Higher-Rank Fields, Currents, and Higher-Spin Holography

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+ work in progress

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HS AdS/CFT correspondence

General idea of HS duality Sundborg (2001), Witten (2001)

AdS_4 HS theory is dual to $3d$ vectorial conformal models

Klebanov, Polyakov (2002), Petkou, Leigh (2005), Sezgin, Sundell (2005); Giombi and Yin (2009);

Maldacena, Zhiboedov (2011,2012); MV (2012); Giombi, Klebanov; Tseytlin (2013,2014) ...

AdS_3/CFT_2 correspondence Gaberdiel and Gopakumar (2010)

Analysis of HS holography helps to uncover the origin of AdS/CFT

Unfolded Dynamics

Covariant first-order differential equations 1988

$$dW^{\Omega}(x) = G^{\Omega}(W(x))\,, \qquad G^{\Omega}(W) = \sum_{n=1}^{\infty} f^{\Omega}{}_{\Lambda_1\ldots\Lambda_n} W^{\Lambda_1}\wedge\ldots\wedge W^{\Lambda_n}$$

$d > 1$: Compatibility conditions

$$G^{\Lambda}(W)\wedge\frac{\partial G^{\Omega}(W)}{\partial W^{\Lambda}} \equiv 0$$

Manifest (HS) gauge invariance under the gauge transformation

$$\delta W^{\Omega} = d\varepsilon^{\Omega} + \varepsilon^{\Lambda}\frac{\partial G^{\Omega}(W)}{\partial W^{\Lambda}}\,, \qquad \varepsilon^{\Omega}(x) : (p_{\Omega}-1)-\ \text{form}$$

Geometry is encoded by $G^{\Omega}(W)$: unfolded equations make sense in any space-time

$$dW^{\Omega}(x) = G^{\Omega}(W(x))\,, \quad x \to X = (x,z)\,, \quad d_x \to d_X = d_x + d_z\,, \quad d_z = dz^u\frac{\partial}{\partial z^u}$$

X-dependence is reconstructed in terms of $W(X_0) = W(x_0, z_0)$ at any X_0

Classes of holographically dual models: different G

$3d$ conformal equations

Rank-one conformal massless equations Shaynkman, MV (2001)

$$\Big(\frac{\partial}{\partial x^{\alpha\beta}} \pm i\frac{\partial^2}{\partial y^\alpha \partial y^\beta}\Big)C_j^\pm(y|x) = 0\,, \qquad \alpha,\beta = 1,2\,, \quad j = 1,\dots\mathcal{N}$$

Bosons (fermions) are even (odd) functions of y: $C_i(-y|x) = (-1)^{p_i} C_i(y|x)$

Rank-two equations: conserved currents

$$\left\{\frac{\partial}{\partial x^{\alpha\beta}} - \frac{\partial^2}{\partial y^{(\alpha}\partial u^{\beta)}}\right\} J(u,\, y|x) = 0$$

Gelfond, MV (2003)

$J(u,\, y|x)$: **generalized stress tensor. Rank-two equation is obeyed by**

$$J(u,\, y\,|x) = \sum_{i=1}^{\mathcal{N}} C_i^-(u+y|x)\, C_i^+(y-u|x)$$

Primaries: $3d$ **currents of all integer and half-integer spins**

$$J(u,0|x) = \sum_{2s=0}^{\infty} u^{\alpha_1}\dots u^{\alpha_{2s}} J_{\alpha_1\dots\alpha_{2s}}(x)\,, \qquad \tilde{J}(0,y|x) = \sum_{2s=0}^{\infty} y^{\alpha_1}\dots y^{\alpha_{2s}} \tilde{J}_{\alpha_1\dots\alpha_{2s}}(x)$$

$$J^{asym}(u,y|x) = u_\alpha y^\alpha J^{asym}(x)$$

$$\Delta J_{\alpha_1\dots\alpha_{2s}}(x) = \Delta \tilde{J}_{\alpha_1\dots\alpha_{2s}}(x) = s+1 \qquad \Delta J^{asym}(x) = 2$$

Conservation equation: $\frac{\partial}{\partial x^{\alpha\beta}}\frac{\partial^2}{\partial u_\alpha \partial u_\beta} J(u,0|x) = 0$

Extension to $Sp(2M)$-invariant space

Rank-one unfolded equation (2001)

$$\left(\xi^{AB}\frac{\partial}{\partial X^{AB}} \pm i\sigma_-\right) C^{\pm}(Y|X) = 0\,, \qquad \sigma_- = \xi^{AB}\frac{\partial^2}{\partial Y^A \partial Y^B}\,,$$

Y^A - auxiliary commuting variables

X^{AB} matrix coordinates of $\mathcal{M}_M$, $X^{AB} = X^{BA}$ $(A, B = 1, \ldots, M = 2^n)$

Fronsdal (1985), Bandos, Lukierski, Sorokin (1999), MV (2001)

$\xi^{MN} = dX^{MN}$ are anti-commuting differentials $\xi^{MN}\xi^{AD} = -\xi^{AD}\xi^{MN}$

Rank-one primary (dynamical) fields : $\sigma_- C(X|Y) = 0$: $C(X)\,,\ C_A(X)Y^A$

Unfolded equations $\Rightarrow$ dynamical equations (2001)

$$\frac{\partial}{\partial X^{\mathbf{AE}}}\frac{\partial}{\partial X^{\mathbf{BD}}}C(X) - \frac{\partial}{\partial X^{\mathbf{BE}}}\frac{\partial}{\partial X^{\mathbf{AD}}}C(X) = 0 \qquad \text{Klein-Gordon–like}\,,$$

$$\frac{\partial}{\partial X^{\mathbf{BD}}}C_{\mathbf{A}}(X) - \frac{\partial}{\partial X^{\mathbf{AD}}}C_{\mathbf{B}}(X) = 0 \qquad \text{Dirac–like}$$

Extension to higher ranks and higher dimensions

Rank-$\mathbf{r}$ unfolded equations: $\mathbf{r}$ twistor variables Y_i^A $i, j, \ldots = 1, \ldots, \mathbf{r}$

$$\left(\xi^{AB}\frac{\partial}{\partial X^{AB}} \pm i\sigma_-^{\mathbf{r}}\right) C^{\pm}(Y|X) = 0\,, \qquad \sigma_-^{\mathbf{r}} = \xi^{AB}\sum_{j=1}^{\mathbf{r}} \frac{\partial^2}{\partial Y_j^A \partial Y_i^B}\,\delta_{ij}\,,$$

A rank-r field in $\mathcal{M}_M \sim$ a rank-one field in $\mathcal{M}_{\mathbf{r}M}$ with coordinates X_{ij}^{AB}.

$$Y_i^A \to Y^{\widetilde{A}}\,, \qquad \widetilde{A} = 1\ldots\mathbf{r}M$$

Embedding of $\mathcal{M}_M$ into $\mathcal{M}_{\mathbf{r}M}$: $X^{AB} \longrightarrow \tilde{X}^{\tilde{A}\tilde{B}}$

$$X_{11}^{AB} = X_{22}^{AB} = \ldots = X_{rr}^{AB} = X^{AB}$$

The map $\mathcal{M}_M \longrightarrow \mathcal{M}_{\mathbf{r}M}$ preserves $Sp(2M)$

Field-current correspondence: Flato-Fronsdal (1978) for $M = 2$

Alternative interpretation: multi-particle states (=higher-rank field) in lower dimension=single-particle states in higher dimensions

Problem: pattern of the holographic reduction of higher-dimensional models to the lower-dimensional ones

Rank-r fields and equations

Rank-r primary fields: $\sigma^{\mathbf{r}}_{-}\, C(Y|X) = 0, \qquad \sigma^{\mathbf{r}}_{-} = \xi^{AB} \sum_{j=1}^{\mathbf{r}} \frac{\partial^2}{\partial Y_j^A \partial Y_i^B}\, \delta_{ij}$

$$C(Y|X) = \sum_n C^{i_1;\dots;i_n}_{A_1;\dots;A_n}(X) Y^{A_1}_{i_1} \cdots Y^{A_n}_{i_n} \Rightarrow \text{tracelessness: } \delta_{\mathbf{i_1 i_2}} C^{\mathbf{i_1};\mathbf{i_2};\dots}_{\dots}(X) = 0.$$

Since $C^{\dots\mathbf{i_m}\dots\;\mathbf{i_k}\dots}_{\dots\mathbf{A_m}\dots\mathbf{A_k}\dots}(X) = C^{\dots\mathbf{i_k}\dots\;\mathbf{i_m}\dots}_{\dots\mathbf{A_k}\dots\mathbf{A_m}\dots}(X)$, **rank-r primary fields are described by -Young diagrams** $\mathbf{Y}^0[h_1, ..., h_m]$ **obeying** $h_1 + h_2 \le \mathbf{r}\,, \qquad h_1 \le M$

Rank-r primary fields $C_{\mathbf{Y}^0}(Y|X)$ **satisfy rank-r dynamical equations**

$$\mathcal{E}^{A_1[\mathbf{r}-h_2+1]\,,\,A_2[\mathbf{r}-h_1+1]\,,\,A_3[h_3],\dots,A_n[h_n]}_{i_1[h_1]\,,\qquad i_2[h_2]\,,\qquad i_3[h_3],\dots,\;i_n[h_n]} \underbrace{\frac{\partial}{\partial Y^{A_1^1}_{i_1^1}} \cdots \frac{\partial}{\partial Y^{A_1^{h_1}}_{i_1^{h_1}}}}_{h_1} \cdots \underbrace{\frac{\partial}{\partial Y^{A_n^1}_{i_n^1}} \cdots \frac{\partial}{\partial Y^{A_n^{h_n}}_{i_n^{h_n}}}}_{h_n}$$

$$\underbrace{\frac{\partial}{\partial X^{A_1^{h_1+1}\;A_2^{h_2+1}}} \cdots \frac{\partial}{\partial X^{A_1^{\mathbf{r}-h_2+1}\;A_2^{\mathbf{r}-h_1+1}}}}_{\mathbf{r}+1-h_1-h_2} C_{\mathbf{Y}^0}(Y|X) = 0\,.$$

The parameter $\mathcal{E}^{\dots}_{\dots}$ **projects to** $\mathbf{Y}^0[h_1, h_2, h_3, \dots, h_n]$ **and to its rank-r two-column dual** $\mathbf{Y}^1[\mathbf{r}+1-h_2, \mathbf{r}+1-h_1, h_3, \dots, h_n]$ **with respect to the lower and upper indices, respectively**

Multi-linear currents

For $\mathbf{r}=2\kappa$, a $(\kappa M-\frac{\kappa(\kappa-1)}{2})$-form

$$\Omega(J)=\mathcal{F}_{i_1[\kappa],\dots,i_N[\kappa]}\mathcal{D}^{A_1[\kappa],\dots,A_N[\kappa]}\underbrace{\frac{\partial}{\partial Y^{A_1^1}_{i_1^1}}\dots\frac{\partial}{\partial Y^{A_1^\kappa}_{i_1^\kappa}}}_{\kappa}\dots\underbrace{\frac{\partial}{\partial Y^{A_N^1}_{i_N^1}}\dots\frac{\partial}{\partial Y^{A_N^\kappa}_{i_N^\kappa}}}_{\kappa}J(Y|X)\Big|_{Y=0}$$

where $N=M+1-\kappa$ $\mathcal{F}$ is described by traceless diagram $\mathbf{Y}[\underbrace{\kappa,\dots\kappa}_{N}]$, and

$$\mathcal{D}^{A_1[\kappa],\dots,A_N[\kappa]}=\epsilon_{D_1^1\dots D_1^M}\dots\epsilon_{D_\kappa^1\dots D_\kappa^M}\xi^{D_1^1D_2^1}\xi^{D_1^2D_3^1}\dots\ \xi^{D_1^{\kappa-1}D_\kappa^1}\xi^{D_1^\kappa A_1^1}\dots\xi^{D_1^M A_N^1}\dots$$
$$\xi^{D_n^nD_{n+1}^n}\xi^{D_n^{n+1}D_{n+2}^n}\dots\xi^{D_n^{\kappa-1}D_\kappa^n}\xi^{D_n^\kappa A_1^n}\dots\xi^{D_n^M A_N^n}\dots\xi^{D_\kappa^\kappa A_1^\kappa}\xi^{D_\kappa^{\kappa+1}A_2^\kappa}\dots\xi^{D_\kappa^M A_N^\kappa}$$

is closed provided that $J(Y|X)$ obeys the rank-$\mathbf{r}=2\kappa$ equations.

The current

$$J_\eta(Y|X)=\eta^{j_1,\dots,j_\mathbf{r}}(\mathcal{A})C_{j_1}(Y_{j_1}|X)\dots C_{j_\mathbf{r}}(Y_{j_\mathbf{r}}|X)$$

where $\mathcal{A}^{1}_{j}{}^{B}(Y_j|X)=2X^{AB}\frac{\partial}{\partial Y_j^A}+Y_j^B,\quad \mathcal{A}^2_j{}_C(Y_j|X)=\frac{\partial}{\partial Y_j^C}$

and $C_j(Y|X)$ – rank-one fields, generates $\mathbf{r}$-linear charge $Q^{\mathbf{r}}_\eta(C)$

Multiparticle algebra: string-like HS algebra (2012)

σ_--cohomology analysis

Rank-$\mathbf{r}$ primary fields and field equations are represented by the cohomology groups $H^0(\sigma_-^{\mathbf{r}})$ and $H^1(\sigma_-^{\mathbf{r}})$, respectively.

Higher $H^p(\sigma_-)$ (and their twisted cousins) are responsible for HS gauge fields and their field equations

General $H^p(\sigma_-)$ via homotopy trick: conjugated operators Ω and Ω^*

$$\Omega := \sigma_-^{\mathbf{r}} = T_{AB}\xi^{AB} \ , \qquad \Omega^* = T^{AB}\frac{\partial}{\partial \xi^{CD}} \, ,$$

$$T_{AB} = \frac{\partial}{\partial Y_i^A}\frac{\partial}{\partial Y_j^B}\delta^{ij} , \quad T^{CD} = Y_i^C Y_j^D \delta^{ij} \, , \quad T_B^A = Y_j^A \frac{\partial}{\partial Y_j^B} \qquad = \mathfrak{sp}(2M)$$

$$\Delta = \{\Omega\, , \Omega^*\} \;= \tfrac{1}{2}\tau_{mk}\tau^{mk} + \nu_B^A \nu_B^A - (M+1-\mathbf{r})\nu_A^A$$

$$\tau_{mk} = Y_m{}^A \frac{\partial}{\partial Y^{kA}} - Y_k{}^A \frac{\partial}{\partial Y^{mA}}$$ are $\mathfrak{o}(\mathbf{r})-$ -generators

$$\nu_B^A = 2\xi^{AD}\frac{\partial}{\partial \xi^{BD}} + T_B^A$$ are $\mathfrak{gl}_M^{tot}$-generators that act on Y_i^A and ξ^{AB}

Δ is semi positive-definite

$$\mathrm{H}(\Omega) \subset \ker \Delta$$

Young diagrams and South-West principle

$$\mathbf{Y}'[B_1\ldots] \subset \mathbf{Y}[h_1\ldots] \otimes (\otimes_n \mathbf{Y}_\delta[1,1]) \otimes \mathbf{Y}_A[a_1,\ldots]$$

where n is a number of $\mathfrak{o}(\mathbf{r})$ metric tensors δ_{ij},

$\mathbf{Y}[h_1\ldots,h_k]$ $\mathfrak{o}(\mathbf{r})$ YD , $\mathbf{Y}'[B_1\ldots,B_m]$ $\mathfrak{gl}_M$: YD, $\mathbf{Y}_A[a_1,\ldots]$: ξ^{AB} YD

$$\tau_{mk}\tau^{mk} = 2\sum_j h_j(h_j - \mathbf{r} - 2(i-1))\,, \qquad \nu^A_B\nu^A_B = -\sum_i B_i(B_i - M - 1 - 2(i-1))\,,$$

$$\Rightarrow \qquad \Delta = -\sum_i B_i(B_i - 2(i-1)) + \sum_j h_i(h_i - 2(i-1)) + \mathbf{r}\sum_i (B_i - h_i)\,.$$

$\chi^a(\mathcal{S}(i,j)) = i - j + a\,,\quad a \in \mathbb{R}$,

$\mathcal{S}(i,j)$ – a sell on the intersection of $j-th$ row and $i-th$ column

$$\mathbf{Y} = \bigcup_{\mathcal{S}(i,j)\in\mathbf{Y}} \mathcal{S}(i,j) \qquad \chi^a(\mathbf{Y}) = -\frac{1}{2}\sum_i h_i(h_i - 2i + 1 - 2a)$$

Δ semi-positive $\Rightarrow$ $\min(\Delta)$ is reached when all cells of $\mathbf{Y}'$ are maximally south-west. This allows us to find $H^p(\sigma_-^{\mathbf{r}})\ \forall p$

Higher-differential forms are relevant to the nonlinear field equations and invariant Lagrangians for multiparticle theory

Invariant functionals

Unfolded equations

$$dF(W) = QF(W)\,,$$

$F(W)$ is an arbitrary function of W

$$Q = G^{\Omega}\frac{\partial}{\partial W^{\Omega}}\,, \qquad Q^2 = 0$$

Q-closed p-form functions $L_p(W)$ are d-closed, giving rise to the gauge invariant functionals represented by Q-cohomology (2005)

$$S = \int_{\Sigma^p} L_p$$

So defined L_p is d-closed in any space-time realization

S is gauge invariant in any space-time

Nonlinear HS Equations

One-form $\mathcal{W} = d_x + dx^\nu W_\nu + dZ^A S_A$ **and zero-form** B:

$$\mathcal{W} \star \mathcal{W} = i(dZ^A dZ_A + F(B, dz^\alpha dz_\alpha \star k\kappa, d\bar{z}^{\dot\alpha} d\bar{z}_{\dot\alpha} \star \bar{k}\bar{\kappa}))$$

$$\mathcal{W} \star B - B \star \mathcal{W} = 0$$

HS star product

$$(f \star g)(Z,Y) = \int dS dT f(Z+S, Y+S) g(Z-T, Y+T) \exp -iS_A T^A$$

$$[Y_A, Y_B]_\star = -[Z_A, Z_B]_\star = 2iC_{AB}\,, \qquad Z - Y \,:\, Z + Y \text{ normal ordering}$$

Inner Klein operators:

$$\kappa = \exp iz_\alpha y^\alpha\,, \qquad \bar\kappa = \exp i\bar{z}_{\dot\alpha}\bar{y}^{\dot\alpha}\,, \qquad \kappa \star f(y,\bar{y}) = f(-y,\bar{y}) \star \kappa\,, \qquad \kappa \star \kappa = 1$$

Nontrivial equations are free of the space-time differentials d

Action is not known but probably is not needed

Generating functional

$C(y,\bar{y}|x) = B(0,y,\bar{y}|x)$: $d=4$ **rank-one field as** $d=3$ **rank-two currents**

$\omega(y,\bar{y}|x) := W(0,y,\bar{y}|x)$: $d=4$ **gauge field as** $d=3$ **conformal gauge field**

$$C(y,\bar{y}|x) = D^s\omega(y,\bar{y}|x)$$

Quadratic functional

$$S = \frac{1}{2}\int d^3x \langle e_0 e_0 \omega C\rangle$$

Non-linear on-shell Lagrangian L **results from the extended HS system**

$$\mathcal{W}\star\mathcal{W} = i(dZ^A dZ_A + F(\mathcal{B}, dz^\alpha dz_\alpha \star k\kappa, d\bar{z}^{\dot\alpha} d\bar{z}_{\dot\alpha}\star\bar{k}\bar{\kappa}))\,, \qquad \mathcal{W}\star\mathcal{B} - \mathcal{B}\star\mathcal{W} = 0$$

$$\mathcal{W} = d_x + dx^\nu W_\nu + dZ^A S_A + dx^{\nu_1}dx^{\nu_2}dx^{\nu_3}W_{\nu_1\nu_2\nu_3} + \ldots\,, \quad \mathcal{B} = B + dx^{\nu_1}dx^{\nu_2}B_{\nu_1\nu_2} + \ldots$$

Generating functional

HS fields are meromorphic at $z = 0$ (at least in the lowest orders) 2012

Extension to complex z via unfolded formulation

Generating functional:

$$Z(\omega) = exp i\, S\,, \qquad S = \oint_{S^1} \int_{\partial AdS} L\,,$$

n**-point functions**

$$\langle J(x_1) \dots J(x_n) \rangle = \frac{\delta^n}{\delta\omega(x_1) \dots \delta\omega(x_n)} Z(\omega) \Big|_{\omega=0}$$

Conclusions

All $Sp(2M)$ invariant fields, currents and field equations in $\mathcal{M}_M$

Higher-rank fields as multiparticle states and/or single-particle states in higher dimensions

Construction of invariant functionals in interacting HS theories

Nonlinear terms in $F(B)$ are ruled out by conformal symmetry