M-strings

Strings 2013

Sogang University
Seoul, Korea

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Based mainly on

arXiv:1305.6322 \`M-Strings', with
Babak Haghighat, Amer Iqbal, Can Kozcaz, Guglielmo Lockhart

And partly on

arXiv:1210.3605 \`BPS Degeneracies and Superconformal Index in Diverse Dimensions', with Amer Iqbal

arXiv:1210.5909 \`Superconformal Partition Functions and Non-perturbative Topological Strings', with Guglielmo Lockhart

See also the related talks in this conference and in particular by Schwarz, Kim, Pasquetti and Gukov, Douglas and Jafferis
and also the work [Kim,Kim,Koh,Lee,Lee]

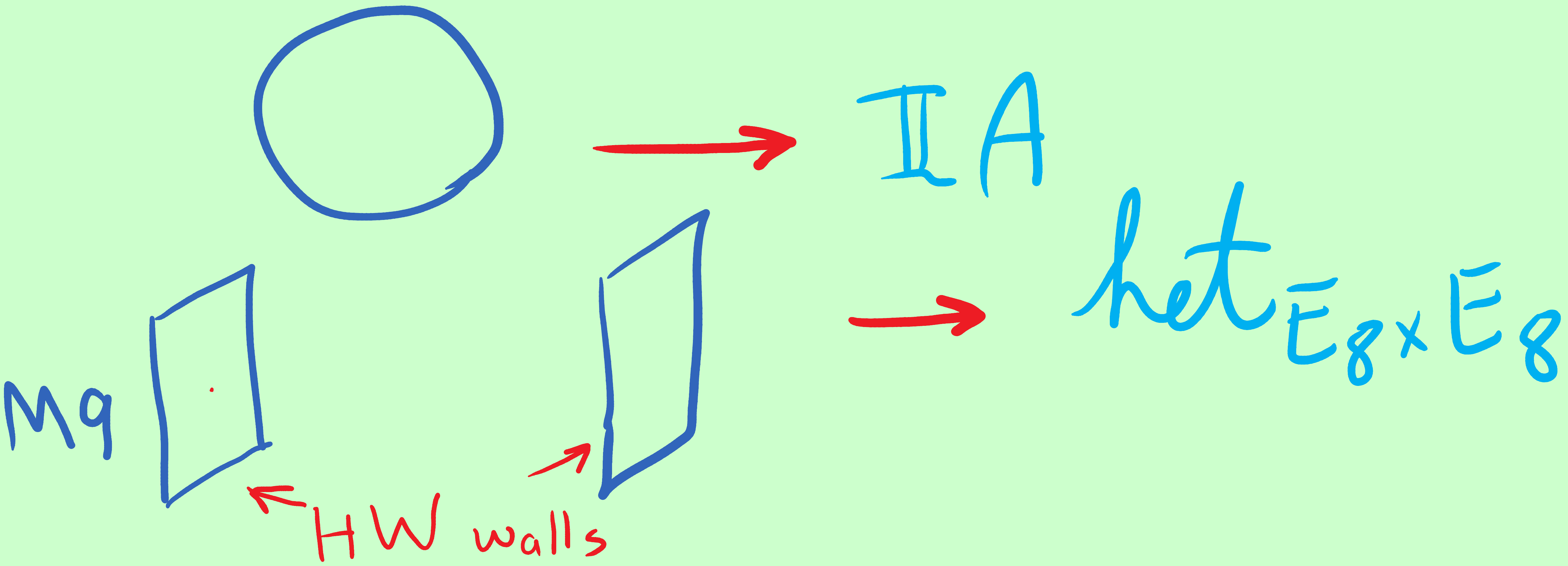
Outline:

1-M5 brane (2,0) theory
2-Suspended M2 branes: M-strings
3-Use Topological Strings: elliptic genus of M-strings (DH states)
4-Interpretation of elliptic genus of M-strings
5-Computation of 6d (2,0) superconformal index from M-strings

M-strings

M2 branes wrapping cycles or intervals can lead, using the duality web, to all the known strings.

Type IIA strings and heterotic strings directly,
and type IIB and type I by duality chain.



M-strings

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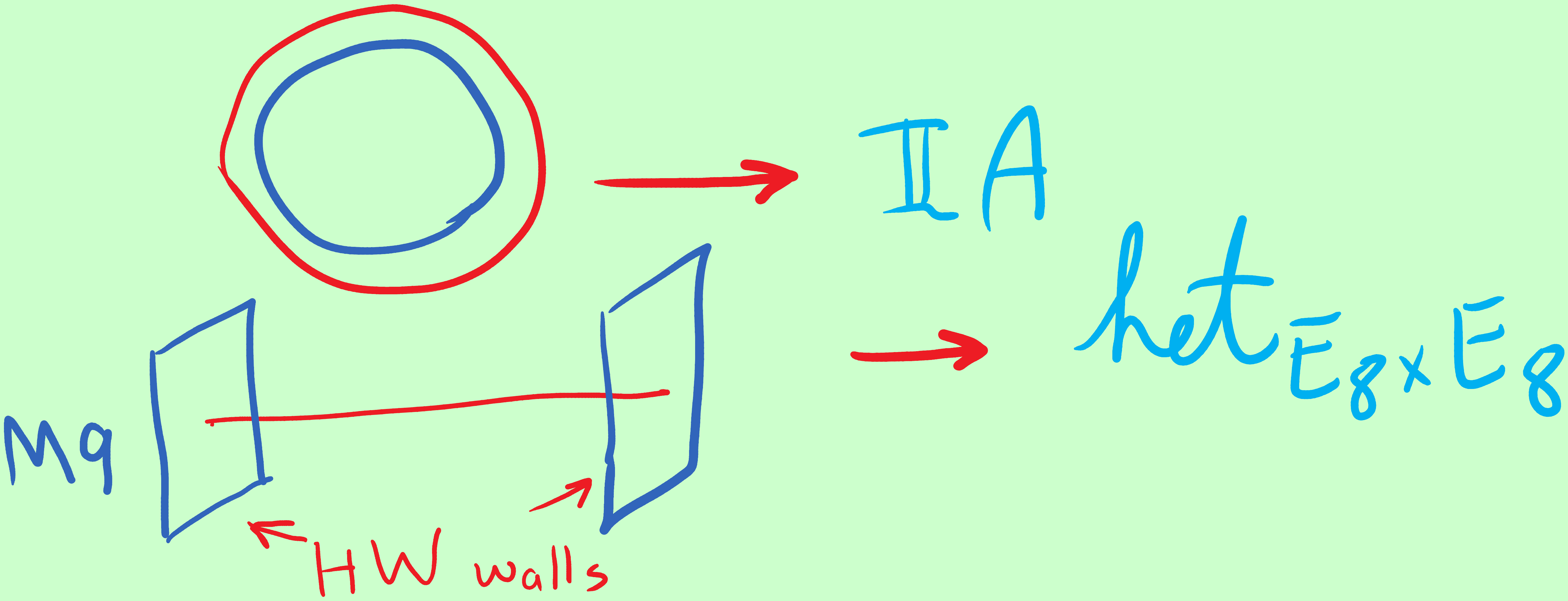
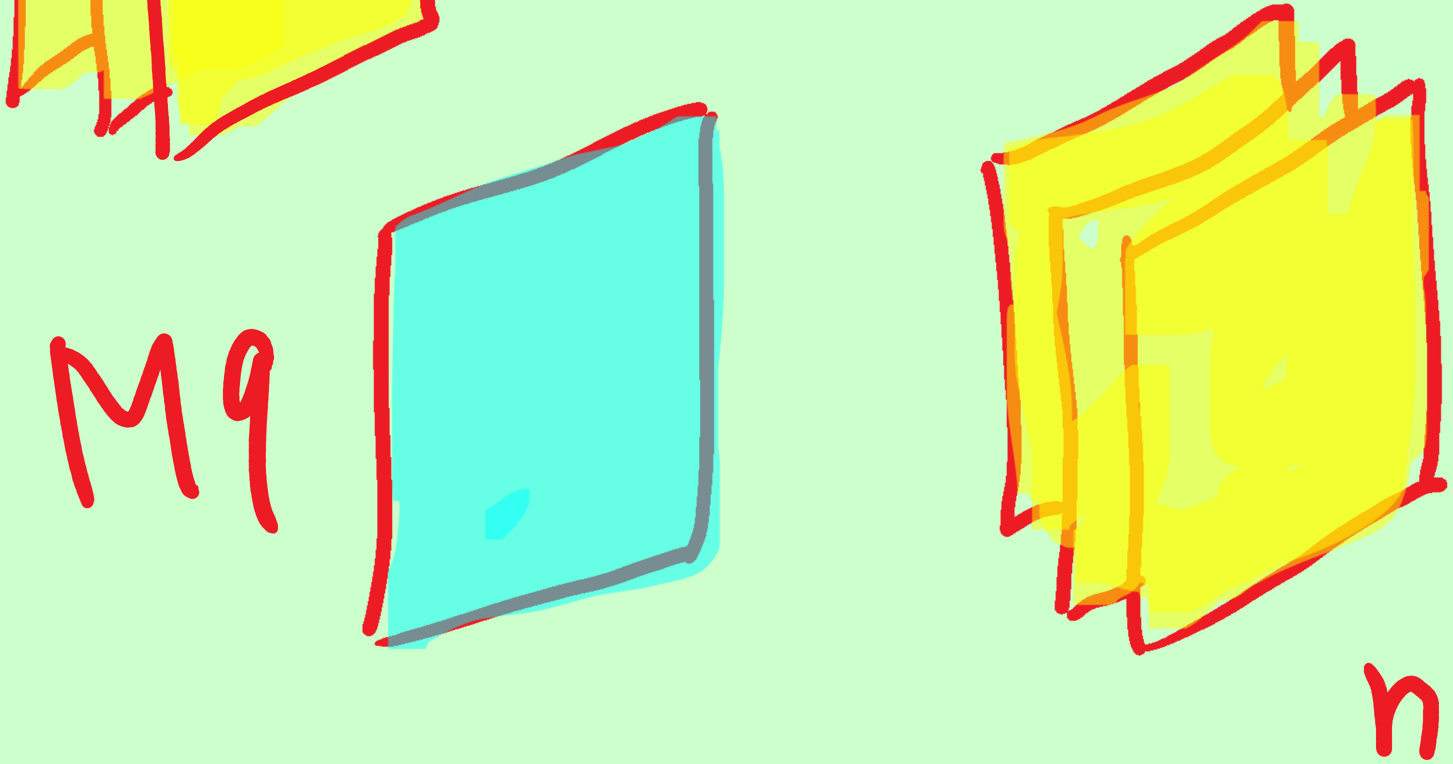


This naturally raises the question of whether the enigmatic 6d (2,0) and (1,0) SCFT's whose existence is signaled by the appearance of tensionless strings lead to some effective perturbative scheme involving light strings?

(2,0):

A_{n-1}

(1,0):

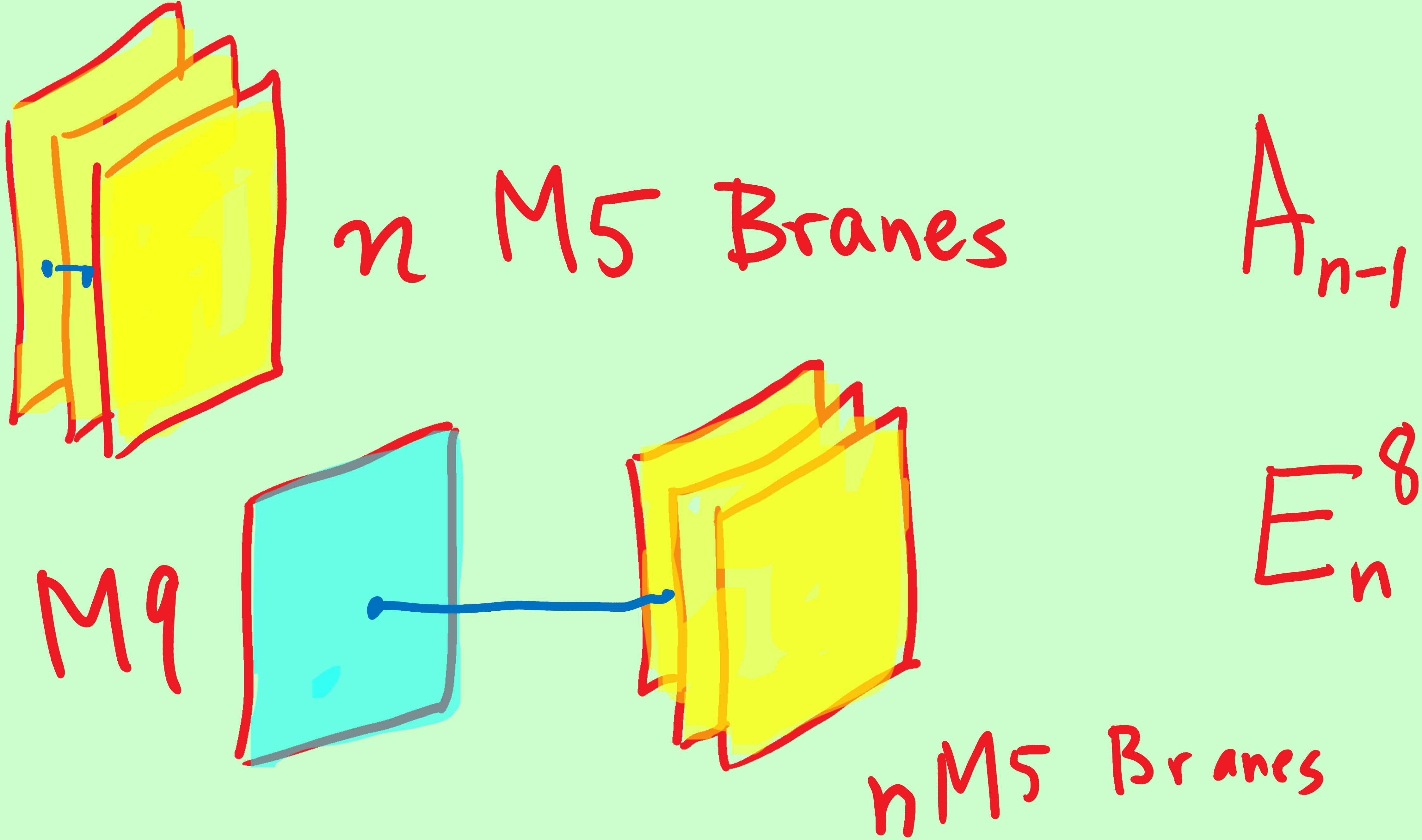


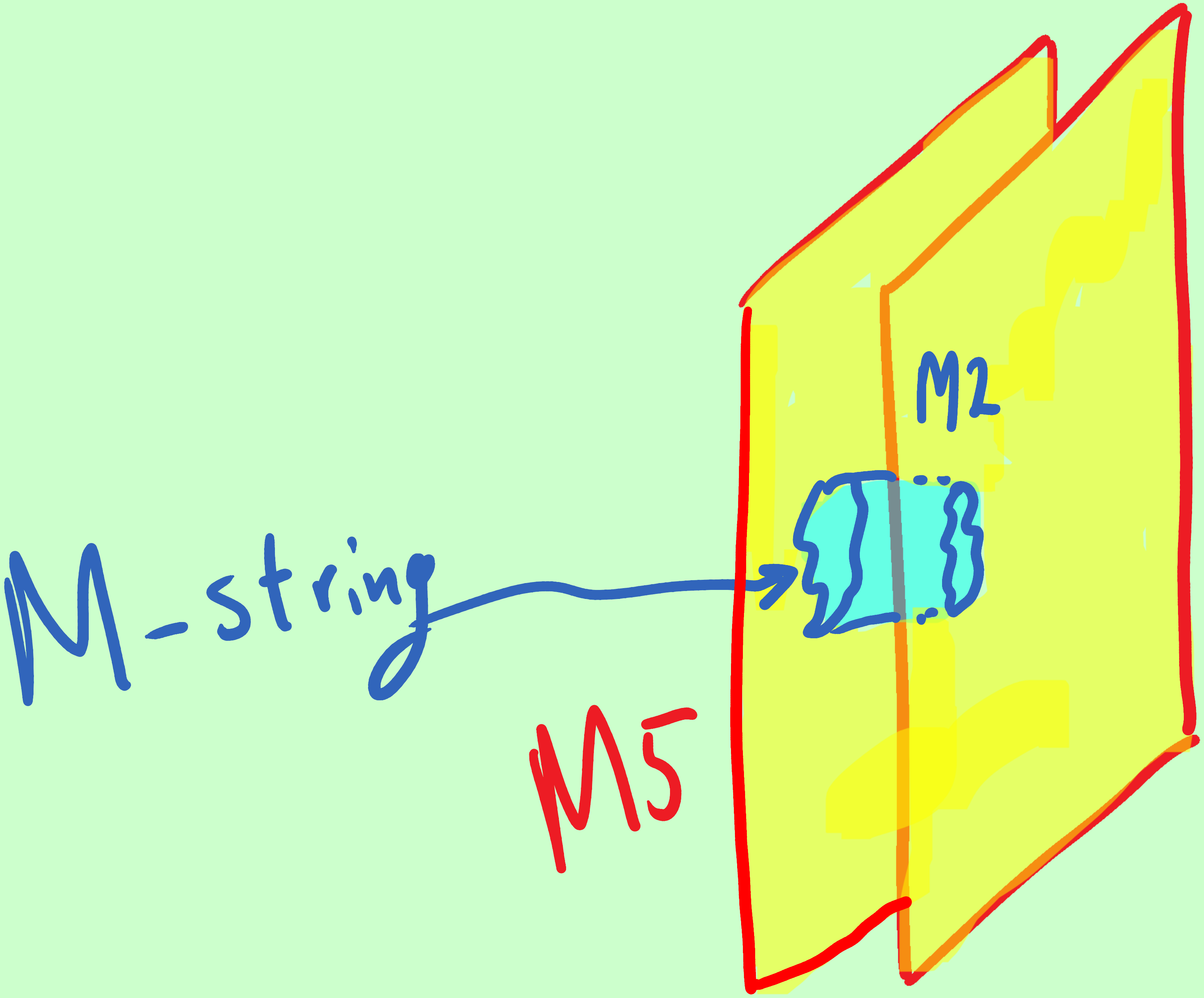
E^{8}_{n}

This naturally raises the question of whether the enigmatic 6d (2,0) and (1,0) SCFT’s whose existence is signaled by the appearance of tensionless strings lead to some effective perturbative scheme involving strings?

(2,0):

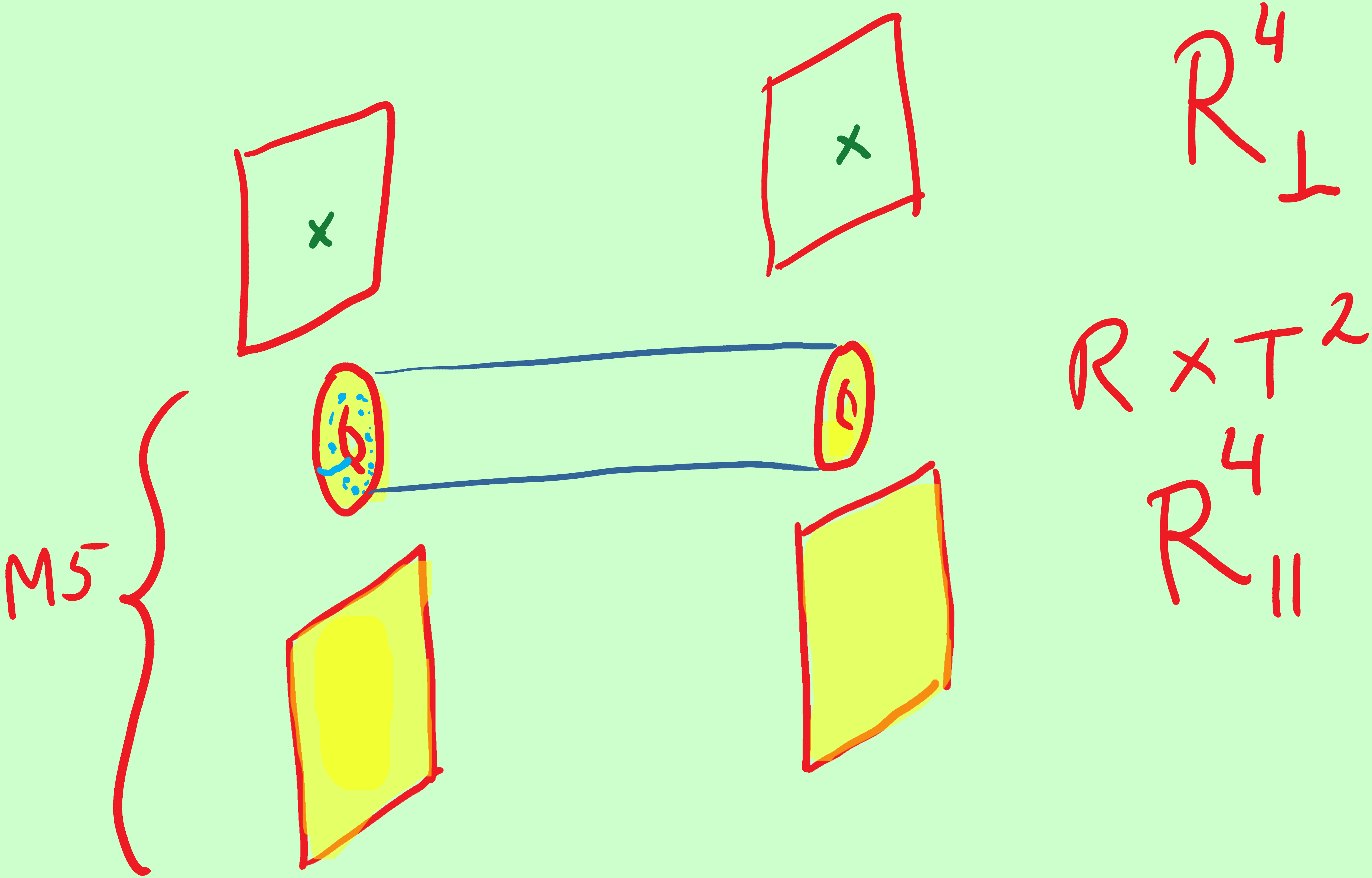
(1,0):



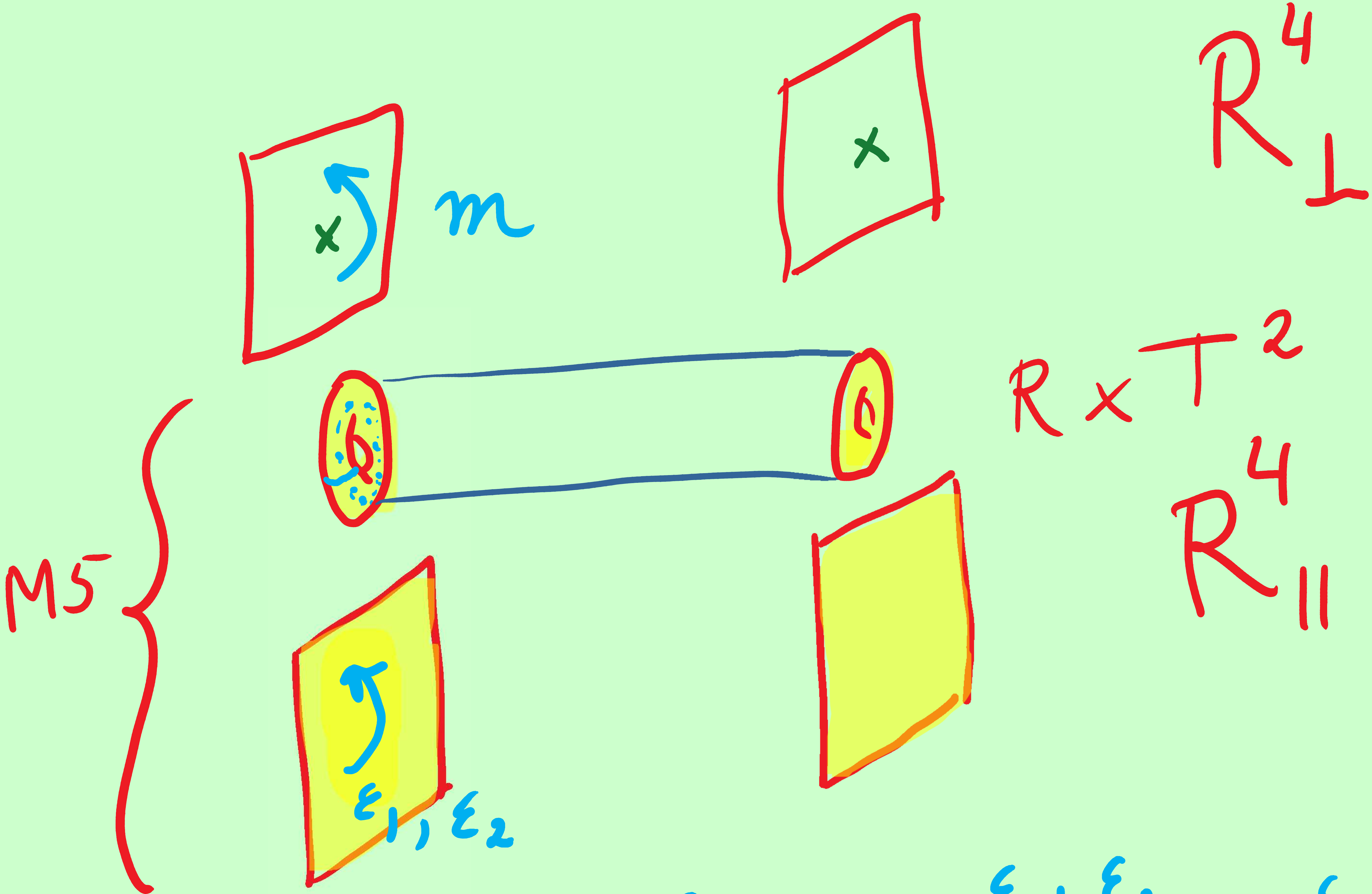


We are interested in computation of the supersymmetric partition function of M-strings on 2-torus with possible twists around cycles:

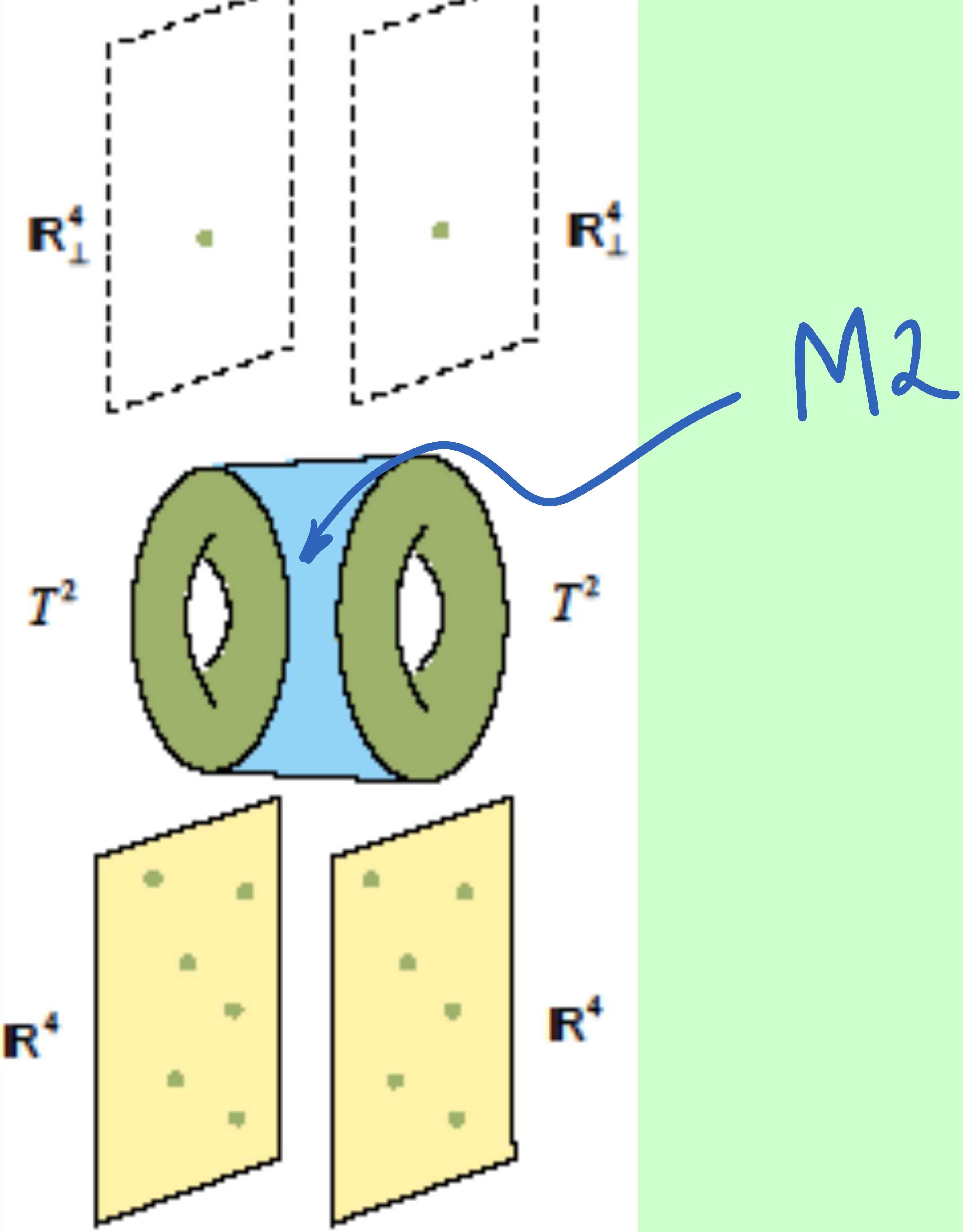
Relevant M5 brane geometry:



Relevant M5 brane geometry:



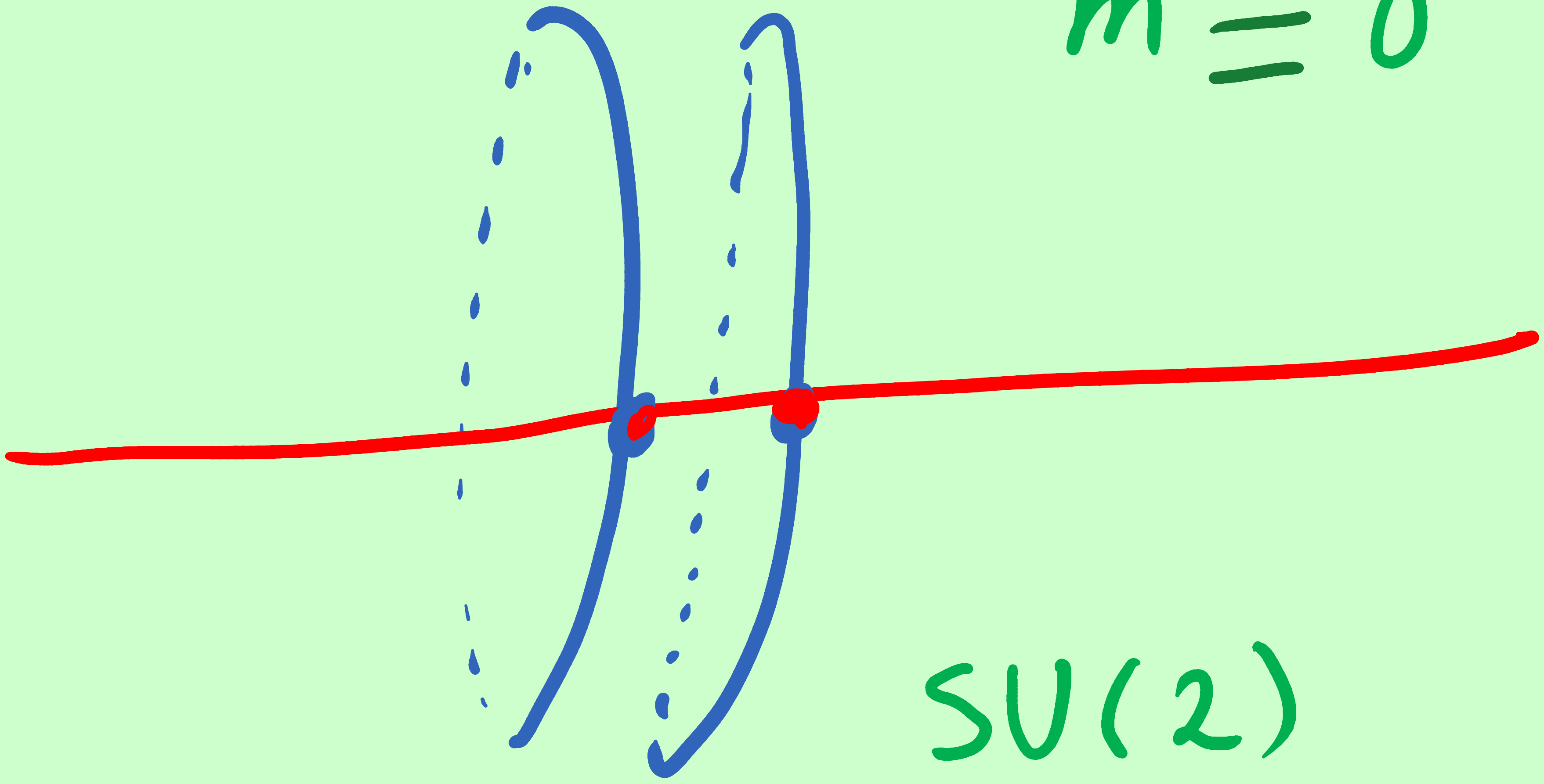
$$SO_\parallel(4)\times SO_\perp(4) \supset \left(\varepsilon_1, \varepsilon_2, m-\frac{\varepsilon_1+\varepsilon_2}{2}, -m-\frac{\varepsilon_1+\varepsilon_2}{2}\right)$$


$\mathbf{R}_\perp^4$
$\mathbf{R}_\perp^4$
M2
T^2
T^2
$\mathbf{R}^4$
$\mathbf{R}^4$

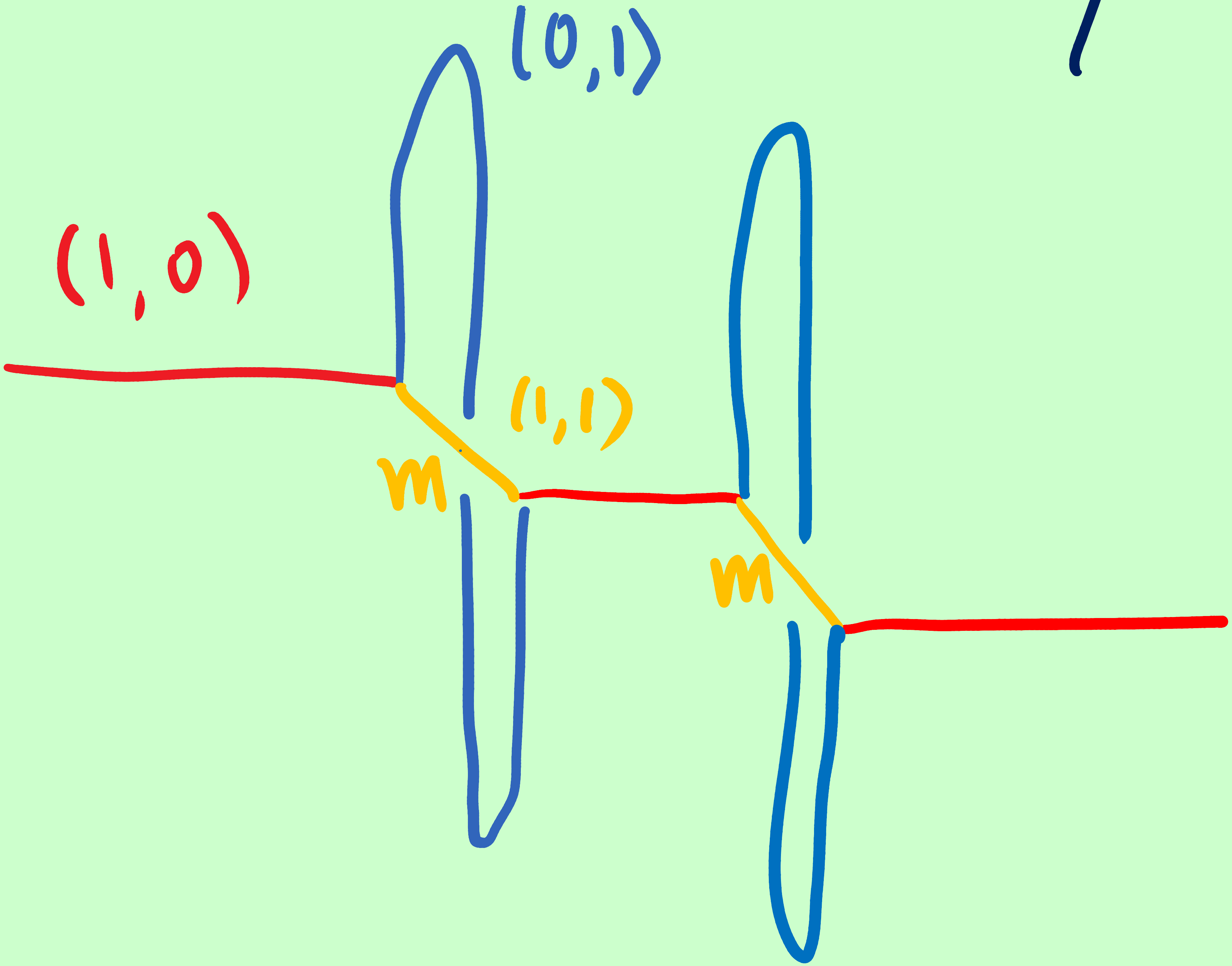
Compactifying the M5 branes on a cirlce leads to a theory which in the IR has SU(N) gauge symmetry.

Adding a mass term leads to the 5d parent of 4d N=2*.

The 5d N=1* theory can be geometrically engineered using elliptic Calabi-Yau or equivalently type IIB (p,q) 5-branes (parallels Witten’s 4d construction):

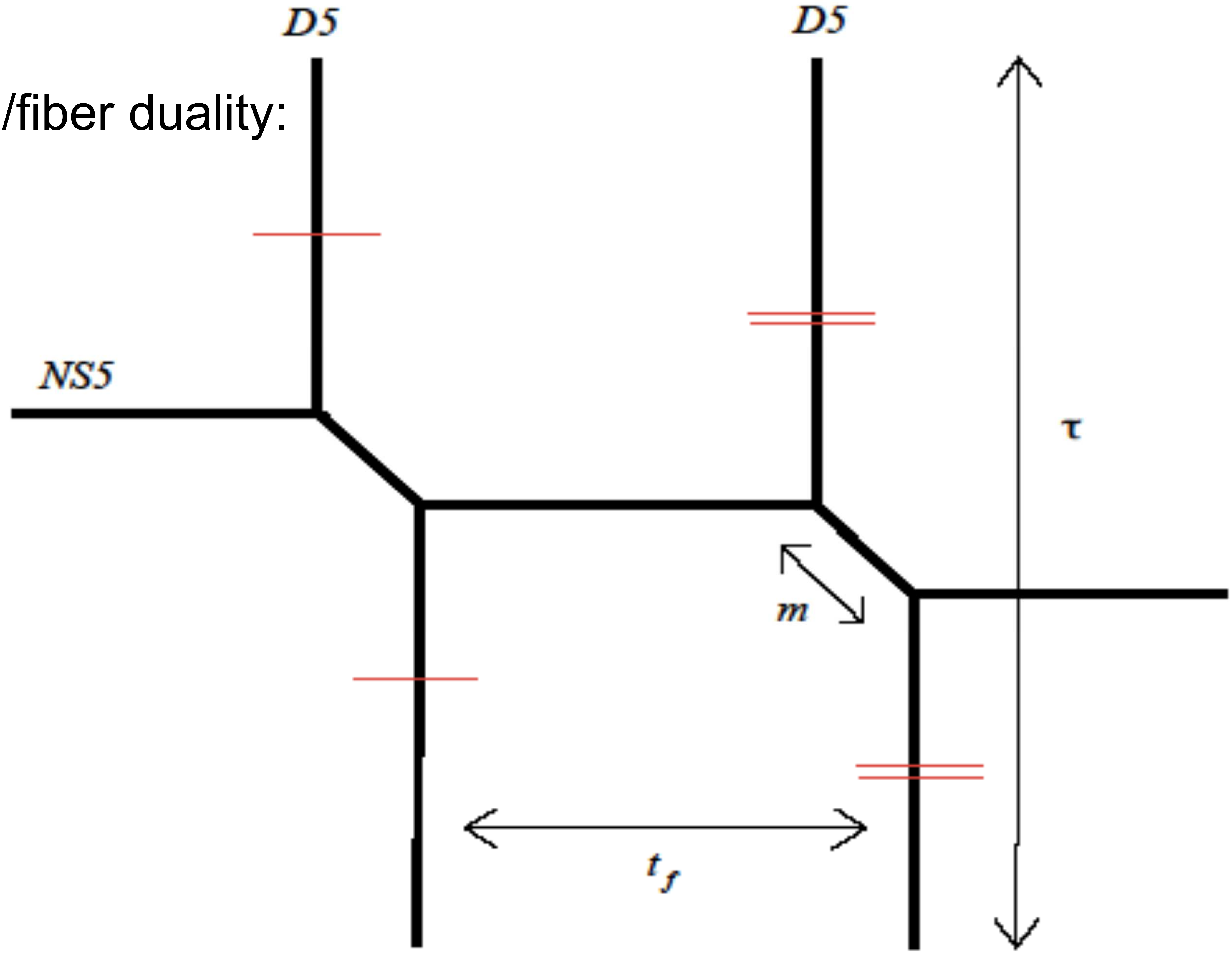

$m = 0$
SU(2)

Toric / (p,q) 5-brane



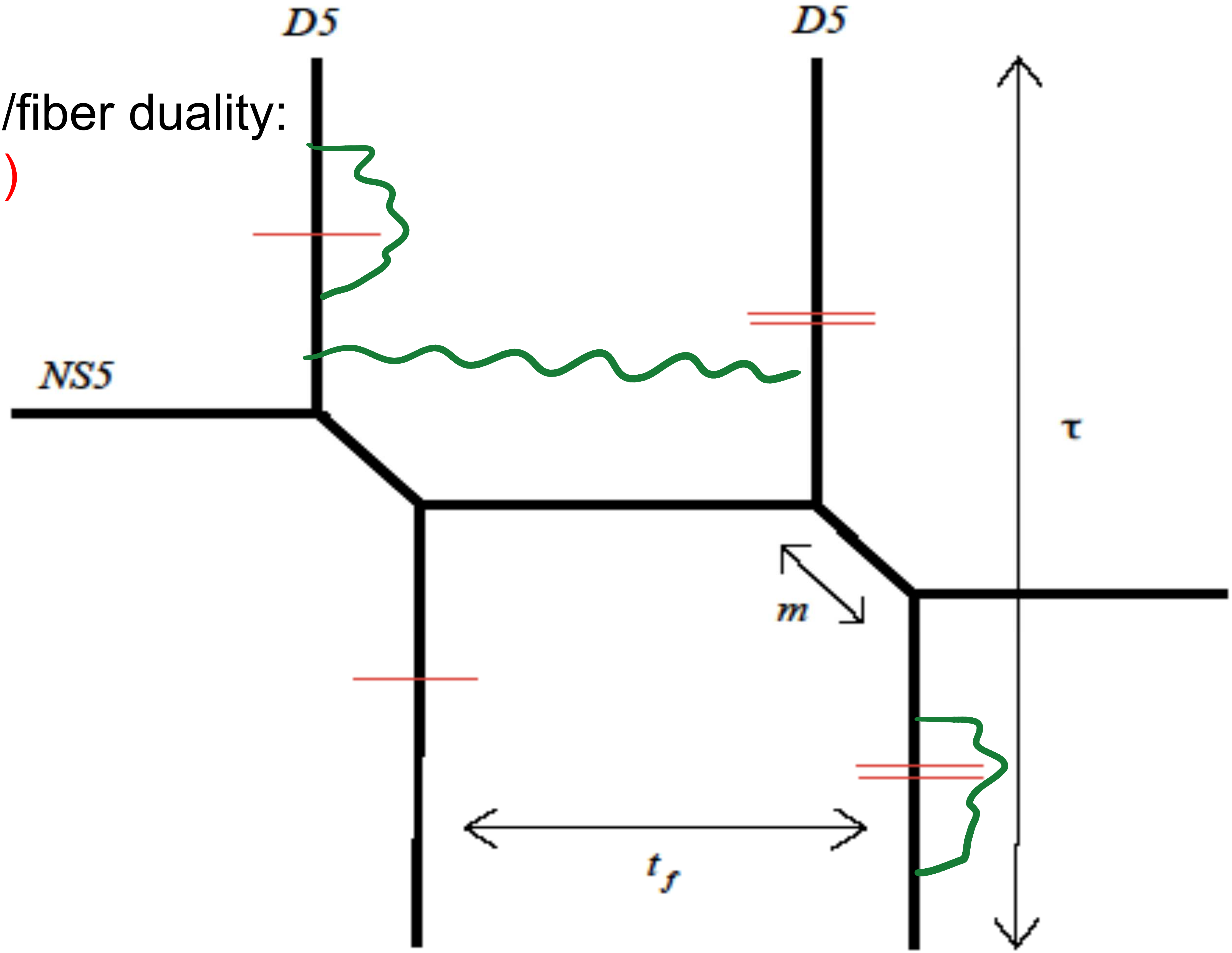
Two M5 branes

Base/fiber duality:



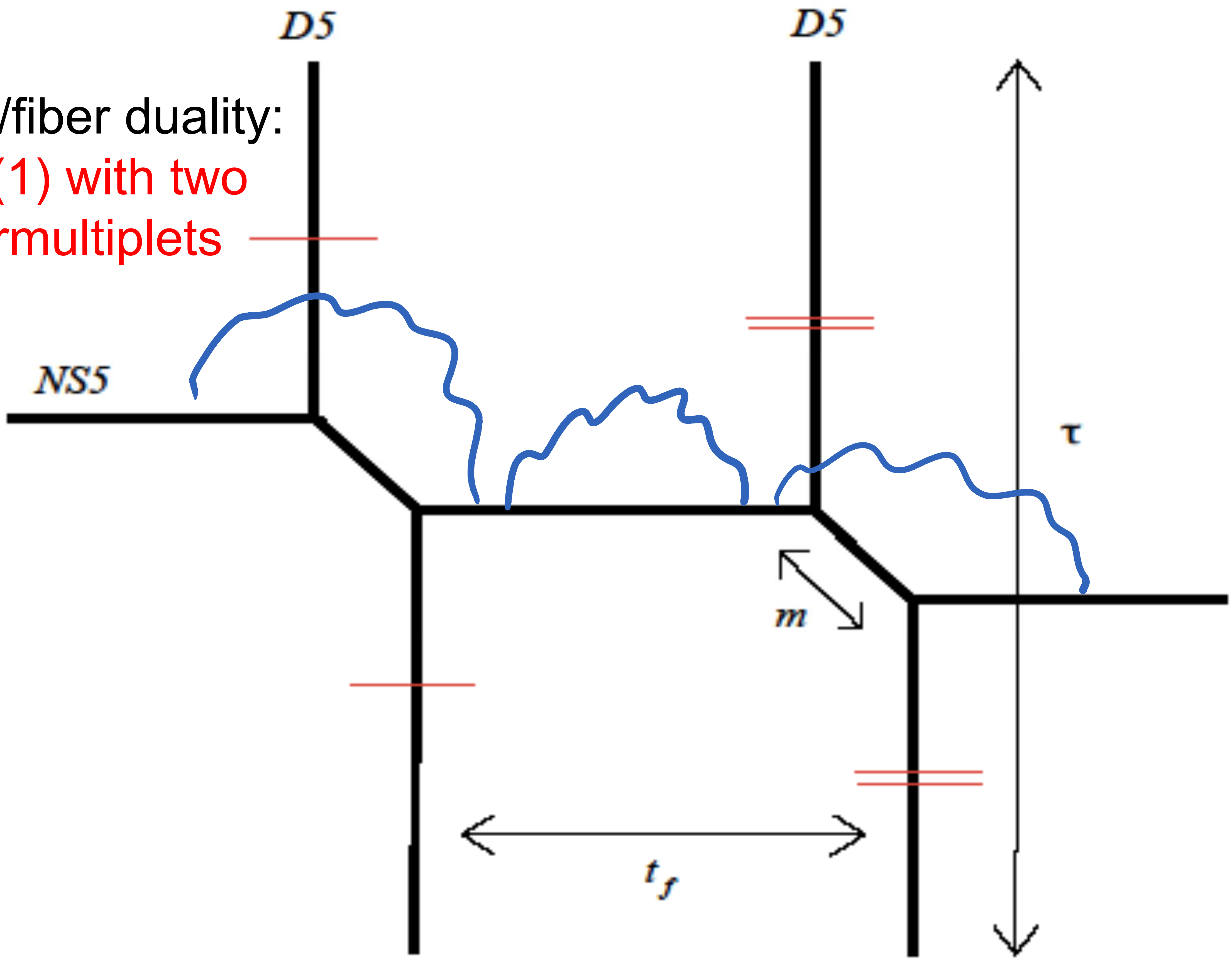
Base/fiber duality:

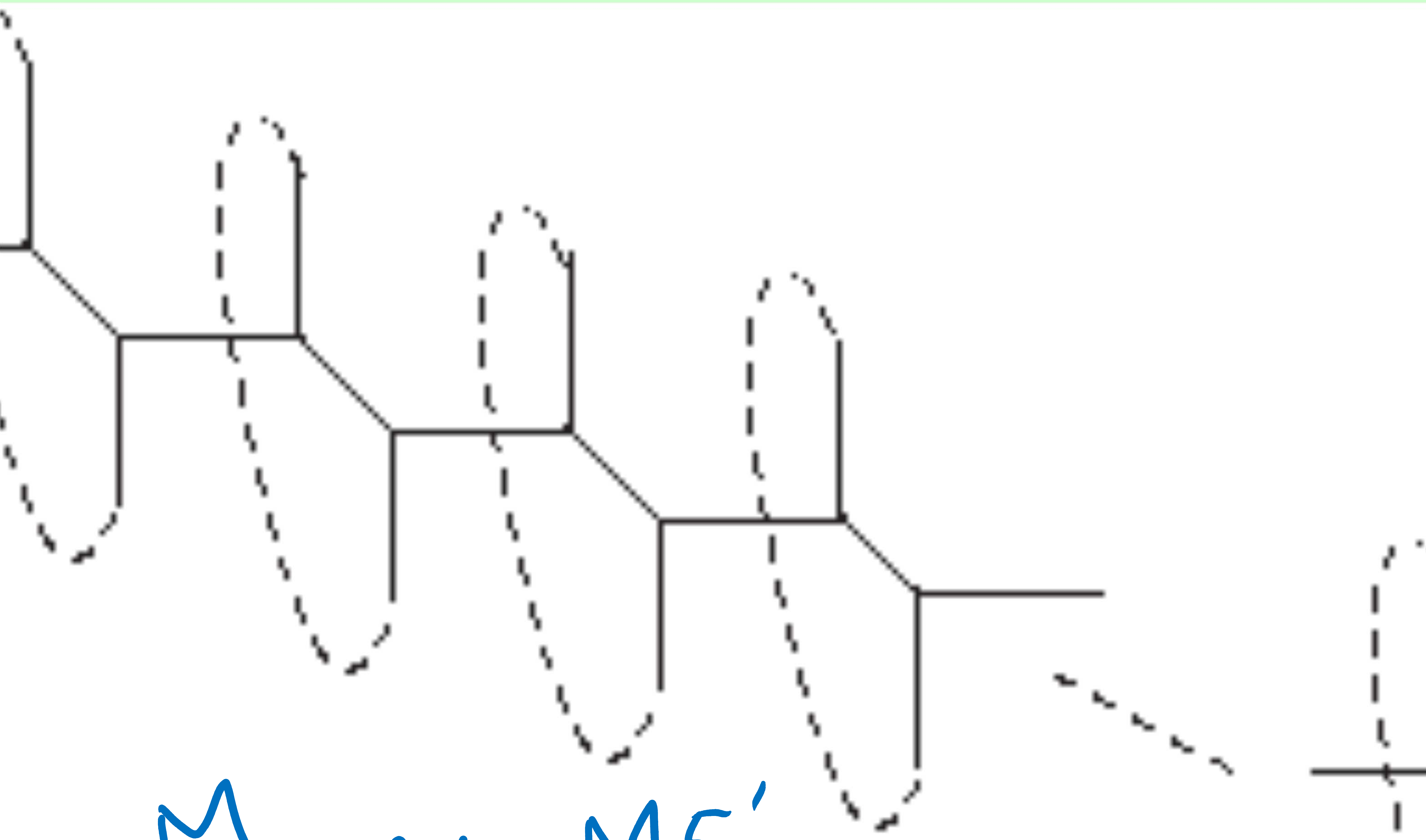
SU(2)



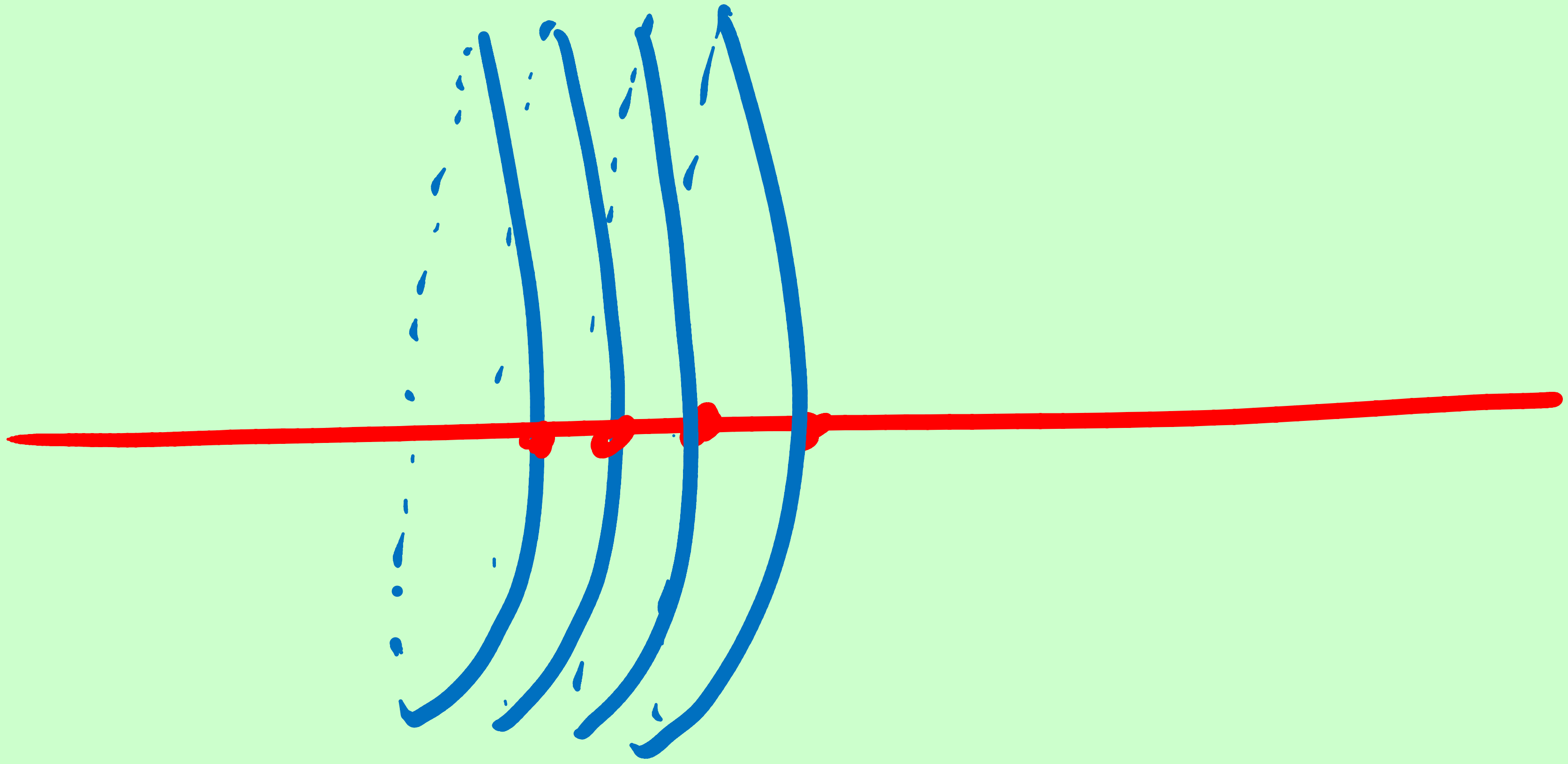
Base/fiber duality:

6d U(1) with two hypermultiplets

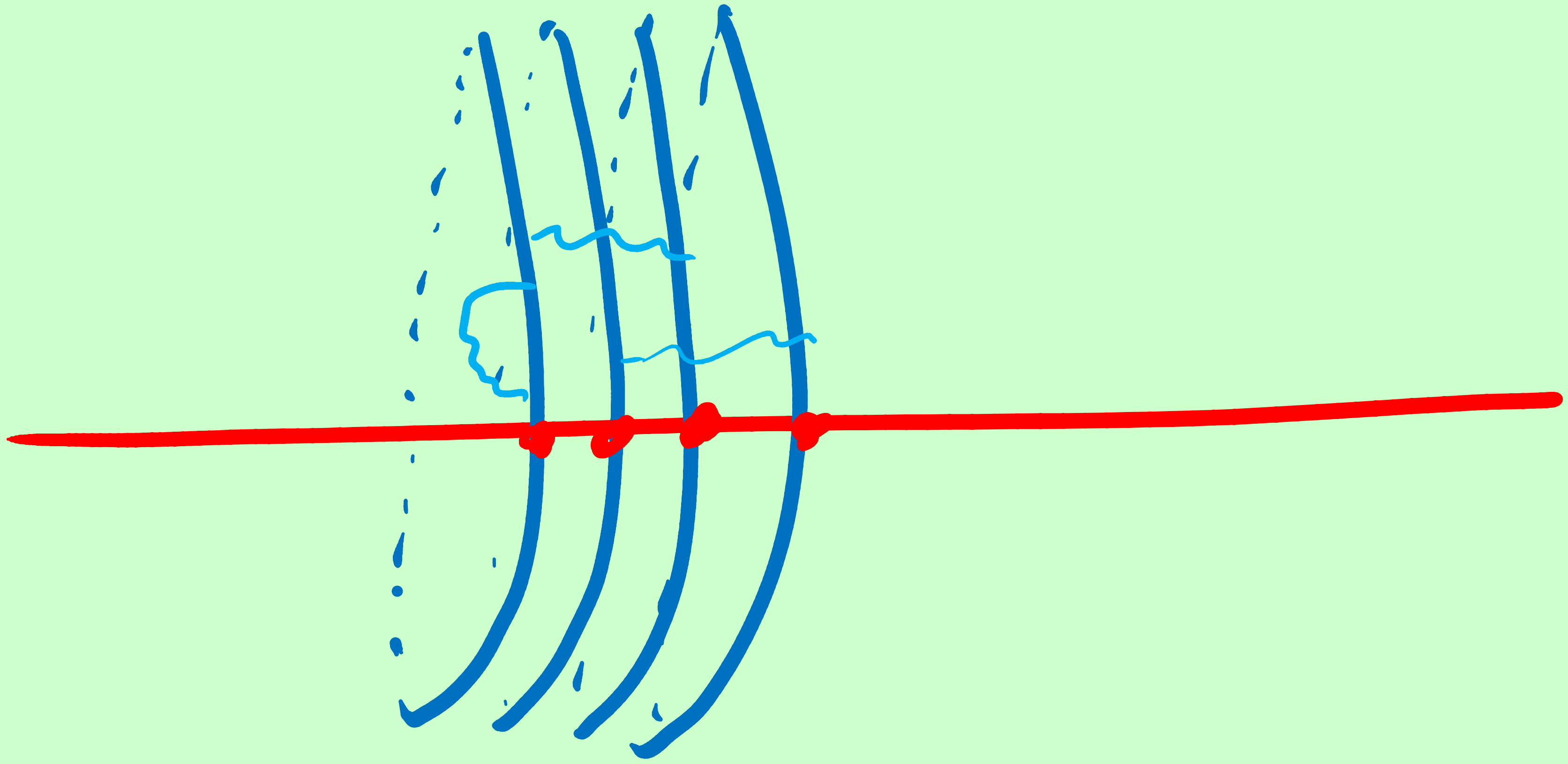




Many M5's

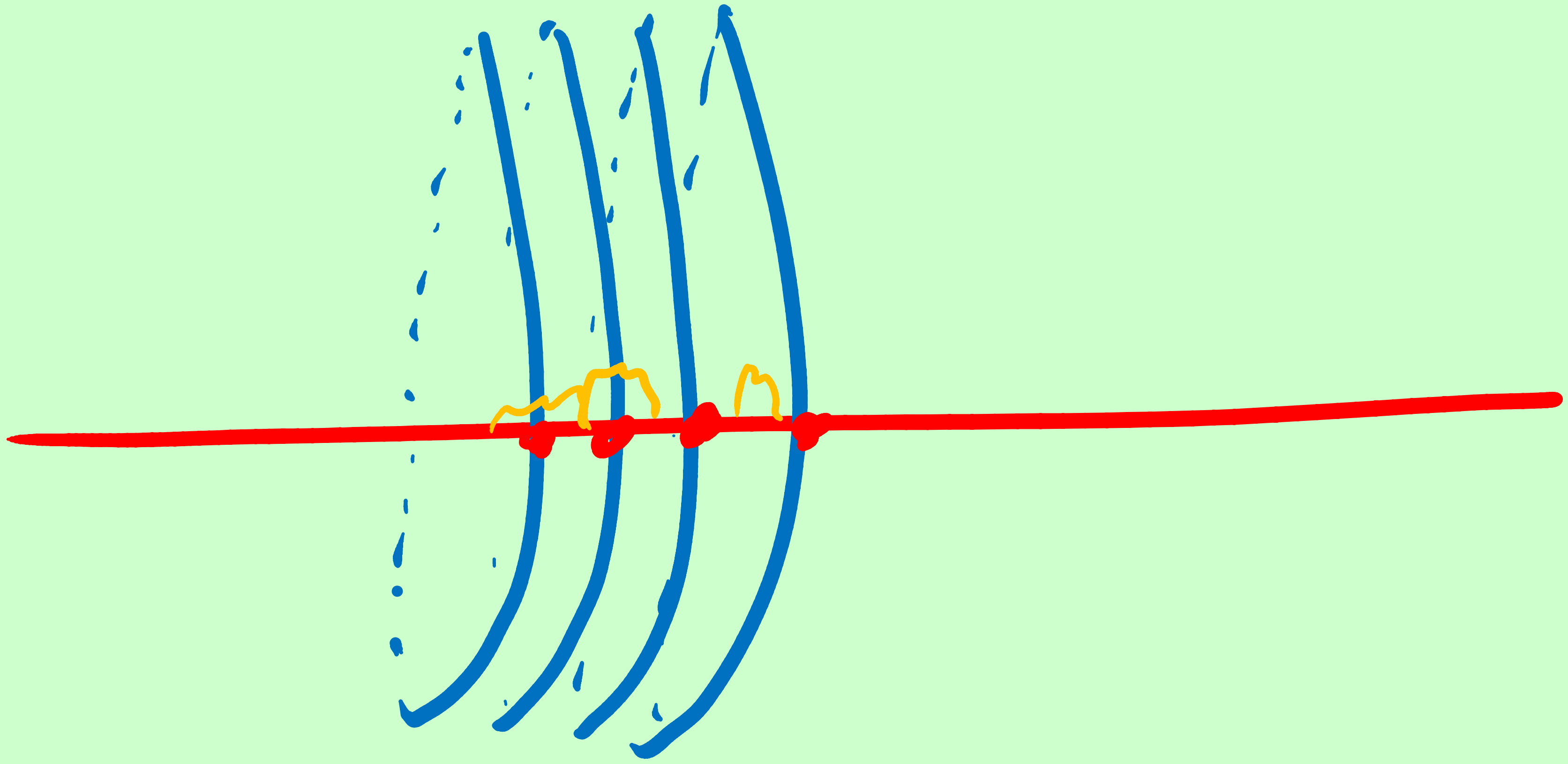


Fiber/Base duality:



Fiber/Base duality:

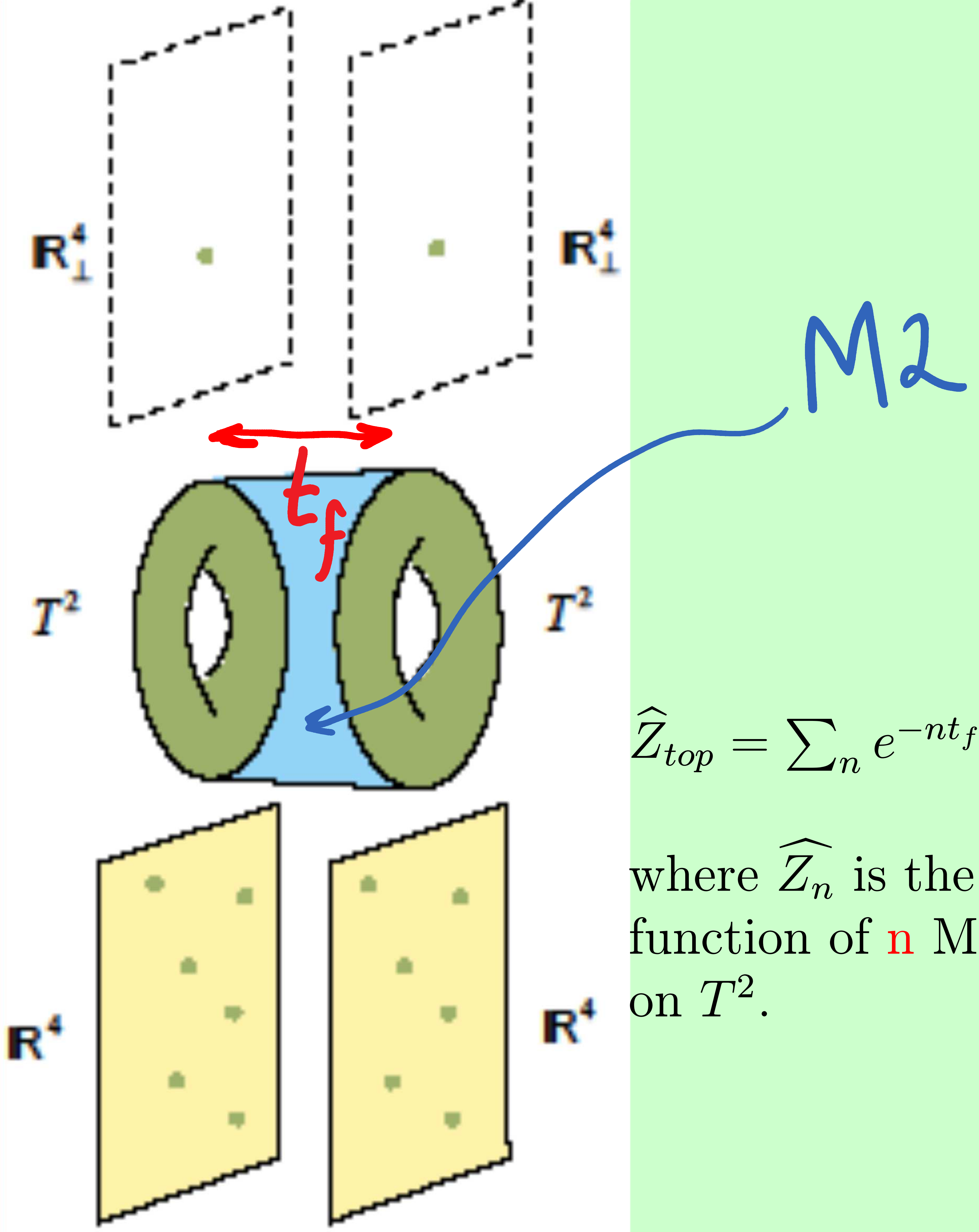
5d SU(N) living on the vertical branes



Fiber/Base duality:

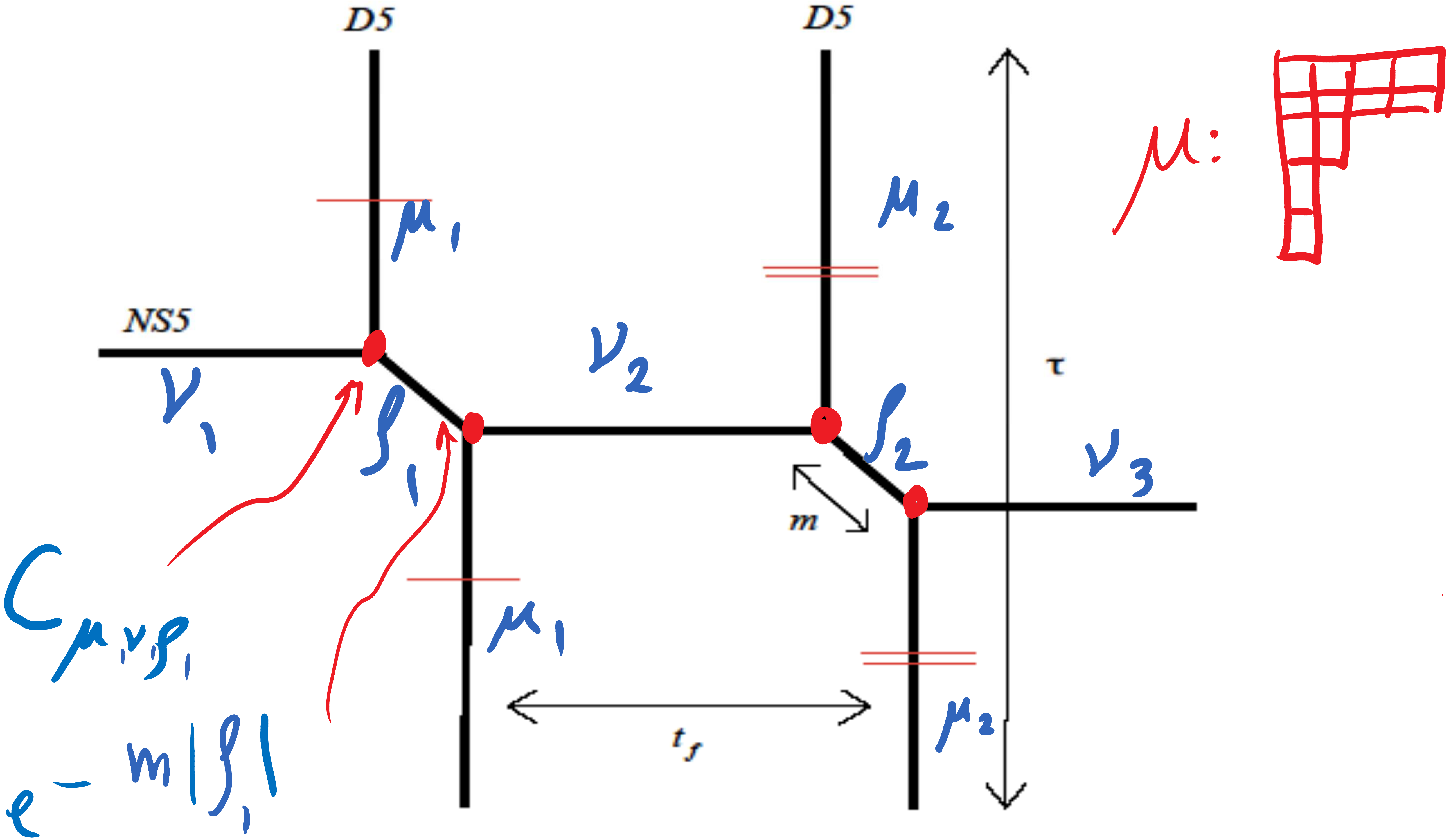
5d SU(N) living on the vertical branes or
6d U(1)x…xU(1) on horizontal brane with
bifundamental matter.

Topological string computes BPS degeneracies and spin of wrapped suspended M2 branes:



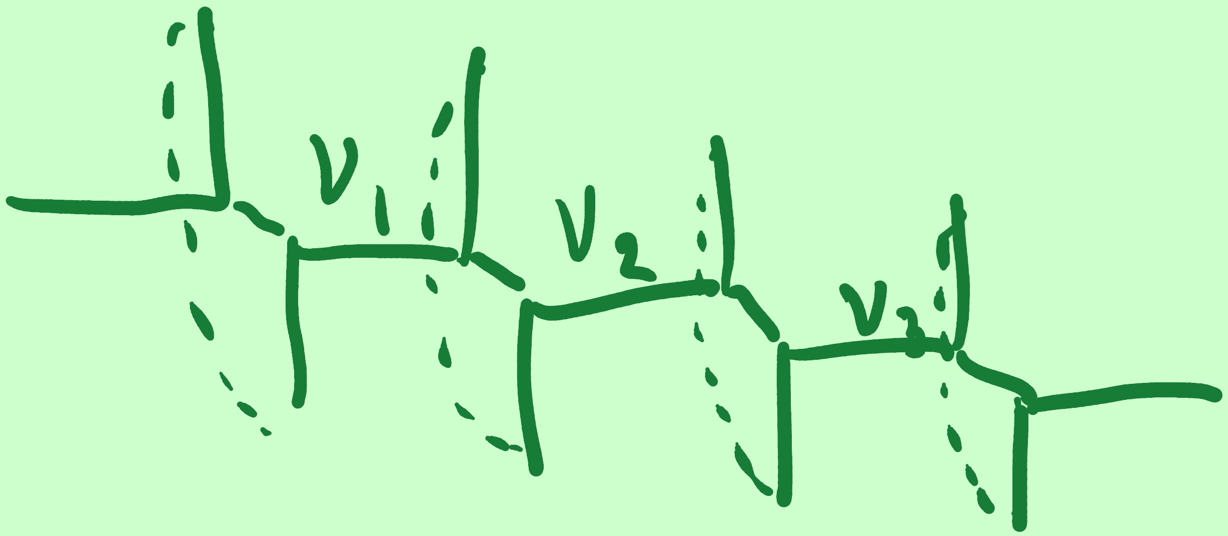
$$\widehat{Z}_{top} = \sum_n e^{-nt_f}\,\widehat{Z_n}$$

where $\widehat{Z_n}$ is the partition function of n M-strings on T^2.



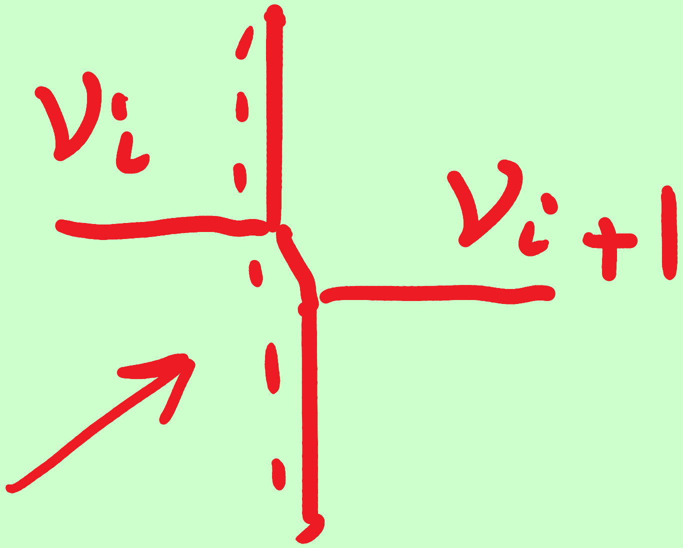
Topological string partition function can be computed using topological vertex formalism.

Dual U(1) perspective suggests summing over vertical partitions, leaving the horizontal partitions (interpreted as U(1) instantons):



$$D^{t}_{\nu_i \, \nu_{i+1}}(\tau, m, \varepsilon_1, \varepsilon_2)$$

Building Block



Basics of M-strings

1-Expect a (4,4) supersymmetric theory

2-General twist leads to (2,0) elliptic genus

$$Z_n(\tau;\underbrace{\varepsilon_1,\varepsilon_2,m}_{\text{maximal susy Cartan } SO(4)\times SO(4)})$$

Dual type IIB on A-D-E → 6d (2,0) theories
with one transverse direction compactified on a circle
(as in little strings [Seiberg] [Aharony,Berkooz,Seiberg][Losev,Moore,Shatashvili])

M-strings ←→ D3 branes wrapped on blown up 2-cycles
(4,4) supersymmetric A-D-E quiver theories

However this does not allow the general twist:

$$SO_\perp(4) \to SO_\perp(3)$$

$$m = \pm \frac{(\epsilon_1 + \epsilon_2)}{2}$$

$$(\epsilon_1 , \epsilon_2 ; \underbrace{-(\epsilon_1 + \epsilon_2) , 0}) \longrightarrow SO_\perp(4)$$

Topological string partition function for $N = 2$ M5 branes gives the answer:

$$\widehat{Z}_n^{(2)} = \sum_{|\nu|=n} \prod_{(i,j)\in\nu} \frac{\theta_1(\tau; z_{ij})\,\theta_1(\tau; v_{ij})}{\theta_1(\tau; w_{ij})\theta_1(\tau; u_{ij})}$$

$$e^{2\pi i\, z_{ij}} = Q_m^{-1}\, q^{\nu_i - j + 1/2}\, t^{-i+1/2}, \qquad e^{2\pi i\, v_{ij}} = Q_m^{-1}\, t^{i-1/2}\, q^{-\nu_i + j - 1/2},$$
$$e^{2\pi i\, w_{ij}} = q^{\nu_i - j + 1}\, t^{\nu_j^t - i}, \qquad e^{2\pi i\, u_{ij}} = q^{\nu_i - j}\, t^{\nu_j^t - i + 1},$$

$$Q_\tau = e^{2\pi i \tau}, Q_m = e^{2\pi i m}, q = e^{2\pi i \epsilon_1}, t = e^{-2\pi i \epsilon_2}$$

We can restrict the parameters so that type IIB dual quiver description applies:

$$m = \pm \frac{\varepsilon_1 + \varepsilon_2}{2}$$

We compare 2 M5 brane case with A_1 quiver:
U(n) (4,4) supersymmetric partition function
with the most general twist which preserve (2,0).
The answer vanishes because of the zero mode in the U(1) of U(n). Deleting that, we can still compare our answer with the elliptic genus for SU(n) theories.

This has been recently computed
[Gadde,Gukov],[Benini,Eager,Hori,Tachikawa]
and our answers agree (to the large order in n that we have checked):

$$\frac{\widehat{Z}_k(\tau,m,\epsilon_1,\epsilon_2)}{\widehat{Z}_1(\tau,m,\epsilon_1,\epsilon_2)} = \sum_{|\nu|=k} \frac{\prod_{(i,j)\in\nu,(i,j)\neq(1,\nu_1)} \theta_1(\tau;z_{ij})\,\theta_1(\tau;v_{ij})}{\prod_{(i,j)\in\nu,(i,j)\neq(\ell(\nu),\nu_{\ell(\nu)})} \theta_1(\tau;w_{ij})\theta_1(\tau;u_{ij})}\,.$$

For example for $SU(2)$ in this limit we get:

$$\begin{aligned}
\frac{\widehat{Z}_2(\tau,m,\epsilon_1,\epsilon_2)}{\widehat{Z}_1(\tau,m,\epsilon_1,\epsilon_2)} &= \frac{\theta_1(\tau;m-\frac{3}{2}\epsilon_1-\frac{1}{2}\epsilon_2)\theta_1(\tau;m+\frac{3}{2}\epsilon_1+\frac{1}{2}\epsilon_2)}{\theta_1(\tau;2\epsilon_1)\theta_1(\tau;\epsilon_1-\epsilon_2)} \\
&+ \frac{\theta_1(\tau;m-\frac{1}{2}\epsilon_1-\frac{3}{2}\epsilon_2)\theta_1(\tau;m+\frac{1}{2}\epsilon_1+\frac{3}{2}\epsilon_2)}{\theta_1(\tau;\epsilon_1-\epsilon_2)\theta_1(\tau;-2\epsilon_2)} \\
&\xrightarrow{m=\pm\frac{\epsilon_1+\epsilon_2}{2}} \frac{\theta_1(\tau;-\epsilon_1)\theta_1(\tau;2\epsilon_1+\epsilon_2)}{\theta_1(\tau;2\epsilon_1)\theta_1(\tau;\epsilon_1-\epsilon_2)} + \frac{\theta_1(\tau;-\epsilon_2)\theta_1(\tau;\epsilon_1+2\epsilon_2)}{\theta_1(\tau;\epsilon_1-\epsilon_2)\theta_1(\tau;-2\epsilon_2)}
\end{aligned}$$

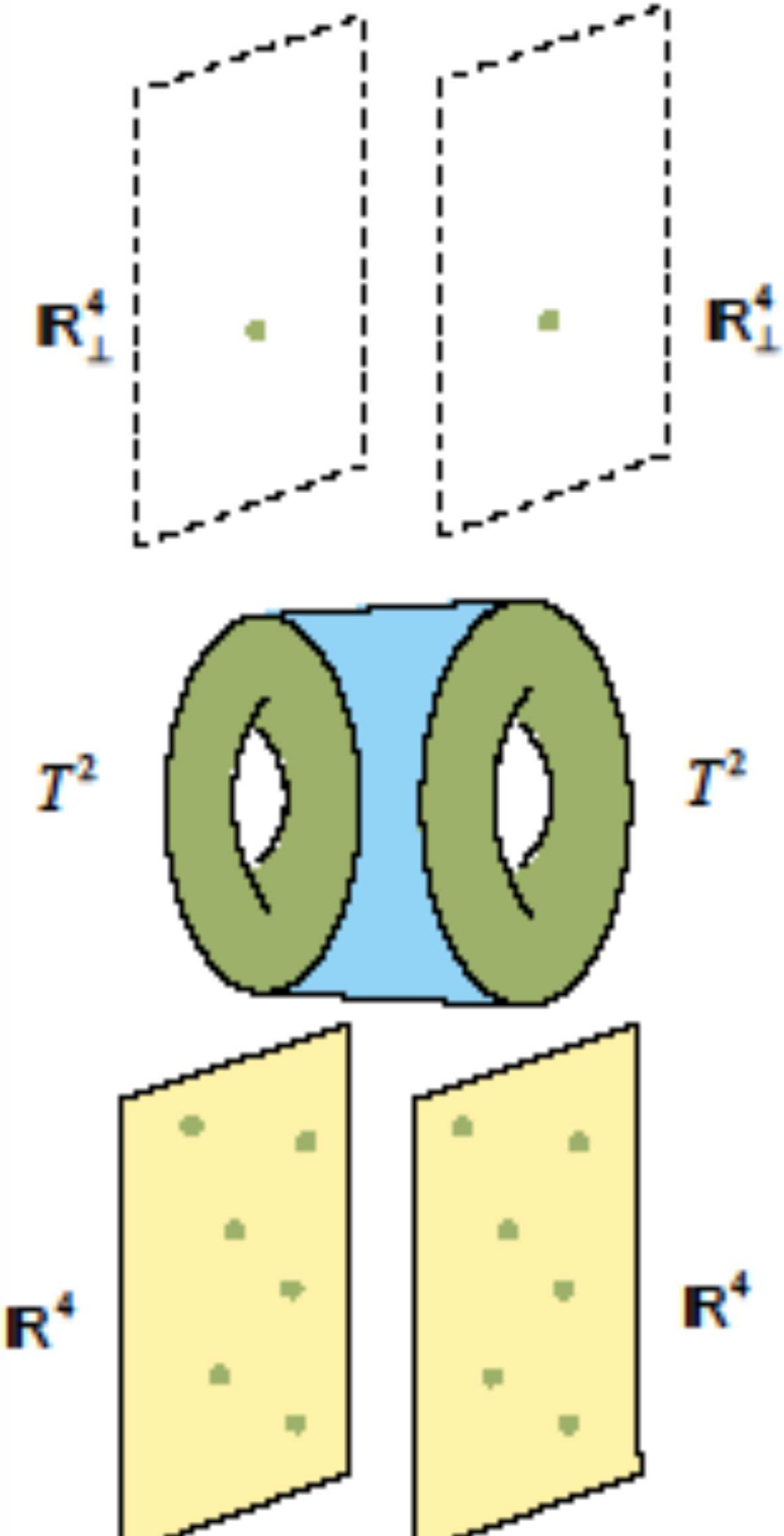
This and the other $SU(N)$ predictions match the index computation:

$$\begin{aligned}
\mathcal{I}^{(N)} &= \sum_{|\nu_1|=N,|\nu_2|=N} \prod_{(i_1,j_1)\in\nu_1,(i_2,j_2)\in\nu_2} \frac{\theta_1(\tau;\epsilon_1(i_2-i_1)+\epsilon_2(j_2-j_1))}{\theta_1(\tau;\epsilon_1(1+i_2-i_1)+\epsilon_2(j_2-j_1))} \\
&\times \prod_{(i_1,j_1)\in\nu_1,(i_2,j_2)\in\nu_2} \frac{\theta_1(\epsilon_1(1+i_2-i_1)+\epsilon_2(1+j_2-j_1))}{\theta_1(\epsilon_1(i_2-i_1)+\epsilon_2(1+j_2-j_1))}\,.
\end{aligned}$$

However the (4,4) quiver theory is not a fully satisfactory answer to what M-strings are:

This description only applies in the special limit and not for general values of m.

We expect n M-strings to be related to (4,4) supersymmetric sigma model on n-fold symmetric product of R^4.



For a single $M2$ brane we take $n = 1$ in the above formula:

$$\widehat{Z}_1^{(2)} = \frac{\theta_1(\tau; -m + (\epsilon_1 + \epsilon_2)/2)\,\theta_1(\tau; m + (\epsilon_1 + \epsilon_2)/2))}{\theta_1(\tau; \epsilon_1)\theta_1(\tau; \epsilon_2)}$$

$$= \prod_{k=1}^{\infty} \frac{(1 - Q_\tau^k Q_m^{\pm 1}\, q^{1/2} t^{-1/2})(1 - Q_\tau^{k-1} Q_m^{\pm 1}\, q^{-1/2} t^{1/2})}{(1 - Q_\tau^k\, q)(1 - Q_\tau^{k-1}\, q^{-1})(1 - Q_\tau^n\, t^{-1})(1 - Q_\tau^{n-1}\, t)}$$

$$= Z_{R^4}$$

Next we consider the case for two M-strings, $n = 2$ and compare it with the partition function of sigma model on symmetric product of two R^4's:

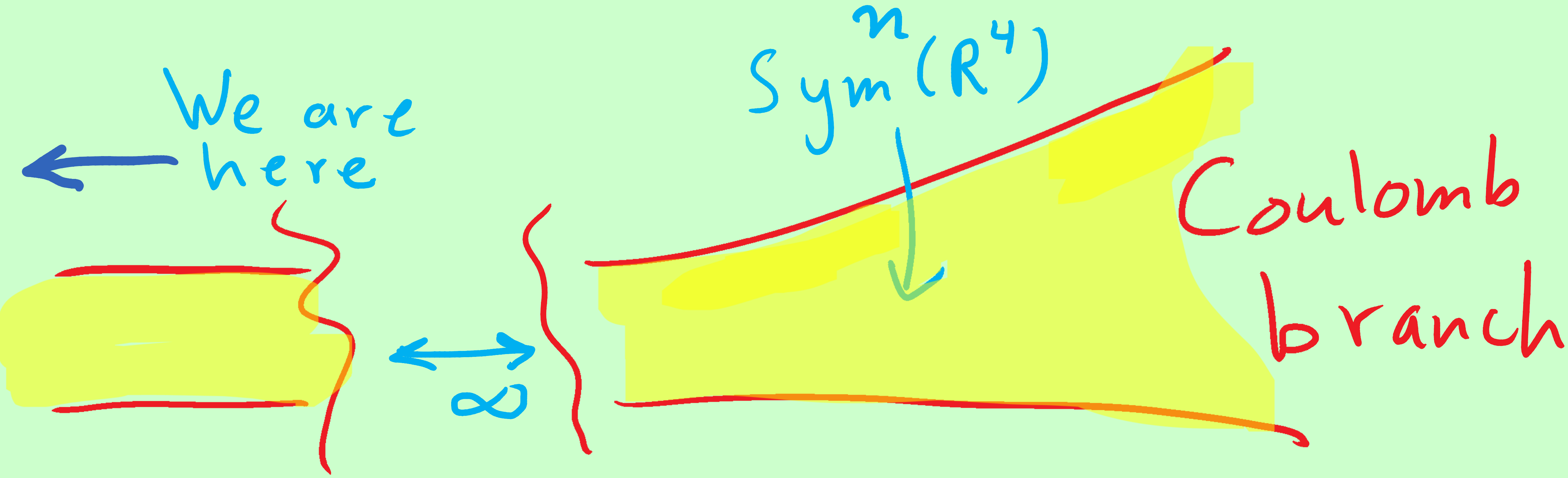
$$Z_{\mathrm{Sym}^2(\mathbb{R}^4)} = \frac{1}{2}\left(\left({}^{1}\underset{1}{\square} + {}^{g}\underset{1}{\square}\right) + \left({}^{1}\underset{g}{\square} + {}^{g}\underset{g}{\square}\right)\right)$$

$$= \frac{1}{2}\Big[\widehat{Z}_1^{(2)}(\tau, \epsilon_1, \epsilon_2, m)^2 + \widehat{Z}_1^{(2)}(2\tau, 2\epsilon_1, 2\epsilon_2, 2m)$$

$$+ \widehat{Z}_1^{(2)}(\tau/2, \epsilon_1, \epsilon_2, m) + \widehat{Z}_1^{(2)}((\tau+1)/2, \epsilon_1, \epsilon_2, m)\Big].$$

$$\widehat{Z}_2^{(2)} \neq Z_{\mathrm{Sym}^2(\mathbb{R}^4)}!$$

The fact that it does not agree with symmetric product is in principle not a contradiction [Witten]:



One can find (using fiber/base duality and Nekrasov's instanton calculus) instead a (4,0) sigma model on symmetric product space which yields the same elliptic genus:

$$M = \mathrm{Hilb}^n(\mathbb{R}^4)$$

$$\begin{cases} V_L = T_{\mathrm{Hilb}} \\ V_R = E + E^* \end{cases}$$

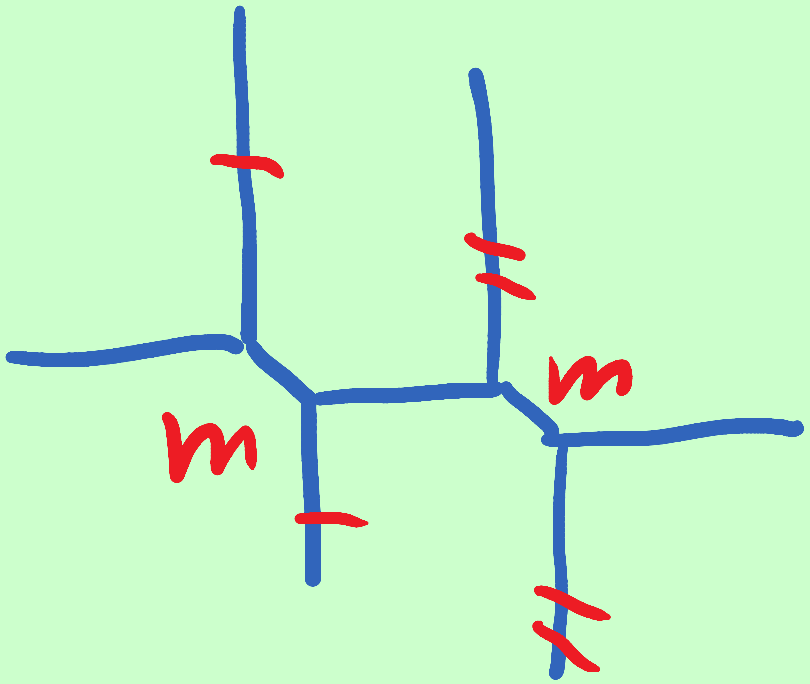
$$\mathrm{Hilb}^n(\mathbb{R}^4) = \mathcal{M}_n^{U(1)\text{ instanton on }\mathbb{R}^4}$$

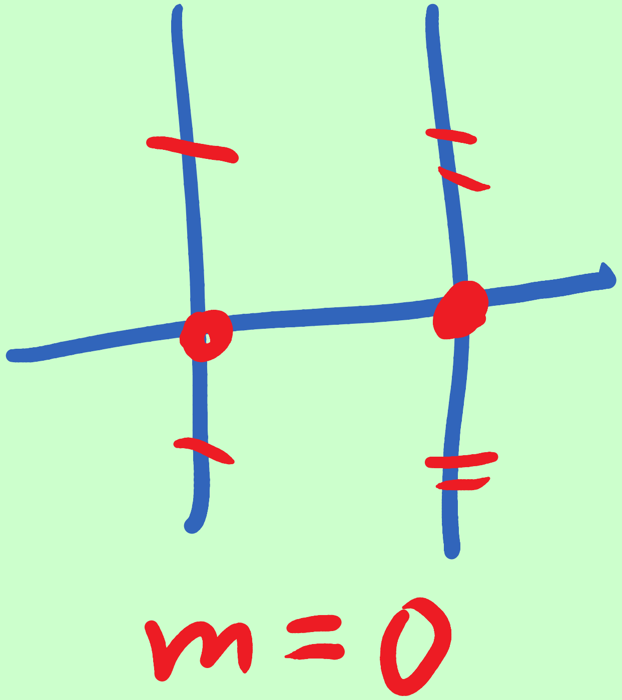
E = Space of Dirac 0-modes

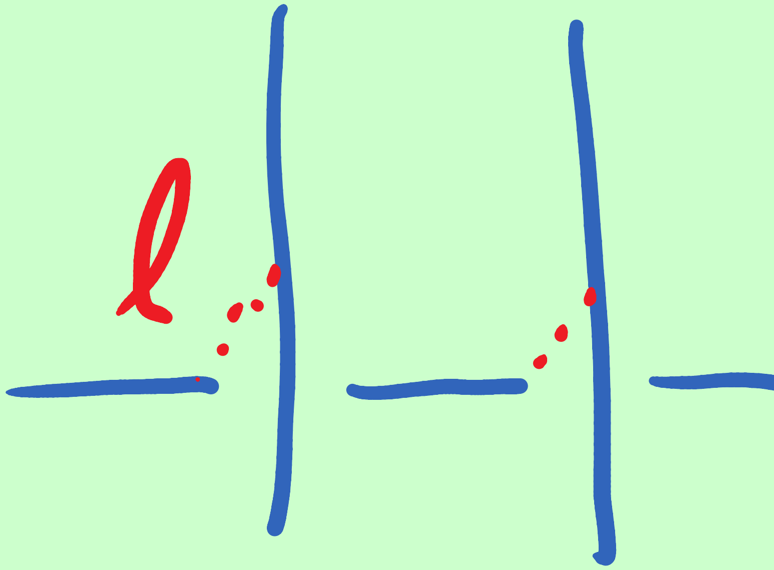
E^* = Conjugate space

$$ch(E+E^*) = ch(T_{Hilb})$$

An explanation of how we get (4,0) SUSY instead of (4,4) and perhaps why Chern Characters are still the same:







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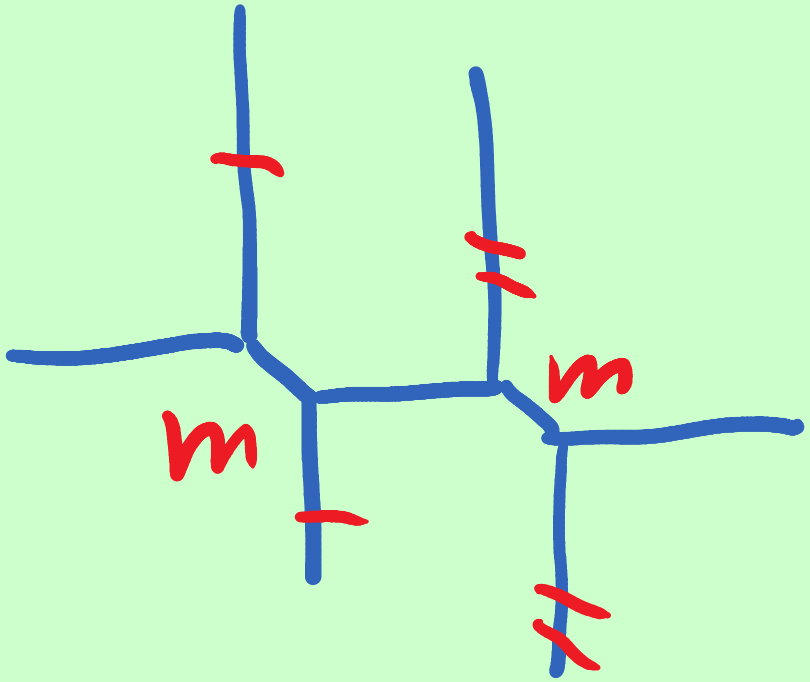
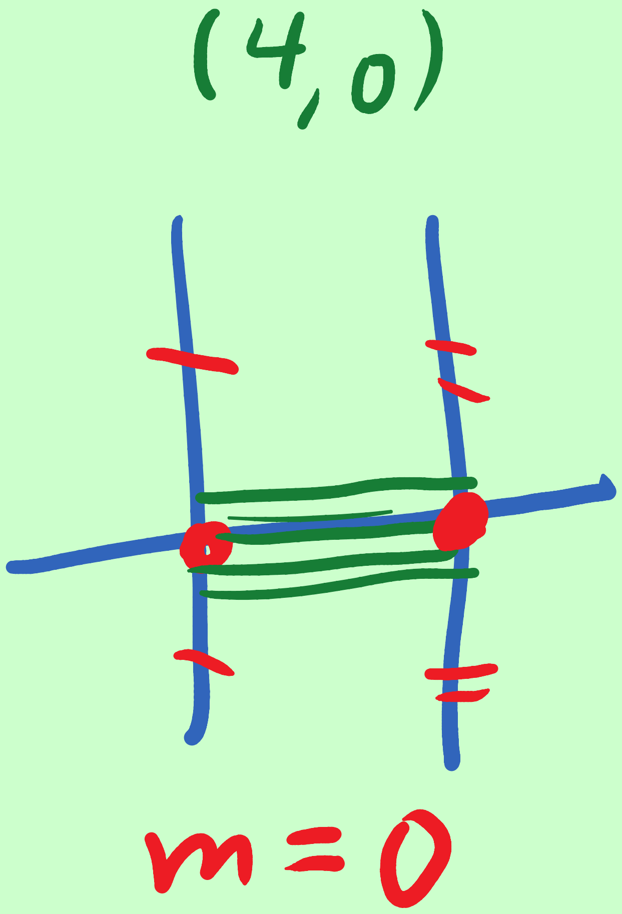
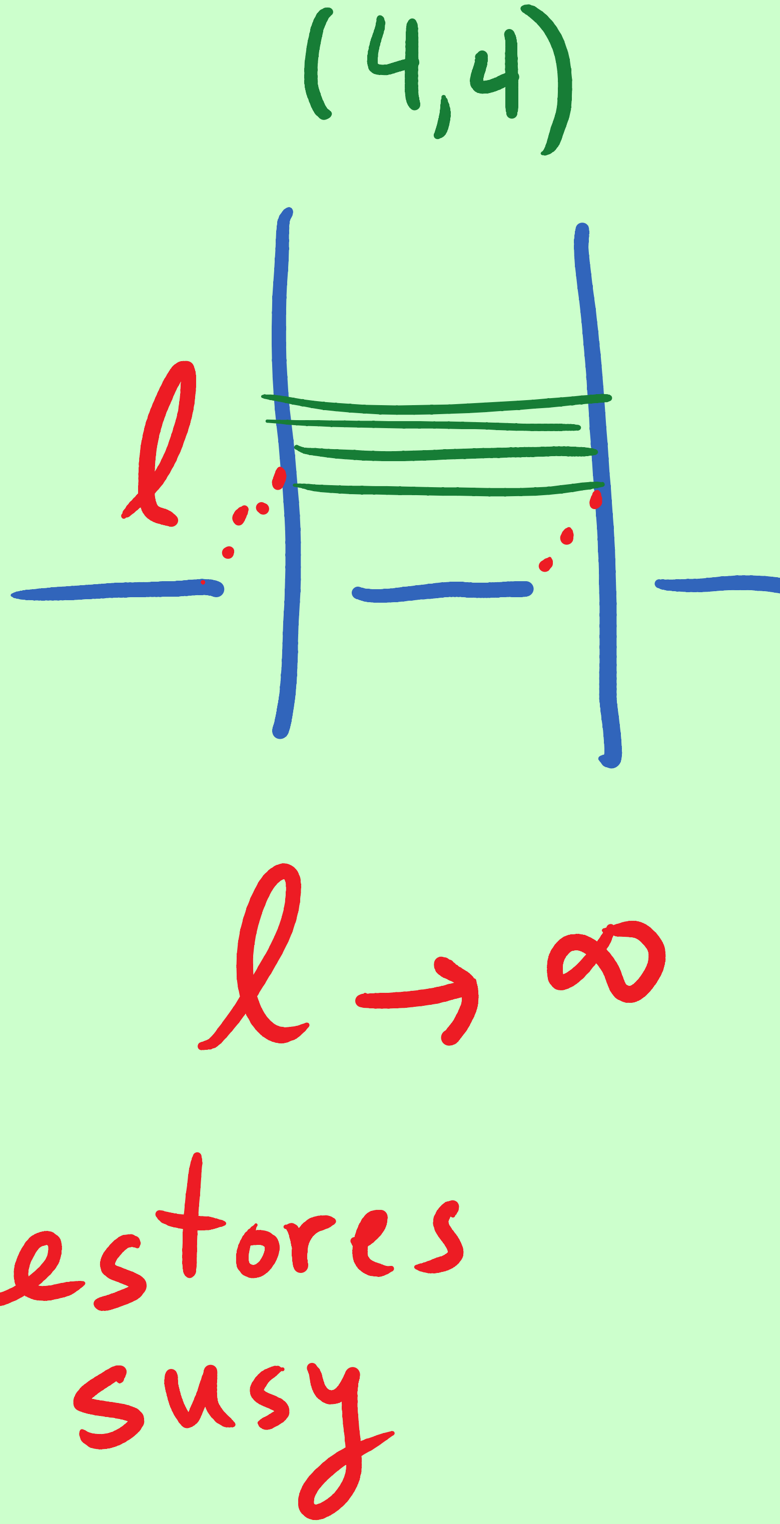
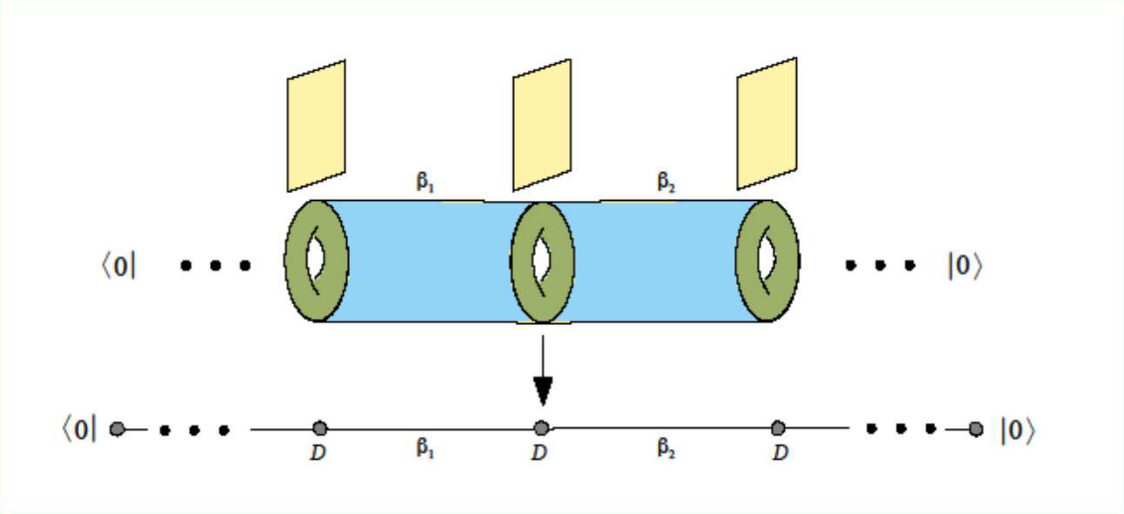






A similar story holds for $N > 2$ M5 branes that can be rephrased as QM:

Hilbert space: Young diagrams ν
Identity operator: $I = \sum_\nu |\nu\rangle\langle\nu^t|$
Hamiltonian: $H = |\nu|$
Domain Wall operator: D



$$\langle\nu^t|D|\mu\rangle = D_{\nu^t\mu}(\tau, m, \epsilon_1, \epsilon_2) = t^{-\frac{\|\mu^t\|^2}{2}}\, q^{-\frac{\|\nu\|^2}{2}} Q_m^{-\frac{|\nu|+|\mu|}{2}}$$

$$\times \prod_{k=1}^{\infty} \prod_{(i,j)\in\nu} \frac{(1 - Q_\tau^k Q_m^{-1}\, q^{-\nu_i+j-\frac{1}{2}}\, t^{-\mu_j^t+i-\frac{1}{2}})(1 - Q_\tau^{k-1} Q_m\, q^{\nu_i-j+\frac{1}{2}}\, t^{\mu_j^t-i+\frac{1}{2}})}{(1 - Q_\tau^k\, q^{\nu_i-j}\, t^{\nu_j^t-i+1})(1 - Q_\tau^{k-1}\, q^{-\nu_i+j-1}\, t^{-\nu_j^t+i})}$$

$$\times \prod_{(i,j)\in\mu} \frac{(1 - Q_\tau^k Q_m^{-1}\, q^{\mu_i-j+\frac{1}{2}} t^{\nu_j^t-i+\frac{1}{2}})(1 - Q_\tau^{k-1} Q_m\, q^{-\mu_i+j-\frac{1}{2}} t^{-\nu_j^t+i-\frac{1}{2}})}{(1 - Q_\tau^k\, q^{\mu_i-j+1} t^{\mu_j^t-i})(1 - Q_\tau^{k-1}\, q^{-\mu_i+j} t^{-\mu_j^t+i-1})}$$

Letting $\beta_a = 2\pi i\, t_{f_a}$ where t_{f_a} denote the separation of the M5 branes.

$$\widehat{Z}^{(N)} = \langle 0|De^{-\beta_1 H} De^{-\beta_2 H} D \cdots e^{-\beta_{N-1} H} D|0\rangle$$

(The partition function of little strings can also be computed instead of vev by taking a trace.)

This picture fits well with:

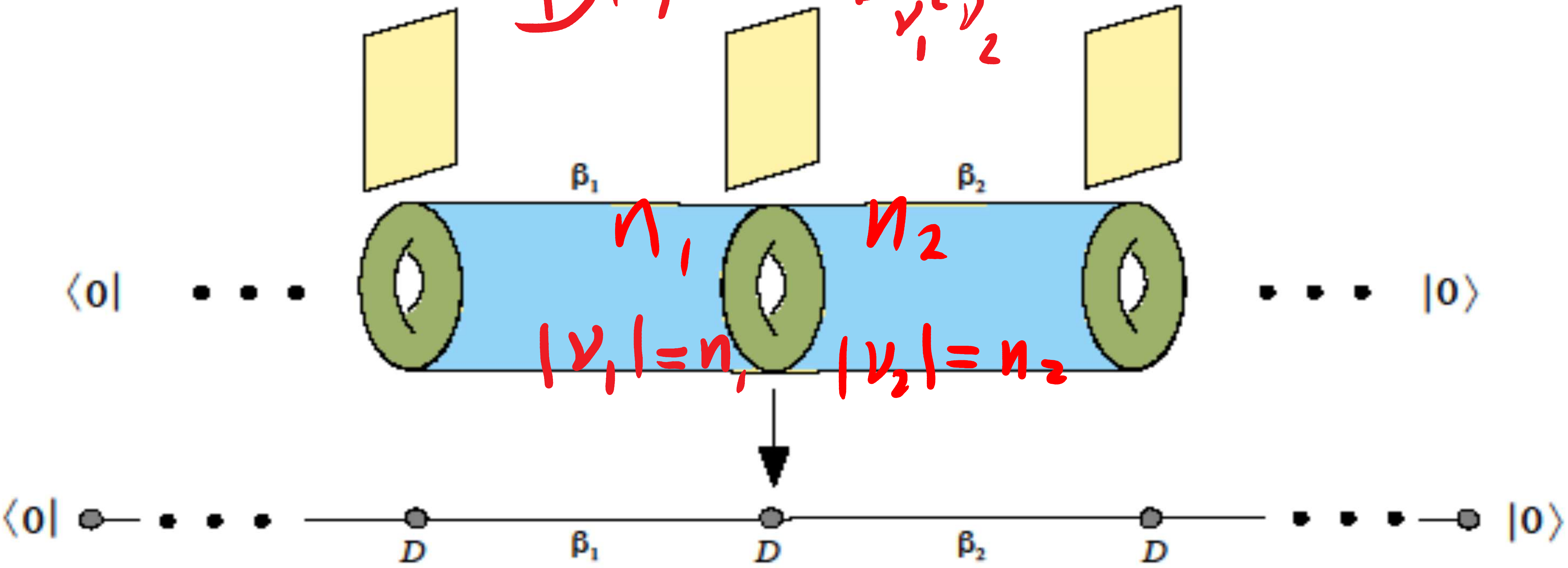
1-For n M2 branes the vacua (at least of the massive twisted theory) are in 1-1 correspondence with Young diagrams of size n, using AdS/CFT and ABJM [Lin,Lunin,Maldacena][Gomis,Rodriguez-Gomez,Van Rammasdonk,Verlinde][Kim,Kim].

2-M5 branes can be viewed as domain walls for M2 branes.

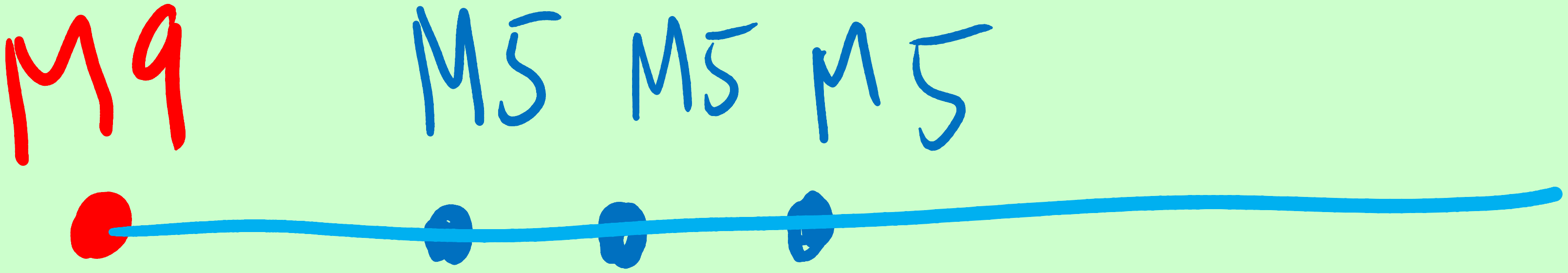
3-If we consider the limit that the area of torus is small we can project to ground states on the left and right.

4-D is the operator which maps one vacuum to the other.

$$D|\nu_1\rangle = D_{\nu_1^t \nu_2}|\nu_2\rangle$$

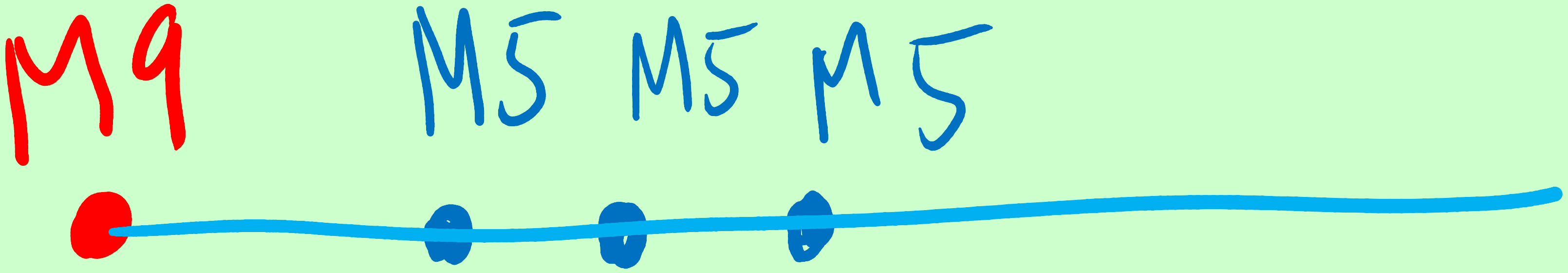


A similar story should hold for (1,0) 6d SCFT and the E-strings:



$$\langle 0|\, \underset{\substack{\uparrow \\ \text{To be determined}}}{D_{M9}}\, e^{-\beta H} \underbrace{D_{M5} \cdots D_{M5}}_{\text{known}} |0\rangle$$

A similar story should hold for (1,0) 6d SCFT and the E-strings:



$$\langle 0 | D_{M9} e^{-\beta H} D_{M5} \cdots D_{M5} | 0 \rangle$$

Hint: $\langle 0 | D_{M9} e^{-\beta H} D_{M9} | 0 \rangle = Z^{het}$

Topological Strings and Spherical Partition Functions

The content of the relation between topological strings and the spherical partition funcitons is that one can compute them as a product over contribution of BPS states, as if they are non-interacting fundamental particles of the theory:

Topological String →BPS states →Partition Functions on Spheres

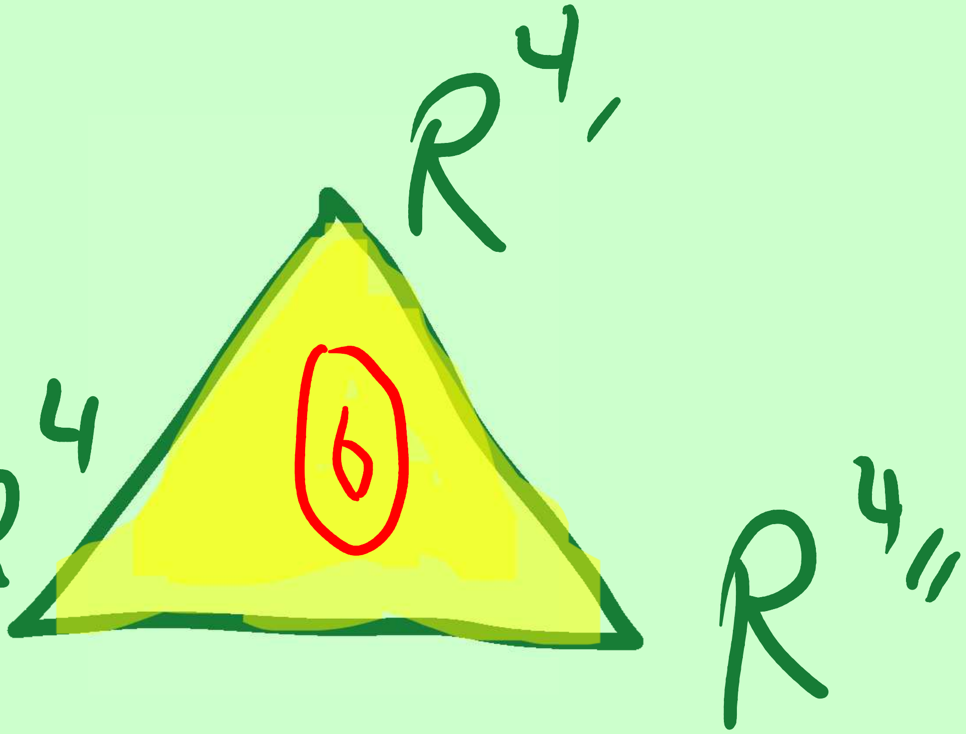
$$Z_{S^\alpha} = \int \prod_{BPS} Z^{BPS}_{S^\alpha}$$

e.g.

$$Z_{CY\times S^5} = \int dt_i \, \overbrace{Z_{top} \, Z'_{top} \, Z''_{top}}^{SL(3,\mathbb{Z})}$$

$$S^5 \xrightarrow{S^1} CP^2$$

$S^1 \times R^4$



See [Lockhart,V],[Kim,Kim,Kim]

and related work [Imamura][Kallen,Qiu,Zabzine][Nieri,Pasquetti,Passerini][Spiridonov]

$$Z' = Z\left(\frac{1}{\varepsilon_1}, \frac{\varepsilon_2}{\varepsilon_1}, \frac{m_\alpha}{\varepsilon_1}, t_i\right)$$

$$Z'' = Z\left(\frac{\varepsilon_1}{\varepsilon_2}, \frac{1}{\varepsilon_2}, \frac{m_\alpha}{\varepsilon_2}, t_i\right)$$

Using elliptic Calabi-Yau's this leads to the computation of superconformal index for (2,0) and (1,0) theories in 6d:

$$Z_{S^1\times S^5}\left(\begin{matrix}(2,0)\\(1,0)\end{matrix}\right)=Z_{S^5}\left(CY_{ell}\right)$$

M-strings → 6d (2,0) Amplitudes?

We can reverse the order: Computation of the elliptic genus of M-strings leads to the computation of the index of the (2,0) theory:

$$\begin{tikzcd}
Z(\text{M-string}) \arrow[r] & Z^{top} \arrow[d, bend left] \\
Z^{M5}_{S^1 \times S^5} & Z^{M5}_{T^2 \times R^4} \arrow[l]
\end{tikzcd}$$

This is an example of how M-strings can be used to compute amplitudes of (2,0) theories. Can this be generalized?