact spectrum of the flux tube excitations. [Basso 2010]
  - All pentagon transitions between any two flux tube eigenstates at any coupling. [Basso,Sever,PV 2013] + [Basso,Sever,PV unpublished/work in progress]

The spectrum and the pentagon transitions are dynamical observables of the color flux tube.
They are not geometrical. The geometry enters in a trivial fashion in the first square bracket.

Questions for the rest of the talk

- What is a nice coordinatization of null polygons from the OPE point of view? I.e, what exactly is τ_i, σ_i, ϕ_i?
- The null polygonal WL is UV divergent because of the cusps. What exactly are we computing?
- What are the flux tube eigenstates ψ_i? How to sum over them?
- How to compute the pentagon transitions at finite coupling?
  - What happens at weak coupling?
  - What happens at strong coupling?

- For a given tessellation in terms of null squares we define the following finite conformal invariant ratio:

$\mathcal{W}_{\text{hep}} \equiv$ 

All cusp divergences drop out of this object. The conformal anomaly of [Drummond, Henn, Korchemsky, Sokatchev] cancels out as well. This is a finite conformal invariant object.

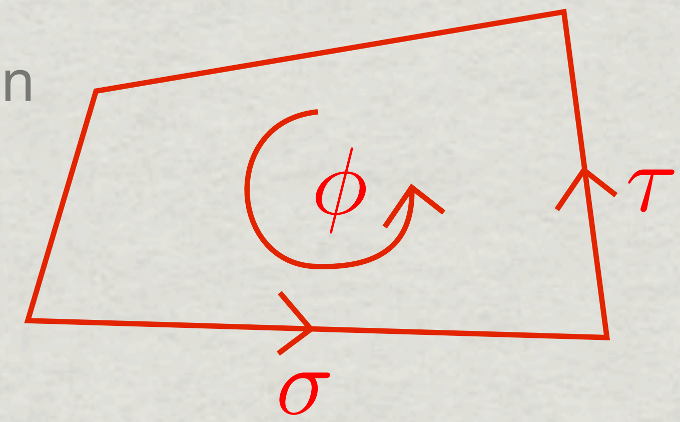
Hence it can only depend on the cross ratios made out of the positions of the cusps of the original polygon. For a *null* polygon with n edges there are 3n-15 such cross-ratios. (for the heptagon in the figure there are 6 cross-ratios.)

Squares and pentagons have no cross-ratios hence they are fixed by conformal symmetry and are given by the BDS ansatz. (This is the analogue of the statement that 2 and 3 point functions are fixed by conformal symmetry in a CFT.)

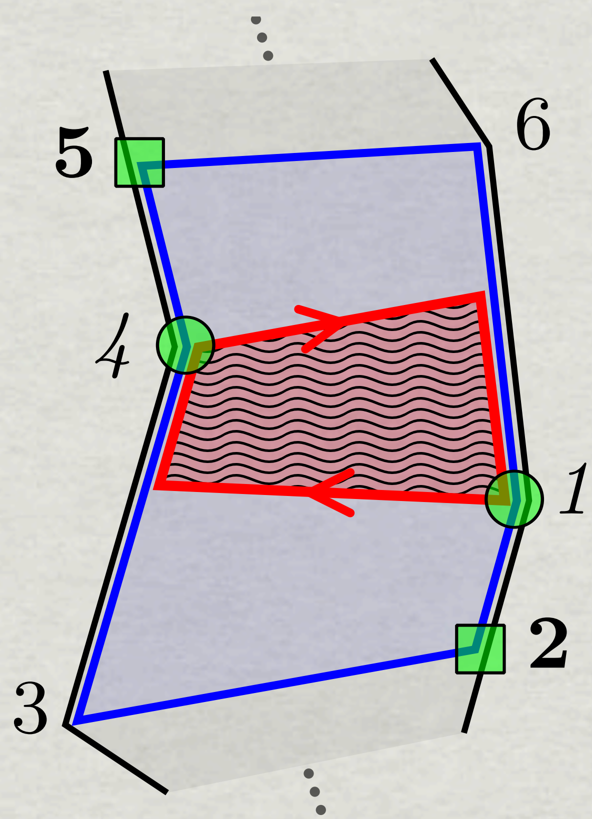
Hence we loose no information in considering this finite ratio.

- To measure some charge of the states flowing from A to B we act with the symmetry generators (corresponding to that charge) on A (or on B). See e.g. the usual OPE for 4pt correlation functions where we act with dilatations on two points to measure what flows from two operators to the other two. (Of course acting on both A and B means doing nothing by definition.)

- Similarly, each middle square in our tessellation has 3 symmetries corresponding to a time translation, a space translation and a rotation of the orthogonal directions. We act with those symmetries on the cusps below each that square. In this way we measure the energy, momentum and angular momentum flowing in each middle square.



Equivalently:



$$u_i = u_{i+3} \equiv \frac{x_{i-1,i+1}^2 x_{i-2,i+2}^2}{x_{i-1,i+2}^2 x_{i+1,i-2}^2}$$

$$\frac{1}{u_2} = 1 + e^{2\tau}$$

$$\frac{1}{u_3} = 1 + (e^{-\tau} + e^{\sigma+i\phi})(e^{-\tau} + e^{\sigma-i\phi})$$

$$\frac{u_1}{u_2 u_3} = e^{2\sigma+2\tau}$$

There are n-5 middle squares so the 3n-15 parameters τ_i, σ_i, ϕ_i coordinatize all conformally inequivalent polygons.

- In other words...

- So far we considered mostly kinematics and thus very general (it applies to any 4D conformal theory). We found that

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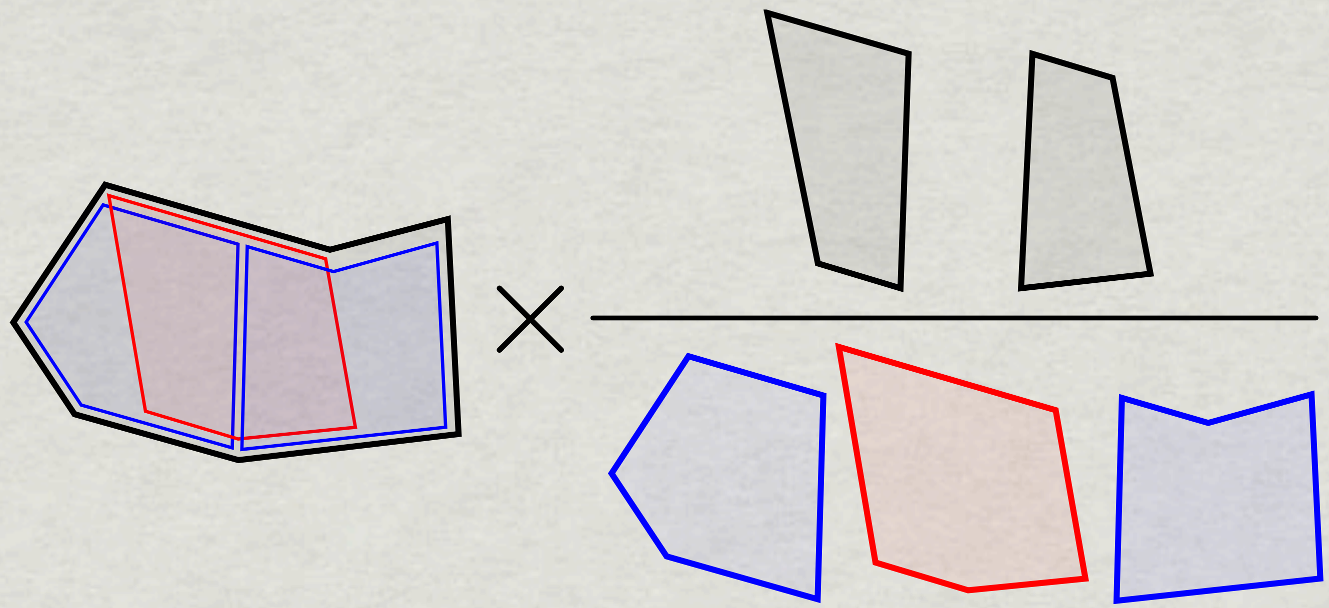


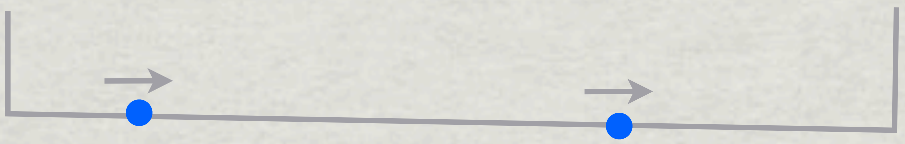
$$= \sum_{\psi_1,\psi_2} \left[\prod_{i=1}^{2} e^{-E_i\tau_i + ip_i\sigma_i + im_i\phi_i} \right] P(0|\psi_1)P(\psi_1|\psi_2)P(\psi_2|0)$$

recall that P does not depend on geometry since all pentagons are conformal equivalent. P is a flux tube observable.

- Now we move to the most interesting part, the dynamics. What are the flux tube eigenstates ψ_i ? How to sum over them? How to compute the pentagon transitions at finite coupling?

  We will now specialize to planar N=4 SYM.

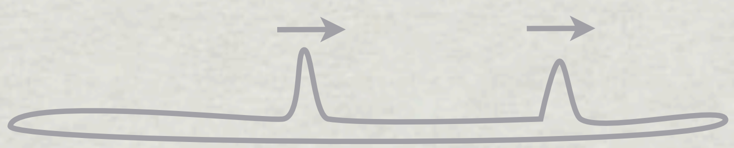
- There are several equivalent descriptions of the excited flux tube:
  - As null Wilson lines with insertions



  - As large spin operators

$$\mathcal{O} = \mathrm{tr}\,(Z\,DDDD\ldots DDDD\,\vec{F}\,DDDD\ldots DDDD\,\vec{F}\,DDDD\ldots DDDD\,Z)$$

  - As an excited GKP [Gubser,Klebanov,Polyakov] folded string.



- These states have a fixed number of excitations with given momenta and we know their spectrum exactly from Integrability [Basso 2010] (we also know how these excitations scatter amongst themselves [Basso,Rej;Fioravanti,Piscaglia,Rossi;Basso,Sever,PV]). Hence

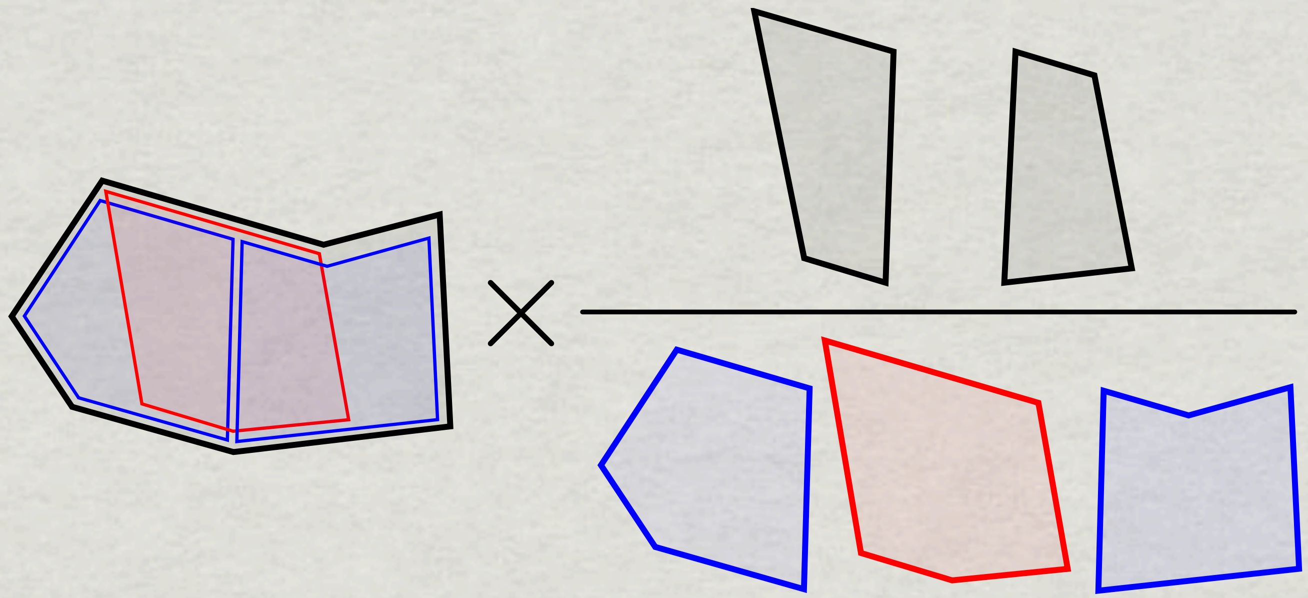
$$\sum_{\psi} = \sum_{\mathbf{a}} \int du_1 \ldots du_N \, \mu(u_1) \ldots \mu(u_N)$$

$$E = \epsilon(u_1) + \cdots + \epsilon(u_N)\,, \qquad p = p(u_1) + \cdots + p(u_N)$$

The vector **a** indicates which kind of excitations we are considering. For example, the state above has two gluonic excitations so that **a**={F,F}.

- Hence we have (boldface indicate vectors)

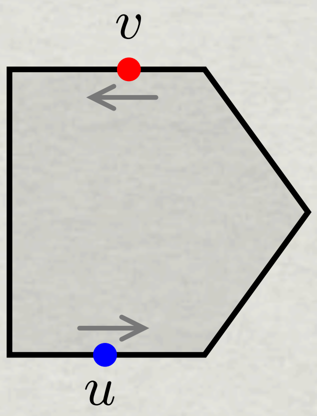
$\mathcal{W}_{\text{hep}} \equiv$



$$= \sum_{\mathbf{a},\mathbf{b}} \int d\mathbf{u}\, d\mathbf{v}\, P_{\mathbf{a}}(0|\mathbf{u})\, e^{-E(\mathbf{u})\tau_1 + ip(\mathbf{u})\sigma_1 + im_1\phi_1} P_{\mathbf{ab}}(\bar{\mathbf{u}}|\mathbf{v})\, e^{-E(\mathbf{v})\tau_2 + ip(\mathbf{v})\sigma_2 + im_2\phi_2} P_{\mathbf{b}}(\bar{\mathbf{v}}|0)$$

- The challenge is now to compute the pentagon transitions between a state with N particles and another state with M excitations.
- The simplest transitions where a single excitation propagates from one square to the next. Multi-particles can be more or less built out of those, see below. This is a manifestation of Integrability for the pentagon transitions.
- Single particle transitions are associated to the lightest states so they are also the most important ones. E.g., they determine the dominant behavior of the WL in the near collinear limit of large tau's.

- We consider for illustration the case where the bottom and the top excitations are gluons. We want to compute the single particle pentagon transition at any coupling



$P(u|v)$

- We postulate three Bootstrap like axioms that this object should satisfy. They take the form of functional equations. We found one solution to these equations which, we conjecture, gives the pentagon transition at any finite coupling.

$$\text{I. } P(u|v) = P(-v|-u)$$

$$\text{II. } P(u|v) = S(u,v)P(v|u)$$

$$\text{III. } P(u^{-\gamma}|v) = P(v|u)$$

- Axiom 1, $P(-u|-v) = P(v|u)$, is the obvious reflection symmetry of the pentagon.

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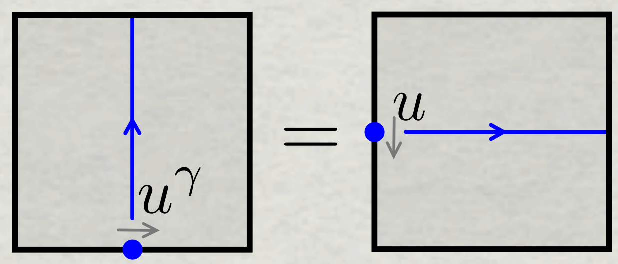
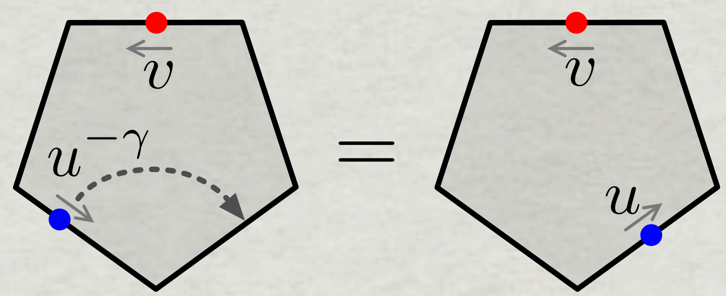
- Axiom 3 comes from the mirror symmetry of the flux tube. There exists a non-perturbative path in the rapidity u which implements the Wick rotation:

$$E(u^\gamma) = ip(u)$$
$$p(u^\gamma) = iE(u)$$



Hence we expect



or $P(u^{-\gamma}|v) = P(v|u)$

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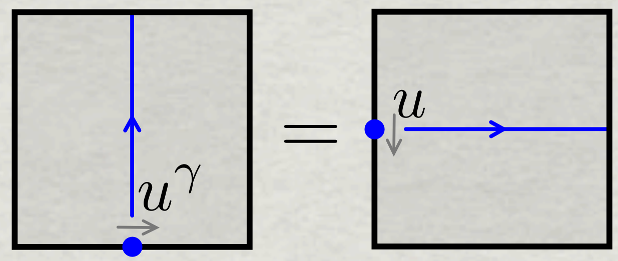

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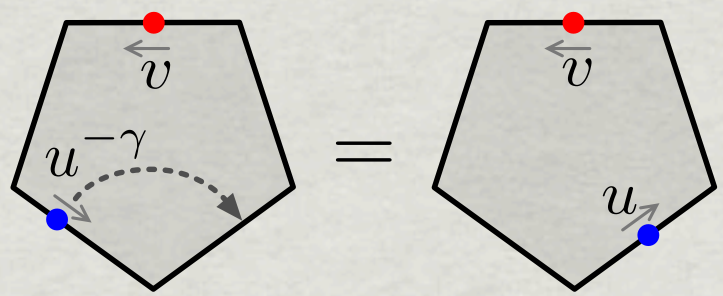
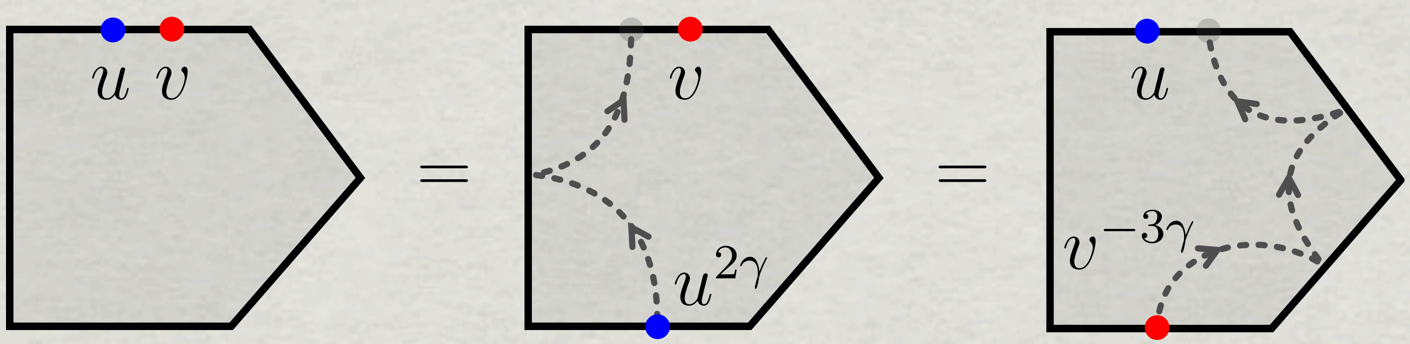


or $P(u^{-\gamma}|v) = P(v|u)$

- Axiom 2, $P(u|v) = S(u,v)P(v|u)$, together with the other two, implies Watson's equation $P(0|u,v) = S(v,u)P(0|v,u)$ where P(0|u,v) is given by



This is a nice self-consistency check and is the main motivation for axiom 2.

- We can solve the bootstrap equations. For example, a solution for the scalar excitations is

$$P(u|v)^2 = \frac{S(u,v)}{g^2(u-v)(u-v+i)S(u^\gamma,v)}$$

It provides a precise connection between the space-time and the flux tube S-matrices. They hold at any coupling.

The flux tube S-matrices can be computed at any coupling using Integrability [Basso,Rej;Fioravanti,Piscaglia,Rossi;Basso,Sever,PV] The main ingredients are solutions to linear integral equations akin to the so-called BES equation for the flux-tube vacuum.

- Multi-particles are built in terms of the single particle transitions:

$$P(\{u_i\}|\{v_j\}) = \frac{\prod\limits_{i,j} P(u_i|v_j)}{\prod\limits_{i>j} P(u_i|u_j) \prod\limits_{i<j} P(v_i|v_j)} \quad \text{x (Group theory matrix part)}$$

We have an algorithm for getting the SU(4) matrix part but so far we only checked it up to a small number of particles (at most eight). For example, for two scalars

$$P(\mathbf{u}|\mathbf{v})_{i_1 i_2}^{j_1 j_2} = P_{\text{dyn}}(\mathbf{u}|\mathbf{v}) \left[\pi_1(\mathbf{u}|\mathbf{v})\delta_{i_1}^{j_1}\delta_{i_2}^{j_2} + \pi_2(\mathbf{u}|\mathbf{v})\delta_{i_1}^{j_2}\delta_{i_2}^{j_1} + \pi_3(\mathbf{u}|\mathbf{v})\delta_{i_1 i_2}\delta^{j_1 j_2}\right]$$

where the matrix functions π_j are simple, rational functions of the rapidities, independent of the coupling.

- We can now compare our representation

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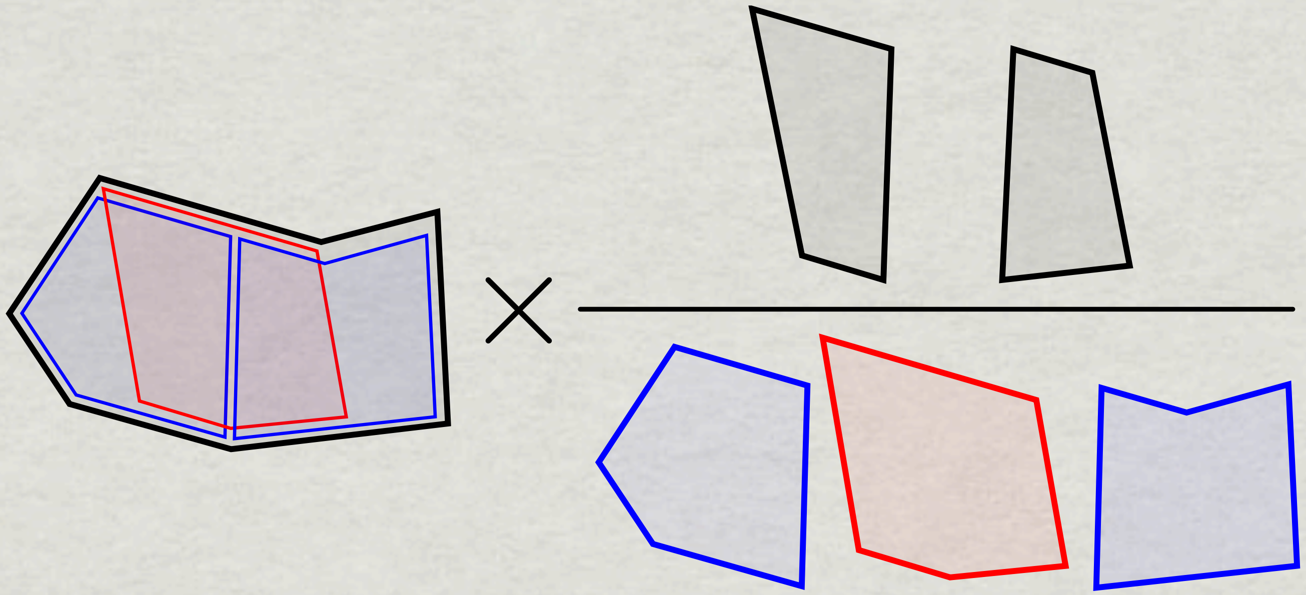
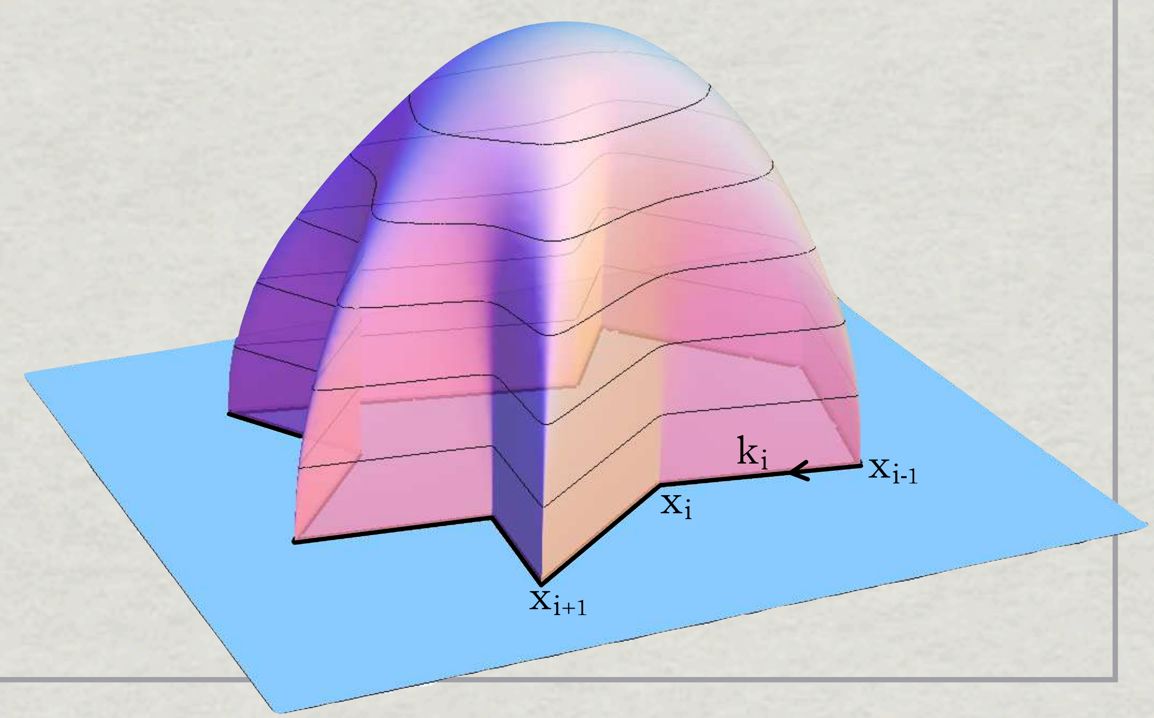


$$= \sum_{\mathbf{a},\mathbf{b}} \int d\mathbf{u}\, d\mathbf{v}\, P_{\mathbf{a}}(0|\mathbf{u})\, e^{-E(\mathbf{u})\tau_1 + ip(\mathbf{u})\sigma_1 + im_1\phi_1} P_{\mathbf{ab}}(\bar{\mathbf{u}}|\mathbf{v})\, e^{-E(\mathbf{v})\tau_2 + ip(\mathbf{v})\sigma_2 + im_2\phi_2} P_{\mathbf{b}}(\bar{\mathbf{v}}|0)$$

with weak and strong coupling results.

- At strong coupling we can derive the Y-system in [Alday,Gaiotto,Maldacena; Alday,Maldacena,Sever,Vieira]
  This involves re-summing infinitely many multi-particles and boundstates.

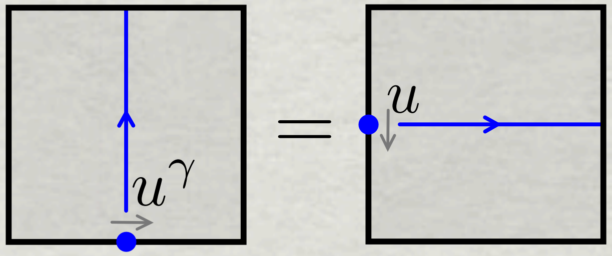
  This is quite exciting since the strong coupling result was begging for such an interpretation.



- At weak coupling we checked the single particle transitions against all available data in the literature for Wilson loops up to 3 loops.
  - MHV Hexagon at 1 Loop [Bern,Dixon,Smirnov], 2 Loops [Del Duca,Duhr,Smirnov; Goncharov,Spradlin,Vergu,Volovich], 3 Loops [Dixon,Drummond,Henn], 4 Loops [Dixon,Duhr,Pennington, to appear]
  - MHV Heptagon at 1 Loop [Bern,Dixon,Smirnov], 2 Loops [Caron-Huot]
  - NMHV Hexagon at 1 Loop [Bern,Dixon,Dunbar,Kosower; Drummond,Henn,Korchemsky,Sokatchev], 2 Loops [Dixon,Drummond,Henn]
  - NMHV Heptagon at 1 Loop [Bourjaily,Caron-Huot,Trnka], 2 Loops [Caron-Huot,He]

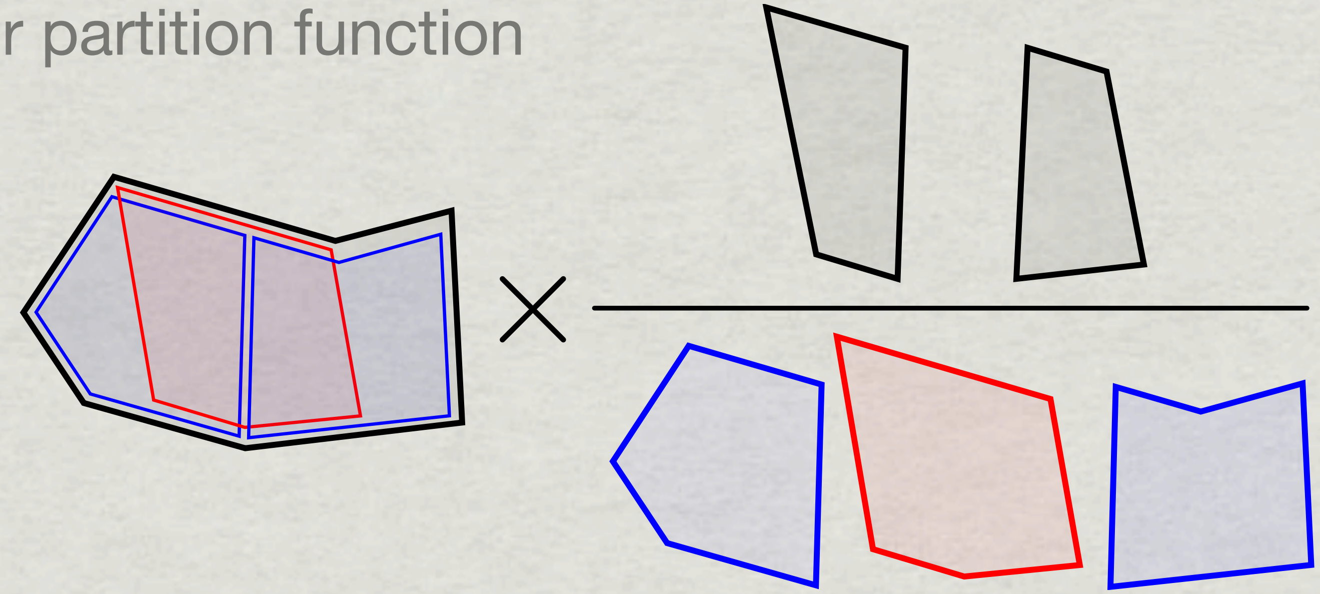
    (MHV amplitudes = Bosonic WL, non-MHV amplitudes = Superloop [Skinner,Mason;Caron-Huot]; OPE still applies [Sever, PV,Wang])
  - We also generated infinitely many higher loop predictions
- Not everything is done:
  - We have conjectures for transitions with **fermions**. But they are not as well motivated since mirror (and crossing) transformation for fermions is not well understood/does not seem to exist.

$$E(u^\gamma) = ip(u)$$
$$p(u^\gamma) = iE(u)$$



  - We seem to be able to compute **matrix part** case by case but a general expression (which would render the re-summation easy) is still lacking.
  - Once we have all the transitions, or at least many transitions, would be nice to think what is the best way to plot the amplitude.

- Does our partition function

$\mathcal{W}_{\text{hep}} \equiv$ 

$$= \sum_{\mathbf{a},\mathbf{b}} \int d\mathbf{u}\, d\mathbf{v}\, P_{\mathbf{a}}(0|\mathbf{u})\, e^{-E(\mathbf{u})\tau_1+ip(\mathbf{u})\sigma_1+im_1\phi_1} P_{\mathbf{ab}}(\bar{\mathbf{u}}|\mathbf{v})\, e^{-E(\mathbf{v})\tau_2+ip(\mathbf{v})\sigma_2+im_2\phi_2} P_{\mathbf{b}}(\bar{\mathbf{v}}|0)$$

re-sum into some beautiful object from the Integrability point of view?

At strong coupling it does!: The Yang-Yang functional and its associated TBA equations.* What about finite coupling? We don't know yet but in any case, we should be able to plot the finite coupling amplitude nevertheless.

- Could be interesting to see how WL data from other conformal gauge theories looks like when OPE decomposed.

* for the experts: the strong coupling result for the Wilson loop contains several contributions but once we compute the finite ratio W everything cancels out except for the nicest of all (from the Integrability point of view) which is the Yang-Yang functional!

감사합니다

Thank you

- Multi-particles

$$P(\{u_i\}|\{v_j\}) = \frac{\prod_{i,j} P(u_i|v_j)}{\prod_{i>j} P(u_i|u_j) \prod_{i<j} P(v_i|v_j)}$$ x (Group theory matrix part)

- The matrix part encodes the SU(4) symmetry of the excitations. For gluons the matrix part is 1. For scalars and fermions it is non-trivial. For example, for two scalars we have

$$P(\mathbf{u}|\mathbf{v})_{i_1 i_2}^{j_1 j_2} = P_{\rm dyn}(\mathbf{u}|\mathbf{v}) \left[\pi_1(\mathbf{u}|\mathbf{v})\delta_{i_1}^{j_1}\delta_{i_2}^{j_2} + \pi_2(\mathbf{u}|\mathbf{v})\delta_{i_1}^{j_2}\delta_{i_2}^{j_1} + \pi_3(\mathbf{u}|\mathbf{v})\delta_{i_1 i_2}\delta^{j_1 j_2}\right]$$

where

$$\pi_1(\mathbf{u}|\mathbf{v}) + \pi_2(\mathbf{u}|\mathbf{v}) = 1\,, \qquad \pi_2(\mathbf{u}|\mathbf{v}) = \frac{(u_1-v_1)(u_2-v_2+i)}{(u_1-u_2-i)(v_1-v_2+i)}$$

$$\pi_2(\mathbf{u}|\mathbf{v}) + \pi_3(\mathbf{u}|\mathbf{v}) = \frac{(u_1-v_1)(u_2-v_2+i)(u_1-v_1-i)(u_2-v_2+2i)}{(u_1-u_2-i)(u_1-u_2-2i)(v_1-v_2+i)(v_1-v_2+2i)}$$

(We believe we have an algorithm for getting the matrix part but so far we only checked it up to a small number of particles, at most 8)

- For the experts, in terms of the momentum twistors appearing in the previous talks we have

$$
\begin{aligned}
e^{2\tau_{2j}} &\equiv \frac{\langle -j, j+1, j+2, j+3\rangle\langle -j-1, -j, -j+1, j+2\rangle}{\langle -j-1, -j, j+2, j+3\rangle\langle -j, -j+1, j+1, j+2\rangle}\,, \\
e^{\sigma_{2j}+\tau_{2j}-i\phi_{2j}} &\equiv \frac{\langle -j-1, -j, j+1, j+2\rangle\langle j, j+1, j+2, j+3\rangle}{\langle -j-1, j+1, j+2, j+3\rangle\langle -j, j, j+1, j+2\rangle}\,, \\
e^{\sigma_{2j}+\tau_{2j}+i\phi_{2j}} &\equiv \frac{\langle -j-2, -j-1, -j, -j+1\rangle\langle -j-1, -j, j+1, j+2\rangle}{\langle -j-2, -j-1, -j, j+2\rangle\langle -j-1, -j, -j+1, j+1\rangle}\,, \\
e^{2\tau_{2j+1}} &\equiv \frac{\langle -j-1, j+1, j+2, j+3\rangle\langle -j-2, -j-1, -j, j+2\rangle}{\langle -j-2, -j-1, j+2, j+3\rangle\langle -j-1, -j, j, j+2\rangle}\,, \\
e^{\sigma_{2j+1}+\tau_{2j+1}-i\phi_{2j+1}} &\equiv \frac{\langle -j-2, -j-1, -j, -j+1\rangle\langle -j-1, -j, j+2, j+3\rangle}{\langle -j-2, -j-1, -j, j+3\rangle\langle -j-1, -j, -j+1, j+2\rangle}\,, \\
e^{\sigma_{2j+1}+\tau_{2j+1}+i\phi_{2j+1}} &\equiv \frac{\langle j+1, j+2, j+3, j+4\rangle\langle -j-1, -j, j+2, j+3\rangle}{\langle -j-1, j+2, j+3, j+4\rangle\langle -j, j+1, j+2, j+3\rangle}\,,
\end{aligned}
$$

- We have a matrix: $$K_{ij} = 2j(-1)^{j(i+1)} \int_0^\infty \frac{dt}{t} \frac{J_i(2gt)J_j(2gt)}{e^t - 1}$$

- And 2 vectors similar to each other. One of them is $\kappa_j(u) \equiv -\int_0^\infty \frac{dt}{t} \frac{J_j(2gt)(J_0(2gt) - \cos(ut)\left[e^{t/2}\right]^{(-1)^{\eta \times j}})}{e^t - 1}$

- Finally we have a matrix of integers $\mathbb{Q}_{ij} = \delta_{ij}(-1)^{i+1} i$ and $\mathbb{M} \equiv (1+K)^{-1} = 1 - K + K^2 - K^3 + \ldots$

- Then we construct four similar functions $f_{1,2,3,4}$. For example $f_4(u,v) = 2\,\kappa(v) \cdot \mathbb{Q} \cdot \mathbb{M} \cdot \kappa(v)$

- The gluon S-matrix which is the non-trivial ingredient in the pentagon transitions, reads

$$S(u,v) = \frac{\Gamma(\frac{3}{2} - iu)\Gamma(\frac{3}{2} + iv)\Gamma(iu - iv)}{\Gamma(\frac{3}{2} + iu)\Gamma(\frac{3}{2} - iv)\Gamma(iv - iu)} F(u,v)$$

where

$$\log F = 2i \int_0^\infty \frac{dt}{t} (J_0(2gt) - 1) \frac{e^{-t/2}(\sin(ut) - \sin(vt))}{e^t - 1} - 2if_1 + 2if_2$$

- The mirror S-matrix uses the other two functions $f_{1,2,3,4}$

- The famous cusp anomalous dimension is nothing but ($4g^2$ times) $[\mathbb{Q} \cdot \mathbb{M}]_{1,1}$

$$\mathcal{W}^{(6134)} = \frac{e^{-\tau}}{2\cosh(\sigma)} \sum_{l=0}^{\infty} g^{2l} \sum_{n=0}^{l} \tau^n F_n^{(l)}(\sigma) + \mathcal{O}(e^{-2\tau})$$

$$\begin{aligned}
F_0^{(0)} &= 1 \\
F_1^{(1)} &= -2\bar{H}_0 - 4H_1 \\
F_0^{(1)} &= -2H_1\bar{H}_0 - 2H_1^2 \\
F_2^{(2)} &= 8H_1\bar{H}_0 + \bar{H}_0^2 + 8H_1^2 + \frac{\pi^2}{3} \\
F_1^{(2)} &= 4\bar{H}_0 H_{0,1} + 12H_1^2\bar{H}_0 + 2H_1\bar{H}_0^2 + \frac{4}{3}\pi^2\bar{H}_0 + 8H_1^3 + 2\pi^2 H_1 - 4\zeta(3) \\
F_0^{(2)} &= 4H_1\bar{H}_0 H_{0,1} + \frac{1}{2}\bar{H}_0^2 H_{0,1} - 4\bar{H}_0 H_{0,1,1} + 2\zeta(3)\bar{H}_0 + 4H_1^3\bar{H}_0 + H_1^2\bar{H}_0^2 + \frac{4}{3}\pi^2 H_1\bar{H}_0 \\
&\quad + \frac{1}{12}\pi^2\bar{H}_0^2 + \frac{1}{6}\pi^2 H_{0,1} + 4H_{0,0,0,1} + 2H_{0,1,0,1} - 4H_1\zeta(3) + 2H_1^4 + \pi^2 H_1^2 + \frac{\pi^4}{36}
\end{aligned}$$

$$\begin{aligned}
-F_0^{(3)} &= \frac{4H_1^6}{3} + 4\bar{H}_0 H_1^5 + 2\bar{H}_0^2 H_1^4 + 2\pi^2 H_1^4 + \frac{2}{9}\bar{H}_0^3 H_1^3 + \frac{38}{9}\pi^2\bar{H}_0 H_1^3 + 16\bar{H}_0 H_{0,1} H_1^3 \\
&- \frac{40}{3}H_{0,0,1}H_1^3 - 16\zeta(3)H_1^3 + \frac{3}{2}\pi^2\bar{H}_0^2 H_1^2 + 5\bar{H}_0^2 H_{0,1}H_1^2 + \frac{5}{3}\pi^2 H_{0,1}H_1^2 + 4\bar{H}_0 H_{0,0,1}H_1^2 \\
&- 48\bar{H}_0 H_{0,1,1}H_1^2 + 40H_{0,0,1,1}H_1^2 + 20H_{0,1,0,1}H_1^2 + \frac{91}{90}\pi^4 H_1^2 + \frac{1}{18}\pi^2\bar{H}_0^3 H_1 + \frac{113}{90}\pi^4\bar{H}_0 H_1 \\
&+ \frac{1}{3}\bar{H}_0^3 H_{0,1}H_1 + 5\pi^2\bar{H}_0 H_{0,1}H_1 - \frac{10}{3}\pi^2 H_{0,0,1}H_1 - 10\bar{H}_0^2 H_{0,1,1}H_1 - \frac{10}{3}\pi^2 H_{0,1,1}H_1 \\
&+ 4\bar{H}_0 H_{0,0,0,1}H_1 - 8\bar{H}_0 H_{0,0,1,1}H_1 + 20\bar{H}_0 H_{0,1,0,1}H_1 + 96\bar{H}_0 H_{0,1,1,1}H_1 + 20H_{0,0,0,0,1}H_1 \\
&- 20H_{0,0,1,0,1}H_1 - 80H_{0,0,1,1,1}H_1 - 20H_{0,1,0,0,1}H_1 - 40H_{0,1,0,1,1}H_1 - 40H_{0,1,1,0,1}H_1 - 32\zeta(5)H_1 \\
&+ 2\bar{H}_0^2\zeta(3)H_1 - 20H_{0,1}\zeta(3)H_1 - \frac{8}{3}\pi^2\zeta(3)H_1 + \frac{7}{60}\pi^4\bar{H}_0^2 + \frac{2}{3}\pi^2\bar{H}_0^2 H_{0,1} + \frac{7}{30}\pi^4 H_{0,1} \\
&+ \frac{1}{3}\pi^2\bar{H}_0 H_{0,0,1} - \frac{1}{3}\bar{H}_0^3 H_{0,1,1} - 5\pi^2\bar{H}_0 H_{0,1,1} - \frac{2}{3}\pi^2 H_{0,0,0,1} + \frac{10}{3}\pi^2 H_{0,0,1,1} \\
&+ 2\bar{H}_0^2 H_{0,1,0,1} + 2\pi^2 H_{0,1,0,1} + 10\bar{H}_0^2 H_{0,1,1,1} + \frac{10}{3}\pi^2 H_{0,1,1,1} - 6\bar{H}_0 H_{0,0,0,0,1} - 4\bar{H}_0 H_{0,0,0,1,1} \\
&- 2\bar{H}_0 H_{0,0,1,0,1} + 8\bar{H}_0 H_{0,0,1,1,1} - 2\bar{H}_0 H_{0,1,0,0,1} - 20\bar{H}_0 H_{0,1,0,1,1} - 20\bar{H}_0 H_{0,1,1,0,1} - 96\bar{H}_0 H_{0,1,1,1,1} \\
&+ 56H_{0,0,0,0,0,1} - 20H_{0,0,0,0,1,1} + 4H_{0,0,0,1,0,1} + 4H_{0,0,1,0,0,1} + 20H_{0,0,1,0,1,1} + 20H_{0,0,1,1,0,1} \\
&+ 80H_{0,0,1,1,1,1} + 4H_{0,1,0,0,0,1} + 20H_{0,1,0,0,1,1} + 20H_{0,1,0,1,0,1} + 40H_{0,1,0,1,1,1} + 20H_{0,1,1,0,0,1} \\
&+ 40H_{0,1,1,0,1,1} + 40H_{0,1,1,1,0,1} + 16\bar{H}_0\zeta(5) - 4\zeta(3)^2 + \bar{H}_0^3\zeta(3) + \frac{4}{3}\pi^2\bar{H}_0\zeta(3) \\
&+ 2\bar{H}_0 H_{0,1}\zeta(3) - 4H_{0,0,1}\zeta(3) + 20H_{0,1,1}\zeta(3) + \frac{11\pi^6}{270} \qquad (190)
\end{aligned}$$

- We can solve the bootstrap equations. A solution for the scalar excitations is

$$P(u|v)^2 = \frac{S(u,v)}{g^2(u-v)(u-v+i)S(u^\gamma,v)}$$

while for gluonic excitations we have (f is a simple function of the so called Zhukowsky variables)

$$P(u,v)^2 = \frac{f(u,v)}{g^2(u-v)(u-v-i)}\frac{S(u,v)}{S(u^\gamma,v)}$$

These formulae establish a precise connection between the space-time and the flux tube S-matrices. They hold at any coupling.

The flux tube S-matrices can be computed at any coupling using Integrability [Basso,Rej;Fioravanti,Piscaglia,Rossi;Basso,Sever,PV] The main ingredients are the solutions to the so called BES [Beisert,Eden,Staudacher] equation describing the flux tube vacuum.

- Multi-particles are built in terms of the single particle transitions:

$$P(\{u_i\}|\{v_j\}) = \frac{\prod_{i,j} P(u_i|v_j)}{\prod_{i>j} P(u_i|u_j)\prod_{i<j} P(v_i|v_j)} \quad \text{x (Group theory matrix part)}$$

(We believe we have an algorithm for getting the matrix part but so far we only checked it up to a small number of particles, at most 8)