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Four point scattering from Amplituhedron

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Object of interest

- Scattering amplitudes in planar $\mathcal{N} = 4$ SYM.
- Huge progress in recent years both at weak and strong coupling.
- Generalized unitarity, Twistor string theory, BCFW recursion relations, Leading singularity methods, Relation between amplitudes and Wilson loops, Yangian symmetry, Strong coupling via AdS/CFT, Symbol of amplitudes, Flux tube S-matrix, Positive Grassmannian and Amplituhedron,...
- Planar $\mathcal{N} = 4$ SYM is integrable: It is believed that scattering amplitudes in this theory should be exactly solvable.
- Long list of people involved in these discoveries....

Integrand

- The amplitude $\mathcal{M}_{n,k,L}$ is labeled by three indices: n - number of particles, k - $SU(4)$ R-charge, L is the number of loops.
- Integrand: Well-defined rational function to all loop orders in planar limit: sum of all Feynman diagrams prior to integration.

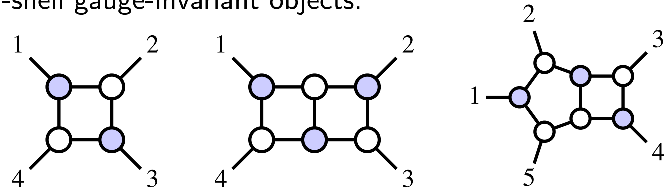
$$\mathcal{M}_{n,k,L} = \int d^4\ell_1 d^4\ell_2 \ldots d^4\ell_L \; \mathcal{I}_{n,k,L}$$

- It is completely fixed by its singularities: locality (position of poles) and unitarity (residues on these poles).
- This is an object of our interest: there is a purely geometric definition of this object which does not make any reference to field theory – *Amplituhedron*.
- There is also a strong evidence of similar structures in the integrated amplitudes.

Positive Grassmannian and On-shell diagrams

[Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, JT, 1212.5605]

- Different expansion of scattering amplitudes using fully on-shell gauge-invariant objects.



  given by gluing together on-shell 3pt amplitudes.
- Explicitly constructed for Yang-Mills theory, and found the expansion of the amplitude in planar $\mathcal{N}=4$ SYM but these objects exist in any QFT.
- On-shell diagrams make the Yangian symmetry of planar $\mathcal{N}=4$ SYM manifest, not local in space-time.
- Direct relation between on-shell diagrams and Positive Grassmannian $G_+(k,n)$.

Positive Grassmannian and On-shell diagrams

[Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, JT, 1212.5605]

- $G_+(k,n)$: $(k \times n)$ matrix mod $GL(k)$

$$C = \begin{pmatrix} * & * & \dots & * \\ \vdots & \vdots & \dots & \vdots \\ * & * & \dots & * \end{pmatrix}$$

  where all maximal minors are positive, $(a_{i_1} a_{i_2} \dots a_{i_k}) > 0$.
- Stratification: cell of $G_+(k,n)$ of dimensionality d given by a set of constraints on consecutive minors.
- For each cell of dimensionality d we can find d positive coordinates x_i, and associate a logarithmic form

$$\Omega_0 = \frac{dx_1}{x_1} \dots \frac{dx_d}{x_d}$$

- The particular linear combination of on-shell diagrams (cells of $G_+(k,n)$) is provided by recursion relations.
- Idea: they glue together into a bigger object.

Amplituhedron

[Arkani-Hamed, JT, 1312.2007]

- We can define Amplituhedron $A_{n,k,L}$ which is a generalization of positive Grassmannian.
- For tree-level $L = 0$, it is a map: $G_+(k, n) \to G(k, k+4)$ defined as

$$Y = C \cdot Z \qquad \text{where } Z \in M_+(k+4, n)$$

- There is a generalization for the loop integrand which involves new mathematical objects.
- In addition we also describe L lines $\mathcal{L}_1, \ldots, \mathcal{L}_L$.

$$C \in G_+(k, n), \qquad D_{i_1 \ldots i_m} \in G(k+2m, n)$$

where D is combination of C and m lines $\mathcal{L}_i$. Then we do the same map,

$$Y = C \cdot Z, \qquad \mathcal{Y} = D \cdot Z$$

Amplituhedron

[Arkani-Hamed, JT, 1312.2007]

- The amplitude is then given by the form with *logarithmic singularities* on the boundaries of this space.
- Logarithmic singularities: if the boundary is characterized by $x = 0$, it is just $\Omega \to \frac{dx}{x}\Omega_0$.
- This is a purely bosonic form but we can extract a supersymmetric amplitude from it: instead of (Z, η) we have one $(4 + k)$-dimensional bosonic variables.
- Two ways how to calculate the form:
  - Fix it from the definition (it is unique).
  - Triangulate the space: for each term in the triangulation we have trivial form

$$\Omega = \frac{dx_1}{x_1}\frac{dx_2}{x_2}\dots\frac{dx_d}{x_d}$$

    and we sum all pieces. On-shell diagrams via recursion relations provide a particular triangulation.

Four-point amplitudes

- The number of Feynman diagrams grows extremely rapidly. Natural strategy: find a basis of scalar and tensor integrals.
- The calculation of integrand of 4pt amplitudes has a long history
  - 1-loop: Brink, Green, Schwarz (1982)
  - 2-loop: Bern, Rozowski, Yan (1997)
  - 3-loop: Bern, Dixon, Smirnov (2005)
  - 4-loop: Bern, Czakon, Dixon, Kosower, Smirnov (2006)
  - 5-loop: Bern, Carrasco, Johannson, Kosower (2007)
  - 6,7-loop: Bourjaily, DiRe, Shaikh, Spradlin, Volovich (2011)
- Even in a suitable basis there is a fast growth of the number of diagrams – no sign of simplification.

L	1	2	3	4	5	6	7
# of diagrams	1	1	2	8	34	256	2329

The 7-loop result is several millions of terms.

Four-point amplitudes

- BDS ansatz [Bern, Dixon, Smirnov, 2005] for the integrated expression for MHV amplitudes in dimensional regularization

$$M_{n,L} = \exp\left[\sum_{L=1}^{\infty} \lambda^L \left(f^{(L)}(\epsilon) M_{n,1}(L\epsilon) + C^{(L)} + \mathcal{O}(\epsilon)\right)\right]$$

where

$$f^{(L)}(\epsilon) = f_0^{(L)} + \epsilon f_1^{(L)} + \epsilon^2 f_2^{(L)}$$

- The leading IR divergent piece is given by

$$f(\lambda) = \sum_{L=1}^{\infty} f_0^{(L)} \lambda^L$$

is known as cusp anomalous dimension, which also governs the scaling of twist-two operators in the limit of large spin S,

$$\Delta\left(\mathrm{Tr}[Z D^S Z]\right) - S = f(\lambda) \log S + \mathcal{O}(S^0)$$

It satisfies BES [Beisert, Eden, Staudacher, 2006] integral equation which can be solved analytically to arbitrary order.

Four-point amplitudes

- There is a tension between results for the integrand and the integrated answer.
- Integrand is a rational function with infinite complexity for $L \to \infty$ (it must capture all cuts) but the non-trivial part of the integrated result is given by simple functions of coupling.
- Important question: Is there a sign of this simplification at the integrand level? What is the role of integrability?
- The ultimate goal:
  - Describe the Amplituhedron space for integrand, its stratification and topological properties.
  - Try to find the form with log singularities to all loops (if it exists in a closed form).
  - If yes, try to find a way how to extract (perhaps some natural deformation [Beisert, Broedel, Ferro, Lukowski, Meneghelli, Plefka, Rosso, Staudacher,...]) a BES equation – ie. understand the integration process as some kind of geometric map.

Four-point amplitudes from Amplituhedron

[Arkani-Hamed, JT, 1312.7878]

- The definition of the Amplituhedron in case of four point amplitudes at arbitrary L is very simple:
  - Let us have $4L$ positive parameters,

    $$x_i, y_i, z_i, w_i \geq 0 \qquad \text{for } i = 1, 2, \ldots L$$

    which satisfy $L(L-1)/2$ quadratic inequalities.

    $$(x_i - x_j)(w_i - w_j) + (y_i - y_j)(z_i - z_j) \leq 0 \qquad \text{for all pairs } i, j$$

  - The amplitude is then the form with logarithmic singularities on the boundaries of this space.
- In this special case the Z-map is not present and the external data are irrelevant.

One-loop amplitude

- We have four parameters $x_1, y_1, z_1, w_1 \geq 0$
- There is no quadratic condition, the form with logarithmic singularities on the boundaries $(0, \infty)$ is just

$$\Omega = \frac{dx_1}{x_1}\frac{dy_1}{y_1}\frac{dz_1}{z_1}\frac{dw_1}{w_1}$$

- We can solve for parameters x_1, y_1, z_1, w_1 in terms of kinematical variables

$$\Omega = \frac{\langle AB\, d^2 Z_A\rangle\langle AB\, d^2 Z_B\rangle\langle 1234\rangle^2}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB41\rangle} = \frac{d^4\ell\, st}{\ell^2(\ell+p_1)^2(\ell+p_1+p_2)^2(\ell-p_4)^2}$$

Two-loop amplitude

- For $L = 2$ we have $x_1, y_1, z_1, w_1, x_2, y_2, z_2, w_2 \geq 0$ which satisfy quadratic relation

$$(x_1 - x_2)(w_1 - w_2) + (y_1 - y_2)(z_1 - z_2) \leq 0$$

- The form has the form

$$\Omega = \frac{dx_1\, dx_2 \dots dz_2\;\, N(x_1, x_2 \dots z_2)}{x_1 y_1 w_1 z_1 x_2 y_2 w_2 z_2 [(x_1 - x_2)(w_1 - w_2) + (y_1 - y_2)(z_1 - z_2)]}$$

  It is a 8-form with 9 poles – non-trivial numerator.
- There are two different strategies to find this form:
  - Expand it as a sum of terms with 8 poles with no numerator – triangulation. [Arkani-Hamed, JT, 1312.7878]
  - Fix the numerator directly. [JT, in progress]

Fixing the two-loop amplitude

- Example 1: calculate residuum $y_1 = y_2 = x_2 = 0$,

$$\Omega = \frac{dx_1\, dz_1\, dz_2\, dw_1\, dw_2\ \widetilde{N}}{x_1^2 w_1 z_1 w_2 z_2 (w_1 - w_2)} \quad \to \quad \widetilde{N} \sim x_1$$

- Example 2: For $x_2 = w_2 = y_2 = z_2 = 0$ we have

$$x_1 w_1 + y_1 z_1 \leq 0$$

  and therefore the numerator must vanish $\widetilde{N} = 0$.
- These conditions fix completely the numerator up to overall constant to be

$$N = x_1 w_2 + x_2 w_1 + y_1 z_2 + y_2 z_1$$

Topology of Amplituhedron

[Franco, Galloni, Mariotti, JT, in progress]

- Topology of $G_+(k,n)$: Euler characteristic $= 1$, it is a very non-trivial property of the space.
- The $L = 1$ case is just $G_+(2,4)$

dim	4	3	2	1	0
# of boundaries	1	4	10	12	6

  with Euler characteristic $\mathcal{E} = 1$.

- We can count boundaries of $L = 2$,

dim	8	7	6	5	4	3	2	1	0
# of boundaries	1	9	44	144	286	340	266	136	34

  Alternating sum of these numbers gives $\mathcal{E} = 2$.

- There are preliminary results for $L = 3, 4$ which show similar topological properties.
- The non-trivial topology is probably closely related to the complexity of the logarithmic form.

Amplitude at L-loops

- At L-loops we have $4L$ positive parameters $x_i, y_i, z_i, w_i \geq 0$ with

$$(x_i - x_j)(w_i - w_j) + (y_i - y_j)(z_i - z_j) \leq 0 \qquad \text{for all pairs } i, j$$

- We can write the general form

$$\Omega = \frac{dx_1\, dy_1\, \dots dz_L \;\; N(x_1, \dots, z_L)}{x_1 y_1 w_1 z_1 \dots x_L y_L w_L z_L \prod_{ij} [(x_i - x_j)(w_i - w_j) + (y_i - y_j)(z_i - z_j)]}$$

  and fix the numerator from constraints. So far this is too hard to solve in general.
- There are two types of special cases we can solve at this moment:
  - Find the form on certain residues (cuts of the amplitude).
  - Smaller set of positivity conditions.

Cuts of L-loop amplitude

[Arkani-Hamed, JT, 1312.7878]

- There are certain residues of Ω (cuts of the amplitude) we can solve for all L.
- Example: quadratic equations factorize, $z_i = 0$.

$$(x_i - x_j)(y_i - y_j) \leq 0$$

- All x_i, y_j are then ordered. The form is

$$\Omega = \frac{1}{w_1 \dots w_L} \sum_\sigma \Omega_\sigma$$

where

$$\Omega_{1\dots n} = \frac{1}{x_1(x_1 - x_2)\dots(x_{L-1} - x_L)y_L(y_L - y_{L-1})\dots(y_2 - y_1)}$$

Toy model for L-loop amplitude

[JT, in progress]

- Let us consider a reduced version of our problem: $4L$ positive variables $x_i, y_i, z_i, w_i \geq 0$, ordered $i = 1, 2, \ldots n$.
- We impose quadratic conditions only between adjacent indices

$$(x_i - x_{i+1})(w_i - w_{i+1}) + (y_i - y_{i+1})(z_i - z_{i+1}) \leq 0$$

- The form is then

$$\Omega = \frac{dx_1\, dy_1\, \ldots dz_L \;\; N_L(x_1, \ldots, z_L)}{x_1 y_1 w_1 z_1 \ldots x_L y_L w_L z_L \prod_{j=i+1} [(x_i - x_j)(w_i - w_j) + (y_i - y_j)(z_i - z_j)]}$$

  We can fully constrain the numerator N_L and write down the explicit solution for any L.
- The solution has an interesting structure:

$$N_L = N_2(12) N_2(23) \ldots N_2(L1) + \Delta_L$$

  where N_2 is the $L = 2$ numerator.

Conclusion

- The problem of calculating the integrand of four-point amplitudes in planar $\mathcal{N} = 4$ SYM can be reformulated in the context of the Amplituhedron.
- We can easily define the problem but to find the solution to all loop orders is hard. I showed some partial results but the complete solution is still missing.
- There must be a close relation between the topology of the space and non-triviality of the form.
- Four point amplitudes as an ideal test case: if the full perturbative expansion for amplitudes can be solved exactly (despite there is no evidence for it so far) we should see it here.