Strings in $AdS_n \times S^n$ and their deformations

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$AdS_n \times S^n$ type IIB backgrounds:

$AdS_5 \times S^5 + F_5$:	limit of D3
$AdS_3 \times S^3 \times T^4 + F_3, H_3$:	limit of D5-D1 + NS5-NS1
$AdS_2 \times S^2 \times T^6 + F_5$:	limit of D3-D3-D3-D3

most symmetric superstrings:

$AdS_n \times S^n = G/H \to$ supercosets $\hat{G}/H$

$\hat{G}$ = super-isometries

$AdS_5 \times S^5 : \quad \hat{G} = PSU(2,2|4)$

$H = SO(1,4) \times SO(5)$

$AdS_3 \times S^3 : \quad \hat{G} = PSU(1,1|2) \times PSU(1,1|2)$

$H = SO(1,2) \times SO(3)$

$AdS_2 \times S^2 : \quad \hat{G} = PSU(1,1|2)$

$H = SO(1,1) \times SO(2)$

- Integrability: key to solution of $AdS_5 \times S^5$ superstring

integrable deformations? solvable models with no susy?

- known examples: orbifolds; T-duality

e.g. β-deformation and generalizations

[Lunin, Maldacena 05; Frolov, Roiban, AT 05]

- novel example: [Delduc, Magro, Vicedo 13,14]

integrable deformation of $AdS_5 \times S^5$ supercoset σ-model
with q-deformed $U_q[\mathfrak{psu}(2,2|4)]$ symmetry
and q-deformed $AdS_5 \times S^5$ l.c. S-matrix

[Arutyunov, Borsato, Frolov 13]

- should be exactly solvable – deserved detailed study

Aim: study simpler deformed $AdS_2 \times S^2$ and $AdS_3 \times S^3$
and special limits

deformed $AdS_2 \times S^2$ metric

$$ds^2 = \frac{1}{1-\kappa^2\rho^2}\Big[-(1+\rho^2)dt^2 + \frac{d\rho^2}{1+\rho^2}\Big]$$

$$+ \frac{1}{1+\kappa^2 r^2}\Big[+(1-r^2)d\varphi^2 + \frac{dr^2}{1-r^2}\Big]$$

- not conformally flat, non-supersymmetric
- defines integrable σ-model with $U_q[\mathrm{so}(1,2) \times \mathrm{so}(3)]$ symmetry
- embedding into type IIB theory: add RR fluxes $\rightarrow$ conformal model with known superstring action
- integrability determines quantum string spectrum also for $\kappa \neq 0$

GS superstrings in $AdS_n \times S^n$:

alg($\hat{G}$) = alg(H) + odd + alg(G/H) + odd

$J = \hat{g}^{-1} d\hat{g} = J^{(0)} + J^{(1)} + J^{(2)} + J^{(3)}$

$$I = \mathrm{h} \int d^2x \, \mathrm{Str}\Big[\sqrt{g} g^{ab} J_a^{(2)} J_b^{(2)} + \epsilon^{ab} J_a^{(1)} J_b^{(3)}\Big]$$

$$\mathrm{h} = \frac{R^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi}$$

- classical integrability; κ-symmetry; UV finiteness
- expected quantum integrability $\rightarrow$ solution for spectrum
- critical $d = 10$: symmetries if $AdS_n \times S^n \times T^{10-2n}$
- $AdS_5 \times S^5$: solution uses symmetry/integrability

based on l.c. gauge – expansion near BMN vacuum

$\mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2)$ symmetry preserved by vacuum choice:

$\rightarrow$ dispersion relation and l.c. gauge S-matrix

Strategy of solution:

(i) find symmetry algebra in l.c. gauge + central extension

(ii) represent l.c. symmetry on particle states

$\rightarrow$ exact dispersion relation $E = \sqrt{1 + 4\mathrm{h}^2 \sin^2 \frac{p}{2\mathrm{h}}}$

(iii) construct 2-particle S-matrix consistent with:

l.c. symmetry, Yang-Baxter equation, unitarity, crossing;

(iv) check vs perturbative S-matrix from string action

special features: no 2d relativistic invariance in l.c. gauge;

basic states are "dressed" elementary excitations;

non-locality of l.c. gauge fermionic Noether charges:

symmetry of l.c. S-matrix as Hopf algebra

(co-product depending on 2d momentum)

l.c. S-matrix $\to$ asymptotic Bethe Ansatz

$\to$ thermodynamic BA or "quantum spectral curve"

$\to$ exact string spectrum

Recent progress:

• same steps for $AdS_3 \times S^3 \times T^4$

(including "massless" modes from T^4)

[Babichenko, Stefanski, Zarembo 09]

[Borsato, Ohlsson Sax, Sfondrini, Stefanski, Torrielli 11-14]

• $AdS_3 \times S^3 \times T^4$ with WZ term $\sim \mathrm{b} = k\,\mathrm{h}^{-1}$

same symmetry, integrability [Cagnazzo, Zarembo 12]

RR $(\mathrm{b} = 0) \to$ NSNS $(\mathrm{b} = 1)$ $SL(2) \times SU(2)$ WZW

l.c. S-matrix / TBA vs CFT solution [Ooguri, Maldacena 00]

⋆ structure of dispersion relation and l.c. S-matrix

$$E = \sqrt{(1 + \mathrm{b}\, p)^2 + 4\mathrm{h}^2(1 - \mathrm{b}^2) \sin^2 \frac{p}{2\mathrm{h}}}$$

[Hoare, AT 13; Hoare, Stepanchuk, AT 13]

⋆ finite gap equations, 1-loop dressing phases, BA

[Beccaria, Levkovich-Maslyuk, Macorini, AT 12; Abbott 12,13]

[Babichenko, Dekel, Ohlsson Sax '14]

⋆ checks against string perturbation theory

[Sundin, Wulff 12; Engelund,McKeown,Roiban 13; Forini,Hoare,Bianchi 13,14]

• Integrability solution of $AdS_2 \times S^2 \times T^6$?

[Sorokin, Wulff, AT, Zarembo 10; Murugan, Sundin, Wulff 12]

l.c. symmetry algebra and S-matrix

[Hoare, Pittelli, Torrielli, to appear]

Similar integrable/solvable models with no manifest susy?

Yes! deformed integrable $AdS_5 \times S^5$ supercoset model

• q-deformed $PSU(2,2|4)$ as classical symmetry

[Delduc, Magro, Vicedo 14]

• l.c. S-matrix – q-deformed $AdS_5 \times S^5$ S-matrix

[Arutyunov, Borsato, Frolov 13]

• l.c. S-matrix $\rightarrow$ BA $\rightarrow$ TBA:

should be possible to find exact spectrum for any q

[Arutyunov, de Leeuw, van Tongeren 14]

⋆ IIB background: singular, only $U(1)$ isometries, no susy

but string theory has large hidden symmetry

⋆ space-time or dual gauge theory interpretation?

• new solvable model: useful lessons in store?

Origins:

- integrable deformation of PCM [Klimcik 02,08]
- classical q-deformed symmetry of deformed PCM (e.g. squashed S^3) and coset models (with q= real) [Kawaguchi, Matsumoto,Yoshida 11,12; Delduc, Magro, Vicedo 12]
- q-deformed R-matrix of 1d Hubbard model with quantum group $U_q[\mathfrak{psu}(2|2)_c]$ symmetry [Beisert-Koroteev 08]

⋆ leads to deformation of $AdS_5 \times S^5$ l.c. S-matrix

⋆ interpolates between $AdS_5 \times S^5$ S-matrix ($q = 0$) and S-matrix of Pohlmeyer reduced (sine-Gordon type) model for $AdS_5 \times S^5$ string (with $q = e^{i\pi/k}$ =complex) [Beisert 10; Hoare, AT 09-11; Hoare, Hollowood, Miramontes 11]

- Integrable σ-model with q-deformed symmetry having such q-deformed l.c. S-matrix?

Pohlmeyer reduction:

solve Virasoro in terms of "currents" – physical d.o.f.

string and its PR – classically equiv. integrable structures

• $R \times S^2 \rightarrow$ sine-G, $R \times S^3 \rightarrow$ complex sine-G

$AdS^2 \times S^1 \rightarrow$ sinh-G, $AdS_3 \times S^1 \rightarrow$ complex sinh-G

• strings on $G/H \rightarrow H/K$ gWZW + potential

$\rightarrow$ σ-model + integrable potential:

generalized symmetric space sine-Gordon type models

[Bakas, Park, Shin 95; Grigoriev, AT 07]

$AdS_n \times S^n$ supercoset models:

⋆ 2-d Lorentz invariance and UV finiteness:

special UV-finite massive integrable models

⋆ Standard kinetic terms for fermions and hidden 2-d susy

PR for $AdS_5 \times S^5$ superstring:

• gWZW for $\frac{SO(1,4)}{SO(4)} \times \frac{SO(5)}{SO(4)}$ + potential + fermions

[Grigoriev, AT 07; Mikhailov, Schafer-Nameki 07]

parameter: WZW coupling k

• conjectured Lorentz-invariant S-matrix is limit

$$\mathrm{h} \to \infty, \quad q = e^{i\pi/k} \ \text{ of } \ \mathrm{S}(\mathrm{h}, q)$$

– q-deformation of l.c. $AdS_5 \times S^5$ string S-matrix

[Hoare, AT 09-11; Beisert 10; Hoare, Hollowood, Miramontes 11]

$\mathrm{h} \to \infty$: $AdS_5 \times S^5$ PR model with coupling k $(q = e^{i\pi/k})$

$k \to \infty$: $AdS_5 \times S^5$ superstring with tension h $(q = 0)$

• q-deformation of l.c. gauge symmetry algebra $\to$
l.c. S-matrix symmetry: $\mathfrak{psu}(2|2)^{\oplus^2} \subset \mathfrak{psu}(2,2|4)$

$$\to \quad U_q[\mathfrak{psu}(2|2)^{\oplus^2}] \subset U_q[\mathfrak{psu}(2,2|4)]$$

• single-particle states – reps. of $U_q[\mathfrak{psu}(2|2)_c^{\oplus^2}]$

• dispersion relation: interpolates between
$E^2 = 1 + 4\mathrm{h}^2 \sin^2 \frac{p}{2\mathrm{h}}$ and $E^2 = \sec^2 \frac{\pi}{2k} + p^2$

• q-def. symmetry fixes bound-state S-matrix up to phase

[de Leeuw, Regelskis, Matsumoto 12; Beisert, Galleas, Matsumoto '11]

• phase $\theta(p_1, p_2; \mathrm{h}, q)$: from unitarity, crossing and fusion

[Hoare, Hollowood, Miramontes '11]

$q \to 0$: string BES phase; $\mathrm{h} \to \infty$: relativistic PR phase

meaning of interpolating S-matrix? corresponding σ-model?

yes, if take real deformation parameter q

Integrable deformation of group space σ-model

= "Yang-Baxter" σ-model [Klimcik 02,08,14]

PCM: $L_0 = \text{Tr}\,(J_+ J_-), \quad J = g^{-1} dg, \quad g \in G$

$$L_\eta = \text{Tr}(J_+ K J_-)\ , \qquad K = \frac{1}{1 - \kappa R_g}$$

$$R_g(M) = g^{-1} R(g M g^{-1}) g\ , \qquad M \in \text{alg}(G)$$

R=const – solution of classical modified YBE

$R(M), R(N)] - R([R(M), N] + [M, R(N)]) = [M, N]$

e.g. $R = \pm 1$ on positive/negative roots and 0 on Cartans

- classical integrability for any κ: Lax pair
- manifest $G \times G$ symmetry broken to $[U(1)]^r \times G$
- hidden non-local conserved charges:
  Hopf-Poisson $U_q[\text{alg}(G)], \quad q = q(\kappa)$
- simplest example: "squashed" S^3 [Cherednik 81]

Integrable deformed G/H σ-models [Delduc, Magro, Vicedo 13]

$$L_\eta = \mathrm{h}\,\mathrm{Tr}\,(J_+ K J_-)\,, \qquad K = \frac{P_2}{1 - \kappa R_g \cdot P_2}$$

$\mathrm{alg}(G) = \mathrm{alg}(H) +$ coset, $J = J^{(0)} + J^{(2)}$, $P_2(J) = J^{(2)}$

$U_q[\mathrm{alg}(G)]$ symmetry:

⋆ local charges: G broken to $[U(1)]^r$ – Cartans of $\mathrm{alg}(G)$

⋆ non-local charges: for simple roots

$$\{Q_{+\alpha_n}, Q_{-\alpha_m}\} = i\delta_{nm}[C_n]_q\,, \qquad [C]_q = \frac{q^C - q^{-C}}{q - q^{-1}}$$

$$q = e^{-\kappa/\hat{\mathrm{h}}}$$

⋆ σ-model : $G_{mn}(\kappa) = G_{mn}(0) + O(\kappa^2)$, $B_{mn} = O(\kappa)$

⋆ simplest case: deformed S^2 = "sausage" model

[Fateev, Onofri, Zamolodchikov 93]

Deformed $AdS_5 \times S^5$ superstring σ-model:

$$\kappa = 0 : \qquad \frac{\hat{G}}{H} = \frac{PSU(2,2|4)}{SO(1,4) \times SO(5)}$$

Z_4 of $\mathfrak{psu}(2,2|4)$: projectors $P_0 + P_2 + P_1 + P_3 = \mathbf{1}$

GS superstring Lagrangian

$$L_0 = \mathrm{h}\, \pi^{ab}\, \mathrm{Str}\big(J_a\, K_0\, J_b\big)\ , \qquad \pi^{ab} \equiv \sqrt{-g}\, g^{ab} - \epsilon^{ab}$$

$$J_a = g^{-1}\partial_a g\ , \qquad K_0 \equiv P_2 + \tfrac{1}{2}(P_1 - P_3)$$

"η-deformation": $\kappa = \frac{2\eta}{1-\eta^2}$ [Delduc, Magro, Vicedo 13,14]

$$L_\kappa = \mathrm{h}\, \pi^{ab}\, \mathrm{Str}\big(J_a\, K\, J_b\big)$$

$$K = \frac{P_\kappa}{1 - \kappa R_g \cdot P_\kappa}\ , \qquad P_\kappa = P_2 + \frac{1}{1+\sqrt{1+\kappa^2}}(P_1 - P_3)$$

⋆ κ-deformation preserves integrability

⋆ kappa-symmetry; UV finiteness

[checked in BMN expansion and $\kappa = \infty$ and $\kappa = i$ limits]

⋆ classical $U_q[\mathfrak{psu}(2,2|4)]$ with real q

$$q = e^{-\kappa/\hat{\mathrm{h}}} \,, \qquad\qquad \hat{\mathrm{h}} = \mathrm{h}\sqrt{1+\kappa^2}$$

corresponding type IIB background?

GS superstring σ-model:

$$I = \hat{\rm h} \int d^2\sigma \, (L_{\rm b} + L_{\rm f})$$

$$L_{\rm b} = (G_{mn} + B_{mn})\partial_+ x^m \partial_- x^n$$
$$L_{\rm f} = \partial x \, \bar\theta(\nabla + e^{\Phi} F_5 + ...)\theta + ...$$

κ-symmetry of deformed model suggests

background should solve type IIB supergravity equations

undeformed $AdS_5 \times S^5$ metric:

$$ds^2_A + ds^2_S = -(1+\rho^2)dt^2 + \frac{d\rho^2}{1+\rho^2} + \rho^2 dS_3$$
$$+ (1-r^2)d\varphi^2 + \frac{dr^2}{1-r^2} + r^2 dS'_3$$

deformed $AdS_5 \times S^5$ metric: [Arutyunov, Borsato, Frolov 13]

$$ds_A^2 = -h(\rho)dt^2 + f(\rho)d\rho^2 + \rho^2 \left[v(\rho,\zeta) \left(d\zeta^2 + \cos^2\zeta \, d\psi_1^2 \right) + \sin^2\zeta \, d\psi_2^2 \right]$$

$$h = \frac{1+\rho^2}{1-\kappa^2\rho^2}, \quad f = \frac{1}{(1+\rho^2)(1-\kappa^2\rho^2)}, \quad v = \frac{1}{1+\kappa^2\rho^4\sin^2\zeta}$$

$$ds_S^2 = \widetilde{h}(r)d\varphi^2 + \widetilde{f}(r)dr^2 + r^2 \left[\widetilde{v}(r,\theta) \left(d\theta^2 + \cos^2\theta \, d\phi_1^2 \right) + \sin^2\theta \, d\phi_2^2 \right]$$

$$\widetilde{h} = \frac{1-r^2}{1+\kappa^2 r^2}, \quad \widetilde{f} = \frac{1}{(1-r^2)(1+\kappa^2 r^2)}, \quad \widetilde{v} = \frac{1}{1+\kappa^2 r^4\sin^2\theta}$$

B-field: non-zero in both AdS and S 3-sphere parts

$$B_{\psi_1\zeta} = \tfrac{1}{2}\kappa\,\rho^4 v(\rho,\zeta)\sin 2\zeta \;, \qquad B_{\phi_1\theta} = -\tfrac{1}{2}\kappa\, r^4 \widetilde{v}(r,\theta)\sin 2\theta$$

- $SO(2,4) \times SO(6)$ broken to $[U(1)]^3 \times [U(1)]^3$
- no manifest supersymmetry but string model is

integrable – hidden non-local (super) charges: $U_q[PSU(2,2|4)]$

- curvature singularity at $\rho = \kappa^{-1}$, i.e. $0 \leqslant \rho < \kappa^{-1}$
- $\kappa \neq 0$: no boundary; it reappears at $\rho = \infty$ for $\kappa \to 0$
- non-trivial dilaton also singular at $\rho = \kappa^{-1}$:

$$\nabla^2 e^{-\Phi} + \tfrac{1}{4}(R - \tfrac{1}{12} H^2_{mnk}) e^{-\Phi} = 0$$

does not need to separate as $\Phi = \Phi_A(\rho, \zeta) + \Phi_S(r, \theta)$

- RR backgrounds (F_5 and F_3, $F_1 \sim \kappa$) still to be found:

should solve their eqs. and have prescribed stress tensor

$$T_{mn}(F) = e^{2\Phi}\big(R_{mn} + 2D_m D_n \Phi - \tfrac{1}{4} H_{mkp} H_n{}^{kp}\big)$$

• Remarkably, bosonic tree level l.c. S-matrix matches q-deformed S-matrix for $\mathrm{h} \gg 1$ and real q

[Arutyunov, Borsato, Frolov 13]

• despite complicated singular background

known l.c. S-matrix $\rightarrow$ solvability of string spectrum

Many open questions:

• string theory resolves singularity at $\rho = \kappa^{-1}$?

• gauge theory dual? no boundary, no conformal and no 4d Lorentz symmetry; but hidden symmetry suggest point-like limit may be partly misleading

Special limits:

• "maximal deformation" $\kappa \to \infty \quad (q = e^{-1/\mathrm{h}})$

model is T-dual to double Wick rotation of $AdS_5 \times S^5$

$\to \ dS_5 \times H^5$ with imaginary 5-form (non-unitary) [Hull 98]

• "imaginary deformation" $\kappa \to i \quad (q = e^{-i/\hat{\mathrm{h}}})$

with $\mathrm{h} \to 0, \ \ \hat{\mathrm{h}} = \mathrm{h}\sqrt{1+\kappa^2}$ = fixed

⋆ target space background is of pp-wave type

in l.c. gauge same as PR model for $AdS_5 \times S^5$

⋆ q-deformed l.c. gauge S-matrix becomes 2d relativistic

$\to$ S-matrix of PR model for undeformed $AdS_5 \times S^5$

Similar low-dimensional models:

useful first step and interesting on their own

• deformed $AdS_2 \times S^2$: related to FOZ "sausage" model for S^2 and AdS_2

• deformed $AdS_3 \times S^3$: related to symmetric case of Fateev integrable deformation of $SU(2)$ and $SL(2)$ PCM

⋆ $\kappa = \infty$ limit: related to $dS_n \times H^n, \ n = 2, 3$

⋆ $\kappa = i$ limit: related to PR of $AdS_n \times S^n, \ n = 2, 3$: (2,2) super sine-Gordon and (4,4) complex sine-Gordon

$AdS_2 \times S^2 \times T^6$ superstring: closely related supercoset

$$\frac{PSU(1,1|2)}{SO(1,1) \times SO(2)}$$

[Berkovits, Bershadsky, Hauer, Zhukov, Zwiebach 99]

[Sorokin, AT, Wulff, Zarembo 11]

embedding to IIB string: D3-D3-D3-D3 [Klebanov, AT 96]

$$ds^2 = -(1+\rho^2)dt^2 + \frac{d\rho^2}{1+\rho^2} + (1-r^2)d\varphi^2 + \frac{dr^2}{1-r^2} + dT^6$$

$$F_5 = \Omega_2(AdS_2) \wedge \Omega_3(T^6) + * \, , \qquad \Omega_3 = \mathrm{Re}(dz^1 \wedge dz^2 \wedge dz^3)$$

effective 4d theory:

$$L = e^{-2\Phi}\left[R + 4(\partial_m \Phi)^2\right] - \tfrac{1}{4} F_{mn} F^{mn} + \ldots$$

Bertotti-Robinson: $AdS_2 \times S^2$ with $\Phi = 0$ and

$$F_2 = \sqrt{2}(d\rho \wedge dt + dr \wedge d\varphi)$$

Deformed $AdS_2 \times S^2$ metric

deformed supercoset action leads to 4d metric

$$ds_A^2 + ds_S^2 = \frac{1}{1-\kappa^2\rho^2}\Big[-(1+\rho^2)dt^2 + \frac{d\rho^2}{1+\rho^2}\Big]$$

$$+\frac{1}{1+\kappa^2 r^2}\Big[+(1-r^2)d\varphi^2 + \frac{dr^2}{1-r^2}\Big]$$

- ds_S^2 = "sausage" model [Fateev, Onofri, Zamolodchikov 93]

deformation of S^2 stable under RG flow:

$ds^2 = f(y)(d\varphi^2 + dy^2), \qquad \frac{\partial f}{\partial t} \sim R(f) \sim e^{-f}\partial_y^2 f$

$ds^2 = \frac{dy^2+d\varphi^2}{\cosh^2 y + \kappa^2 \sinh^2 y} = \frac{1}{1+\kappa^2 r^2}\Big[(1-r^2)d\varphi^2 + \frac{dr^2}{1-r^2}\Big]$

- curvature of deformed AdS_2 singular at $\rho \to \kappa^{-1}$

$R = 4(1+\kappa^2)\Big[-\frac{1}{1-\kappa^2\rho^2} + \frac{1}{1+\kappa^2 r^2}\Big]$

Full deformed background [Lunin, Roiban, AT]

$U(1) \times U(1)$ invariant background?

A non-trivial 4d solution of

$$\int d^4x\sqrt{G}\left[e^{-2\Phi}(R+4\partial^m\Phi\partial_m\Phi)-\tfrac{1}{2}\partial^m C\partial_m C-\tfrac{1}{4}F^{mn}F_{mn}\right]$$

direct lift to 10d type IIB solution as $M^2_\kappa \times T^6$:

$F_1 = dC$ or F_3 and $F_2 = dC_1$ - reduction of F_5

$$e^\Phi = \left[(1-\kappa^2\rho^2)(1+\kappa^2 r^2)\right]^{-1/2} X^{-1}(\rho, r)$$

$$C = 2\kappa\sqrt{1+\kappa^2}\,\rho\, r\, X(\rho, r)$$

$$C_1 = \sqrt{2}\sqrt{1+\kappa^2}\, X(\rho, r)\sqrt{1+\kappa^2}\,(\rho\, dt + r\, d\varphi)$$

$$X = \frac{1}{\sqrt{1+\kappa^2(r^2-\rho^2+\rho^2 r^2)}}$$

- ⋆ metric is direct product but RR fields + dilaton are not
- ⋆ dilaton is also singular at $\rho \to \kappa^{-1}$: $\qquad e^{\Phi} \to \infty$
- ⋆ $\sqrt{G} e^{-2\Phi}$ = regular:

singularity "resolved" by formal T-duality in t?

$$dr^2 + r^2 d\phi^2 \,, \quad e^{\Phi} = 1 \qquad \leftrightarrow \qquad dr^2 + \frac{1}{r^2} d\widetilde{\phi}^2, \qquad e^{\widetilde{\Phi}} = \frac{1}{r}$$

• special limit: $\kappa \to \infty$

and $\rho, r \to 0$ (or $\mathrm{h} \to \infty$) with $\rho' = \kappa\rho,\ r' = \kappa r$ = fixed

$$ds^2 = \frac{-dt^2 + d\rho'^2}{1-\rho'^2} + \frac{d\varphi^2 + dr'^2}{1+r'^2}$$

formally related to $dS_2 \times H_2$ (with imaginary F_2):

(i) T-duality $t, \varphi \to \widetilde{t}, \widetilde{\varphi}$

(ii) analytic continuation $\widetilde{t} = i\varphi',\ \widetilde{\varphi} = -it'$ (and flip)

$$ds'^2 = -(1+r'^2)dt'^2 + \frac{dr'^2}{1+r'^2} + (1-\rho'^2)d\varphi'^2 + \frac{d\rho'^2}{1-\rho'^2}$$

special limit: $\kappa = i$

$$\kappa^2 = -1 + \epsilon^2 \ , \quad (t, \varphi) = \epsilon^{-1} x^+ \mp \epsilon x^- \ , \quad \epsilon \to 0$$

$$ds^2 = dx^+ dx^- - V(v, \bar{v})(dx^+)^2 + dv d\bar{v} \ , \qquad v = \alpha + i\beta$$

$$V = |\sin v|^2 = \sin^2 \alpha + \sinh^2 \beta \ , \qquad \rho = \tan \alpha, \quad r = \tanh \beta$$

supported by F_5 in 10d or by its T^6 reduction F_2 in 4d

$$F_2 = \big(\cos v \, dv + \cos \bar{v} \, d\bar{v} \big) \wedge dx^+$$

⋆ pp-wave background with integrable l.c. gauge action

[Maldacena-Maoz 02; Russo, AT 02; Bakas, Sonnenschein 02]

⋆ background preserves 4d space-time supersymmetry

⋆ resulting l.c. gauge superstring action same as of PR model for $AdS_2 \times S^2$ superstring [Grigoriev, AT 07] i.e. (2,2) world-sheet supersymmetric sine-Gordon model

⋆ implies relation between l.c. S-matrices in this limit

Deformed $AdS_3 \times S^3$

$AdS_3 \times S^3 \times T^4$ superstring: related to supercoset

$$\frac{PSU(1,1|2) \times PSU(1,1|2)}{SO(1,2) \times SO(3)}$$

6d metric of defomed $AdS_3 \times S^3$ supercoset: sum of

$$ds_A^2 = \frac{1}{1 - \kappa^2 \rho^2}\Big[-(1+\rho^2)dt^2 + \frac{d\rho^2}{1+\rho^2}\Big] + \rho^2\, d\psi^2$$

$$ds_S^2 = \frac{1}{1 + \kappa^2 r^2}\Big[(1-r^2)d\varphi^2 + \frac{dr^2}{1-r^2}\Big] + r^2 d\phi^2$$

full type IIB background: similar form as in $AdS_2 \times S^2$

no ρ, r separation for RR fluxes and dilaton, etc.

S^3_κ metric: $[U(1)]^2$ isometry and Z_2 symmetry

$$\varphi \leftrightarrow \phi\,, \qquad r \leftrightarrow \sqrt{\tfrac{1-r^2}{1+\kappa^2 r^2}}$$

special symmetric case of

2-parameter deformation of $SU(2)$ PCM [Fateev 96]

classically integrable [Lukyanov 12]

Fateev model: deformation of S^3 σ-model that

(i) has $U(1) \times U(1)$ symmetry

(ii) preserved by RG flow

(iii) dual to integrable massive theory of 3 bosons

with exponential potential $e^{\beta\varphi}\cos(\phi_1+\phi_2)+...$

(iv) quantum-integrable: S as product of sG S-matrices

$g \in SU(2)$: parameters ℓ, r

$$L_F = \mathcal{M}(g)\Big[\tfrac{1}{2}\mathrm{Tr}(\partial_a g \partial_a g^{-1}) + \ell\, J_a^3 J_a^3 + \mathrm{r}\, \widetilde{J}_a^3 \widetilde{J}_a^3\Big]$$

$$\mathcal{M}^{-1} = (1+\ell)(1+\mathrm{r}) - \tfrac{1}{4}\ell \mathrm{r}\, \big[\mathrm{Tr}(g\sigma_3 g^{-1}\sigma_3)\big]^2$$

$$J^k = \tfrac{1}{2i}\mathrm{Tr}(g^{-1} dg\, \sigma^k)\ , \qquad \widetilde{J}^k = \tfrac{1}{2i}\mathrm{Tr}(dg g^{-1}\, \sigma^k)$$

$\ell = 0$: squashed 3-sphere

$\ell = \mathrm{r} = \frac{1}{2}(\sqrt{1+\kappa^2} - 1)$: $\frac{SO(4)}{SO(3)}$ coset deformation

[$\kappa = 0$ and $\kappa = \infty$ are IR and UV asymptotics of RG flow in bosonic S^3_κ model]

Special limits of full conformal $(AdS_3 \times S^3\,)_\kappa$ model:

- $\kappa = \infty$:

relation by T-duality to $dS_3 \times H^3$ with imaginary F_3 flux

• $\kappa = i$:

becomes pp-wave background ($\rho \equiv \tan\alpha, \quad r \equiv \tanh\beta$)

$$ds^2 = dx^+dx^- - \big(\sin^2\alpha \; + \; \sinh^2\beta\big)\,(dx^+)^2$$

$$+ \, d\alpha^2 + \tan^2\alpha\, d\psi^2 + d\beta^2 + \tanh^2\beta\, d\phi^2$$

⋆ l.c. gauge: complex sine-G + complex sinh-G

⋆ full 6d background ($u = \sin\alpha\, e^{i\psi}, \quad w = \sinh\beta\, e^{i\phi}$)

$$ds^2 = dx^+dx^- - (|u|^2 + |w|^2)\,(dx^+)^2 + \frac{du\, d\bar{u}}{1 - |u|^2} + \frac{dw\, d\bar{w}}{1 + |w|^2}$$

$$e^{-2\Phi} = (1 - |u|^2)(1 + |w|^2)$$

RR 3-form F_3 with real potential

$$C_2 = i\Big[(1+|w|^2)(u d\bar{u} - \bar{u} du) + (1-|u|^2)(w d\bar{w} - \bar{w} dw)\Big] \wedge dx^+$$

⋆ $M^6 \times T^4$ is embedding of complex sine-G + sinh-G integrable model into 10d type IIB string theory cf. [Bakas, Sonnenschein 02]

⋆ corresponding GS Lagrangian is same as of PR model for $AdS_3 \times S^3$ supercoset [Grigoriev, AT 08]

⋆ unlike pp-wave solutions with flat transverse space no target space supersymmetry but should be hidden $(4, 4)$ world-sheet susy in corresponding GS superstring model in l.c. gauge – as in equivalent PR model for $AdS_3 \times S^3$ superstring [Grigoriev, AT 07; Hoare, AT 11]

[Goykhman, Ivanov11; Hollowood, Miramontes 11]

Summary / Open questions

- interesting new examples of integrable deformations of supercoset models with no manifest susy but with q-deformed symmetry
- limits and low-dimensional analogs of deformed $AdS_5 \times S^5$ supercoset integrable model: superstring embedding of deformations of S^2 and S^3
- corresponding l.c. gauge S-matrix related to q-deformation of S-matrix of $AdS_5 \times S^5$ superstring
- full type IIB background for deformed $AdS_5 \times S^5$ model should be similar to one found in $AdS_2 \times S^2$ case
- $\alpha' \sim \mathrm{h}^{-1}$ corrections? κ is not finitely renormalized?
- meaning of YB-type σ-model deformation at quantum / path integral level? why conformal invariance is preserved for supercosets?

- singularity of metric / dilaton resolved in string theory?
  string loop corrections still not important?
- TBA solution for tree-level string spectrum?
- dual gauge theory interpretation?
  implications of hidden symmetry?
- role of complexification of (deformed) $AdS_5 \times S^5$?
  deeper meaning of relation to $dS_5 \times H^5$? unitarity?
- complex q or $\kappa \to i\kappa$?

σ-model for S-matrix with $q = e^{i\pi/k}$
interpolating between non-relativistic $AdS_5 \times S^5$ S-matrix
and relativistic PR S-matrix

- $\kappa = i$: relation to PR model at Lagrangian level:

deformed model $\to$ pp-wave σ-model $\to$
in l.c. gauge PR model for undeformed supercoset

- generalizations: $AdS_3 \times S^3$ with WZ term, $AdS_4 \times CP^3$