Entanglement Negativity in Conformal Field Theory

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P. Calabrese, J. Cardy and E.T.;	**[1206.3092]**
	[1210.5359]
P. Calabrese, L. Tagliacozzo and E.T.;	**[1302.1113]**
A. Coser, L. Tagliacozzo and E.T.;	**[1309.2189]**
P. Calabrese, A. Coser and E.T.;	**[14xx.xxxx]**

Strings 2014

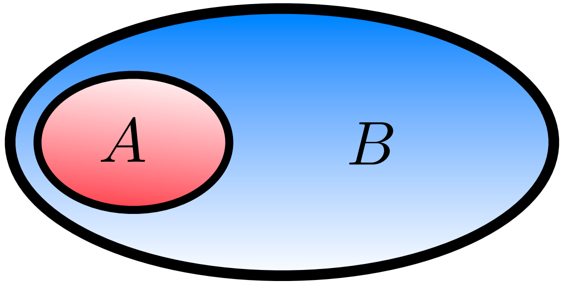
Princeton, June 2014

Outline

- Entanglement in 2D CFT:
  - Motivations for negativity and definitions
  - Entanglement entropies for disjoint intervals
  - Entanglement negativity: pure and mixed states
  - Entanglement negativity after a global quantum quench

Motivations for Negativity

- Ground state $\rho = |\Psi\rangle\langle\Psi|$ and *bipartite* system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$



Reduced density matrix $\rho_A = \mathrm{Tr}_B \rho$

Entanglement entropy $S_A \equiv -\mathrm{Tr}(\rho_A \log \rho_A) = \lim_{n\to 1} \frac{\log(\mathrm{Tr}\rho_A^n)}{1-n} = -\lim_{n\to 1} \frac{\partial}{\partial n} \mathrm{Tr}\, \rho_A^n$

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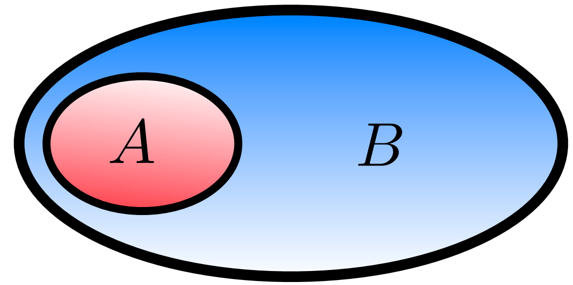

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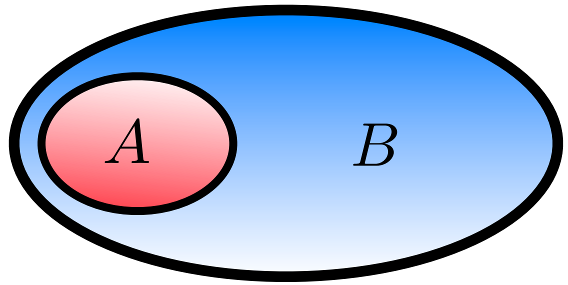

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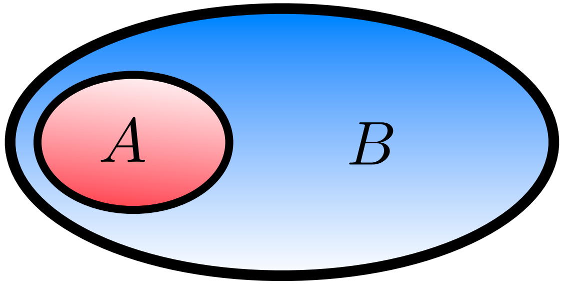

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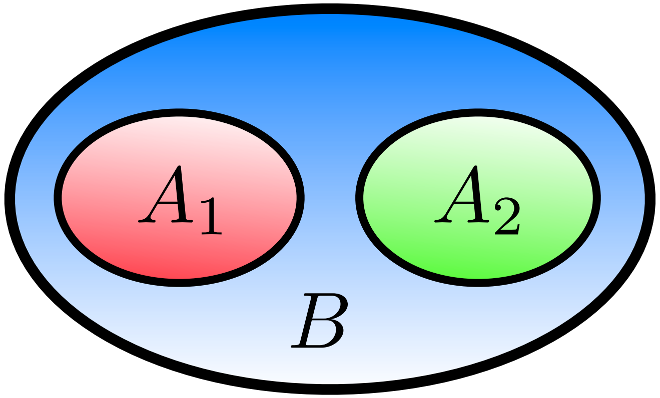

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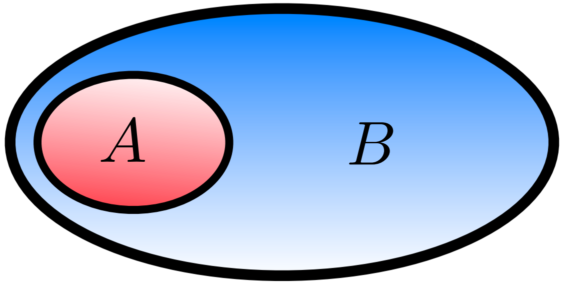

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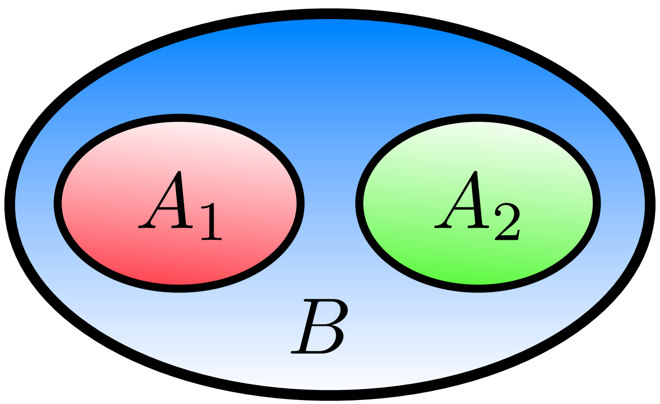
Entanglement between A_1 and A_2?



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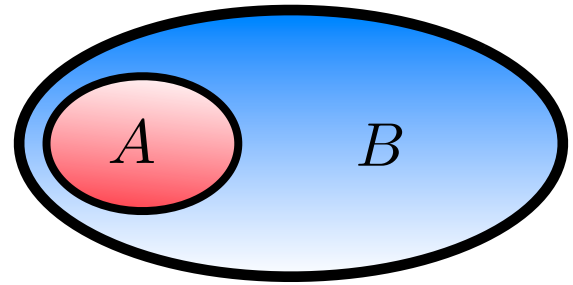

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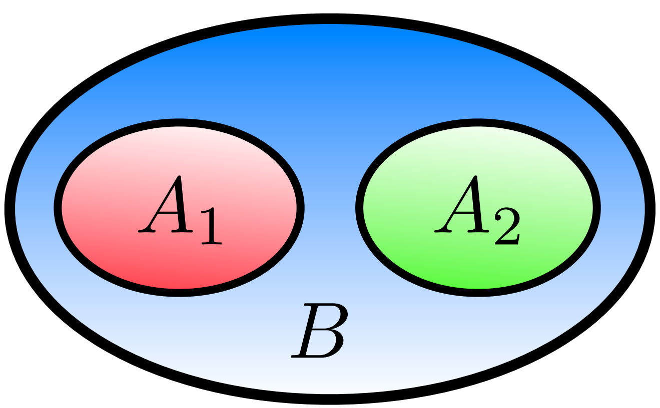

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Entanglement between A_1 and A_2?

- The mutual information $S_{A_1} + S_{A_2} - S_{A_1 \cup A_2}$ gives an upper bound
- A *computable* measure of the entanglement is the *logarithmic negativity* [Vidal, Werner, (2002)]

Partial transpose & Negativity: definitions

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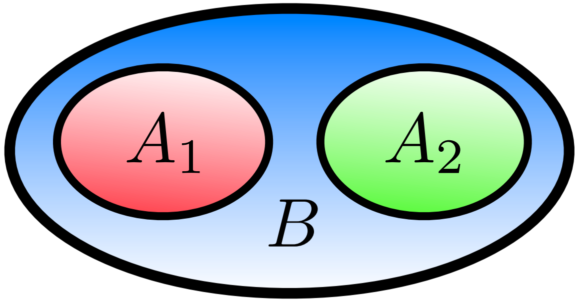

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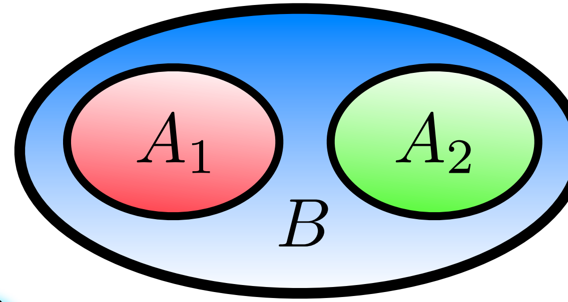

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($|e_i^{(k)}\rangle$ base of $\mathcal{H}_{A_k}$)

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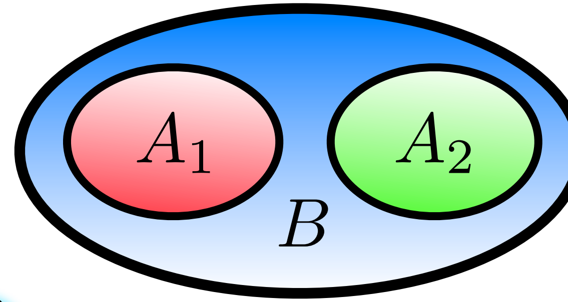

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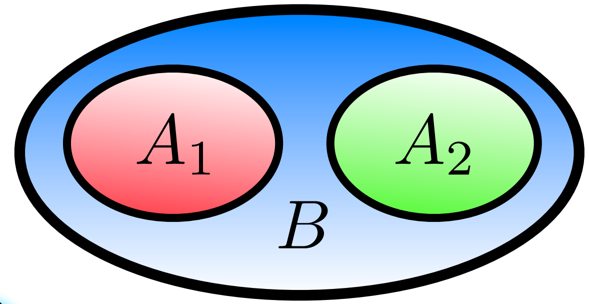

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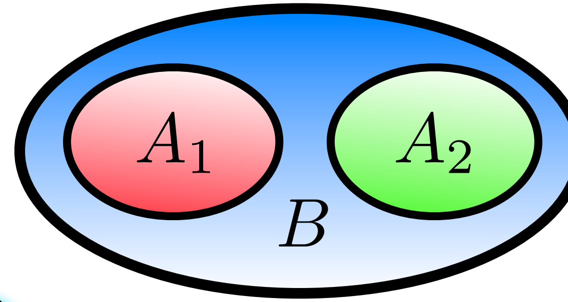

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Replica approach to Negativity

[Calabrese, Cardy, E.T., (2012)]

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- A *parity effect* for $\mathrm{Tr}(\rho^{T_2})^n$

$$\mathrm{Tr}(\rho^{T_2})^{n_e} = \sum_i \lambda_i^{n_e} = \sum_{\lambda_i>0} |\lambda_i|^{n_e} + \sum_{\lambda_i<0} |\lambda_i|^{n_e}$$

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Schmidt decomposition

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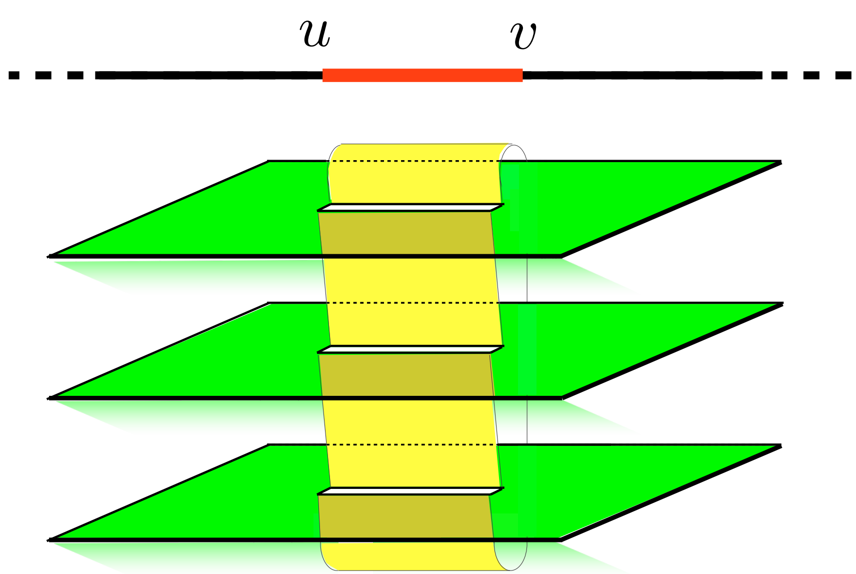
- Taking $n_e \to 1$ we have $\mathcal{E} = 2\log \mathrm{Tr}\rho_2^{1/2}$ (Renyi entropy 1/2)

2D CFT: Renyi entropies as correlation functions



$\mathcal{R}_{1,n}$

$\mathrm{Tr}\rho_A^n$

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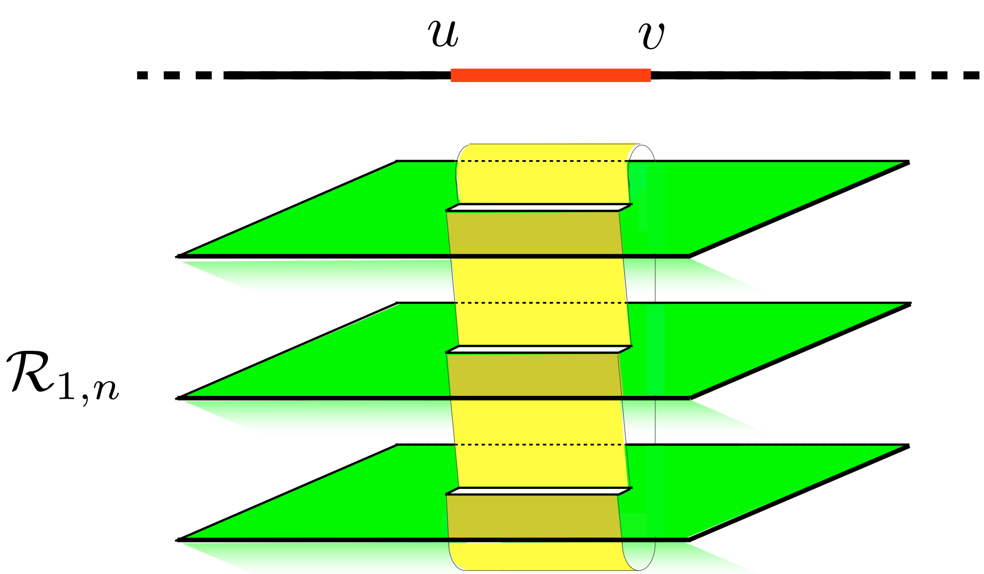
- One interval ($N = 1$): the Renyi entropies can be written as a two point function of *twist fields* on the sphere [Calabrese, Cardy, (2004)]



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$$\Delta_n = \frac{c}{12}\left(n - \frac{1}{n}\right)$$

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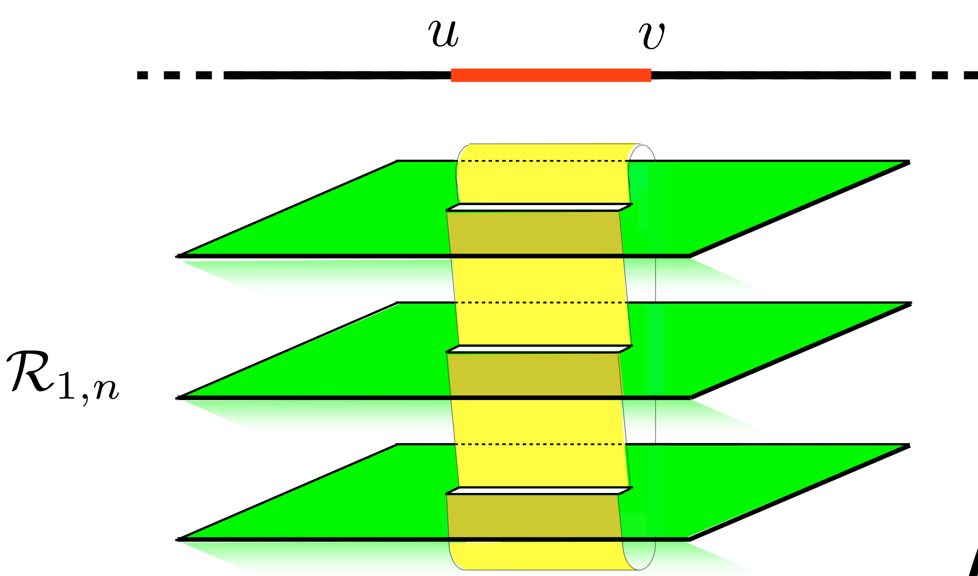

[Holzhey, Larsen, Wilczek, (1994)]

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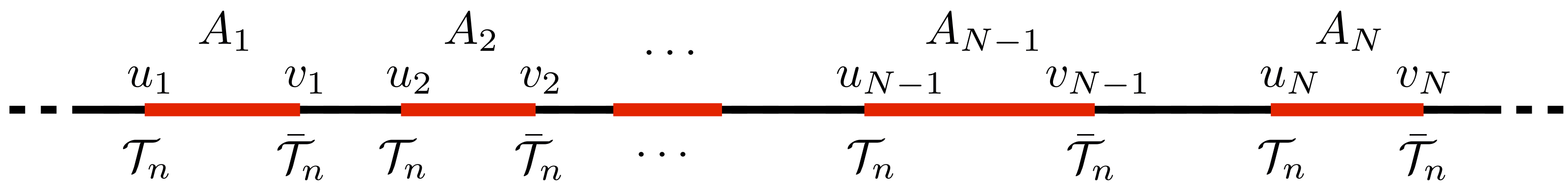

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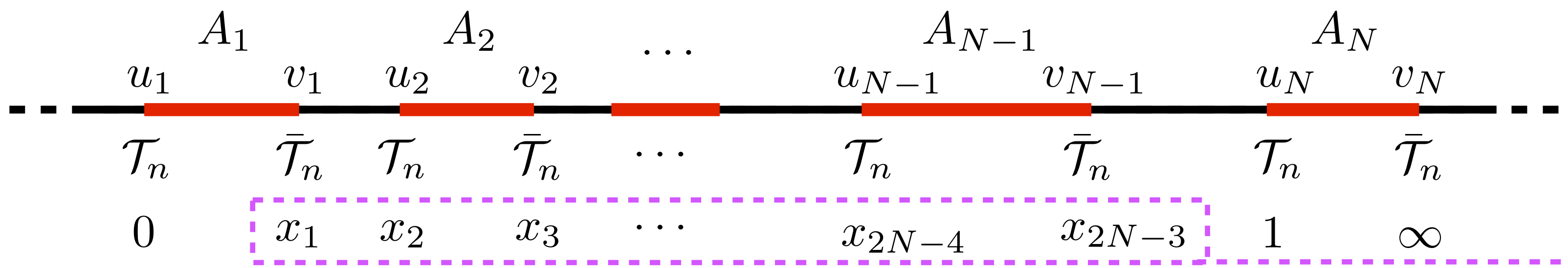

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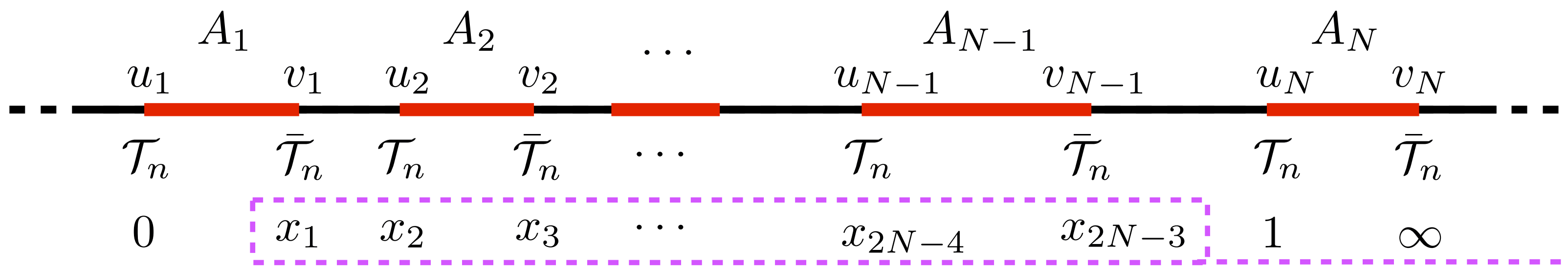

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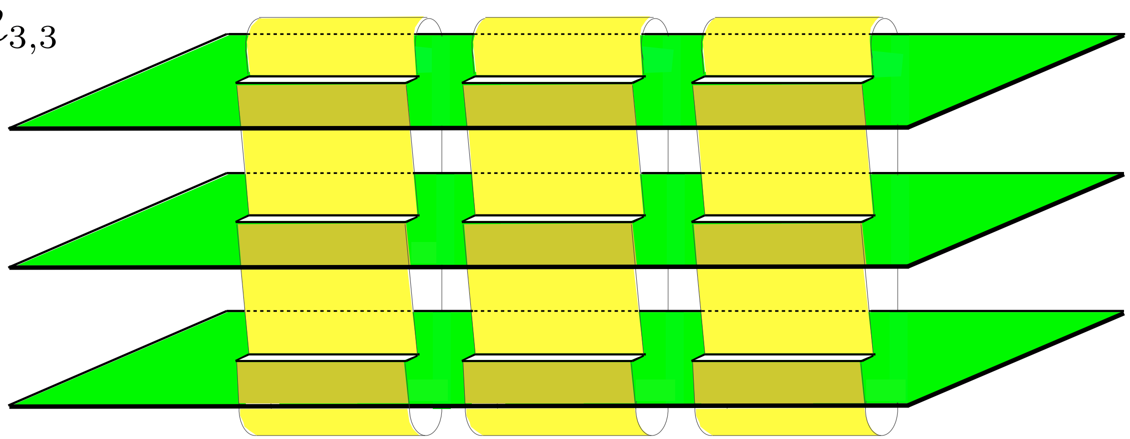
$\mathcal{R}_{3,3}$



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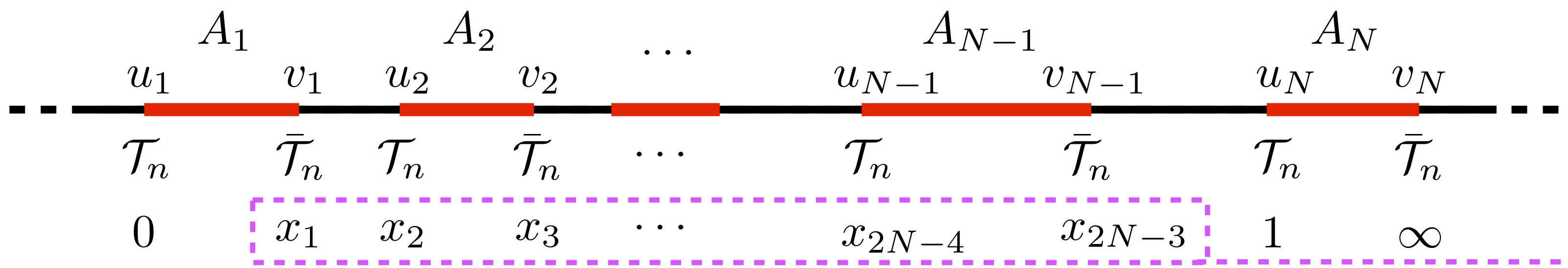

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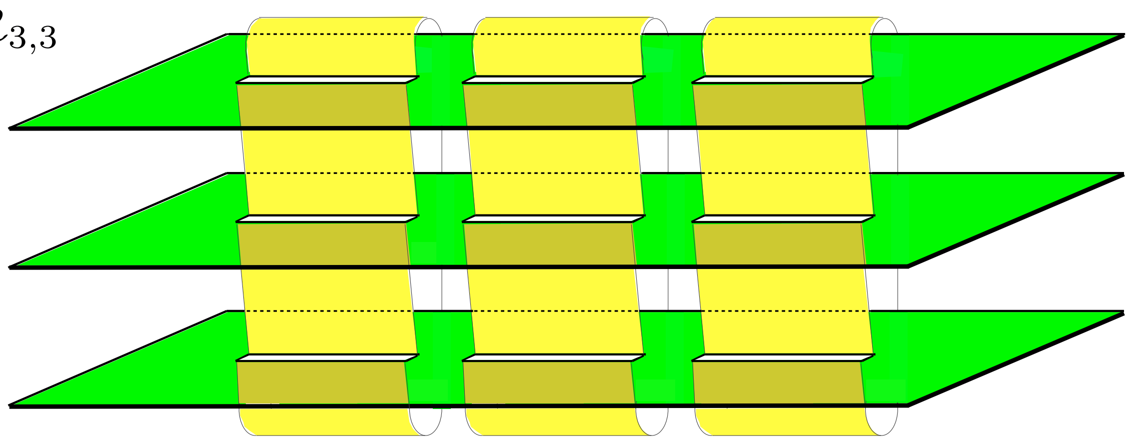

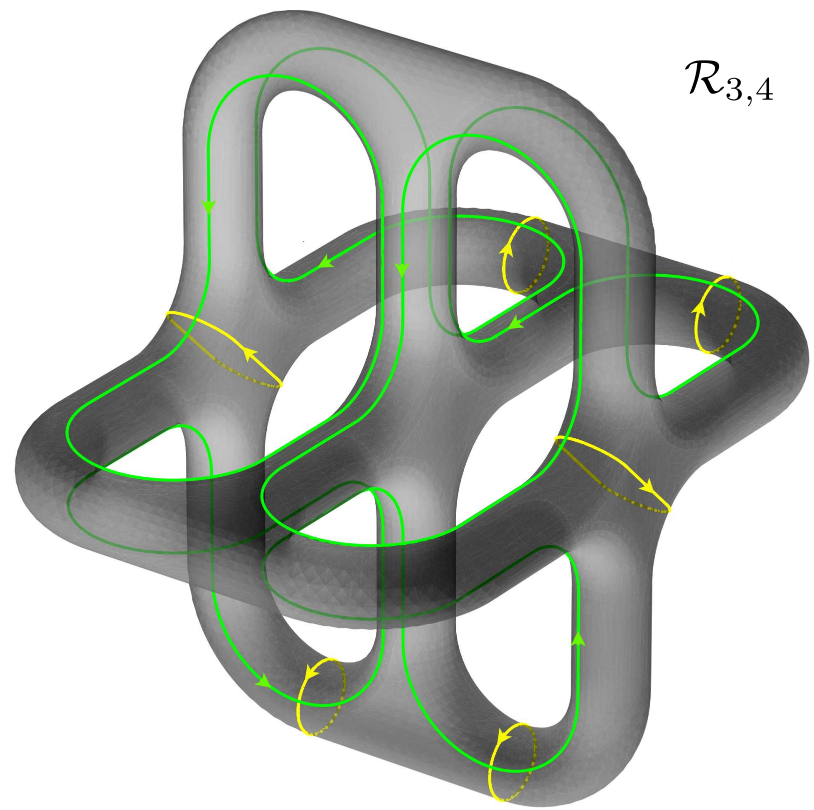

$\mathcal{R}_{3,4}$

Periodic harmonic chain

- Harmonic chain on a circle (critical for $\omega = 0$)

$$H = \frac{1}{2} \sum_{j=1}^{L} \left[\, p_j^2 + \omega^2 q_j^2 + (q_{j+1} - q_j)^2 \, \right]$$

[Peschel, Chung, (1999)] [Botero, Reznik, (2004)]
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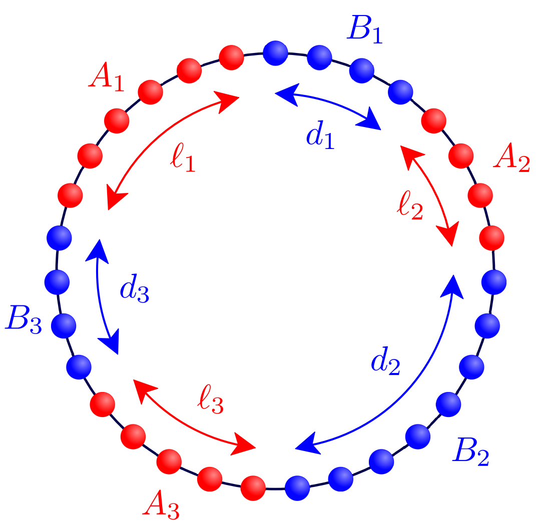

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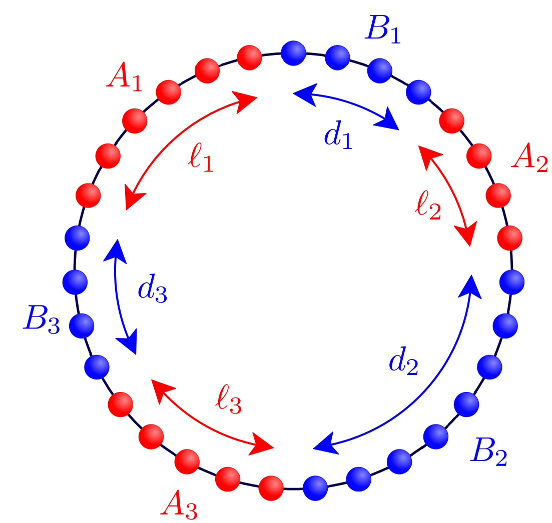


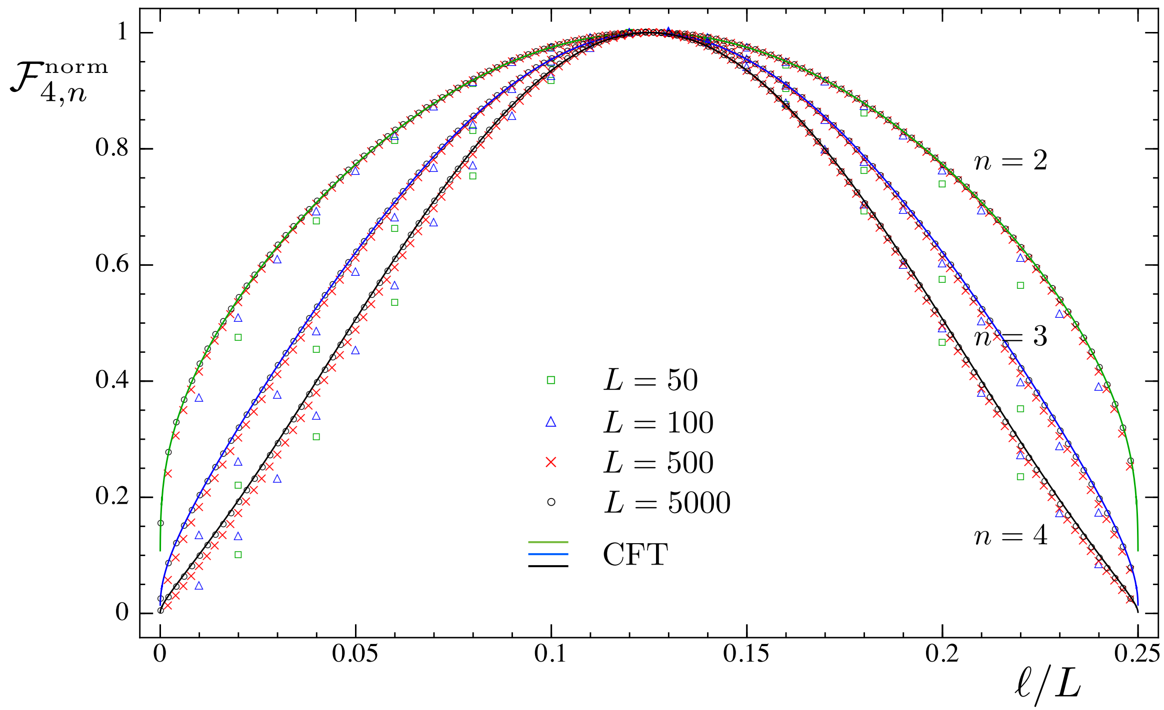
- Decompactification regime

[Dijkgraaf, Verlinde, Verlinde, (1988)] [. . .]
[Coser, Tagliacozzo, E.T., (2013)]

$$\mathcal{F}_{N,n}^{\rm dec}(\boldsymbol{x}) = \frac{\eta^{g/2}}{\sqrt{\det(\mathcal{I})}\, |\Theta(\mathbf{0}|\tau)|^2}$$

- period matrix $\tau = \mathcal{R} + \mathrm{i}\,\mathcal{I}$
  [Enolski, Grava, (2003)]
- Riemann theta function Θ

→ Nasty n dependence



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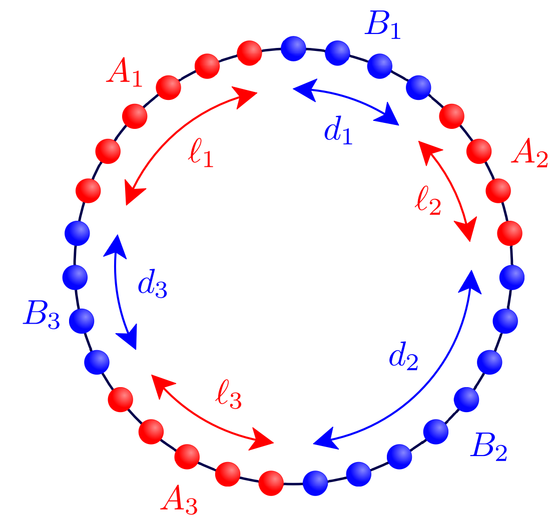

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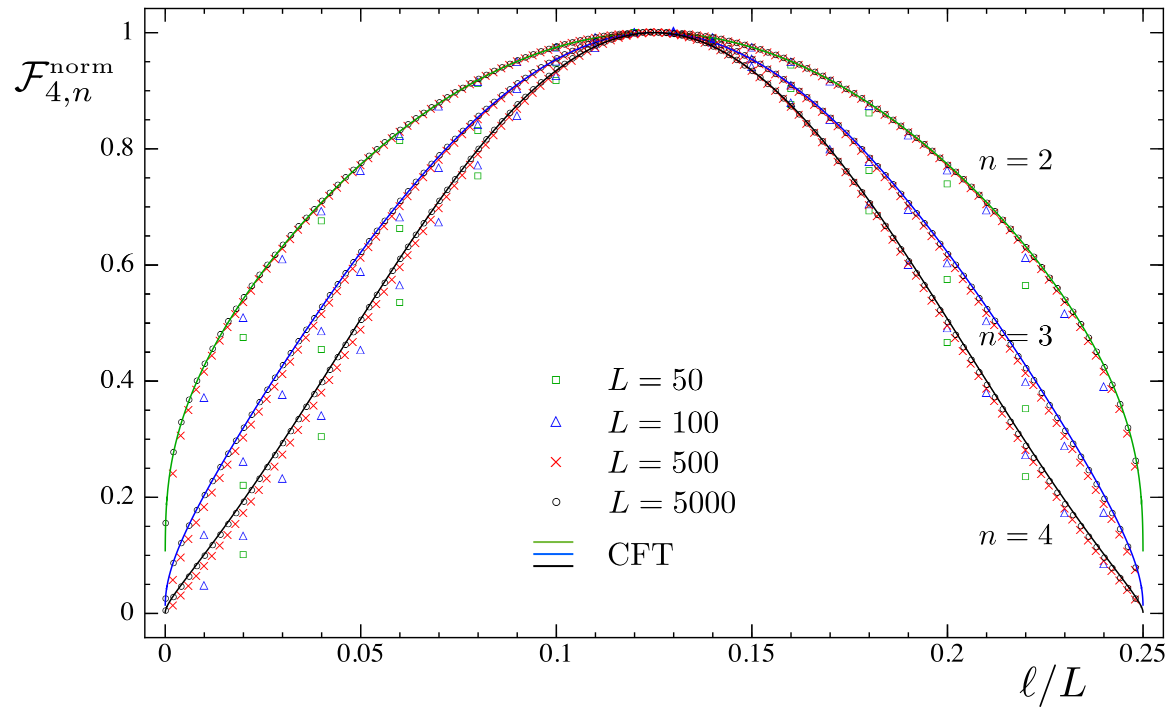
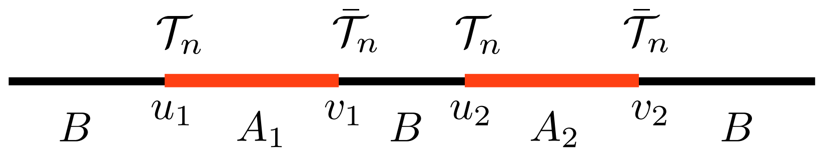
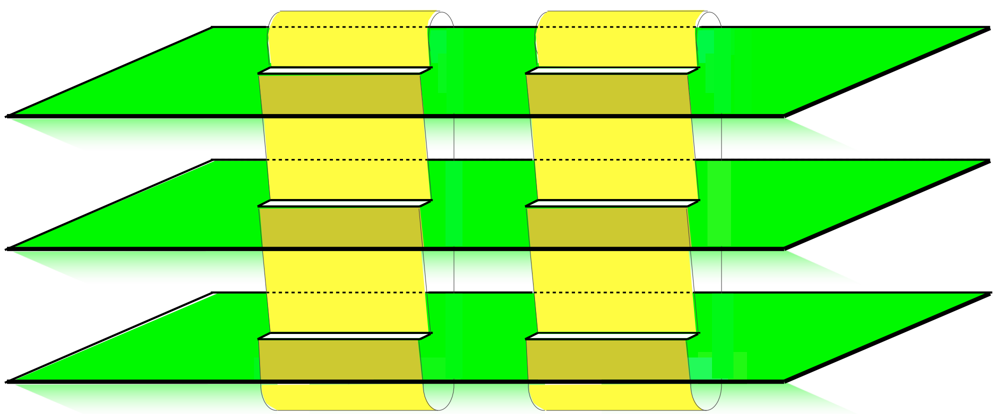


- Numerical checks for the Ising model through Matrix Product States

Partial transposition: two disjoint intervals

$$\mathrm{Tr}\rho^n_{A_1\cup A_2}$$





$$\mathrm{Tr}\rho^n_A \;=\; \langle \mathcal{T}_n(u_1)\bar{\mathcal{T}}_n(v_1)\mathcal{T}_n(u_2)\bar{\mathcal{T}}_n(v_2)\rangle$$

[Caraglio, Gliozzi, (2008)]
[Furukawa, Pasquier, Shiraishi, (2009)]
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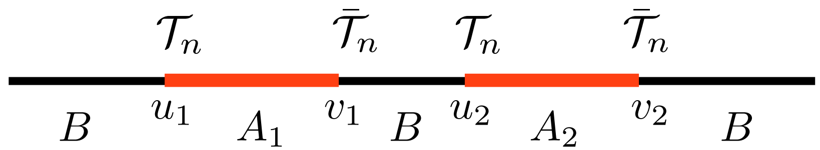

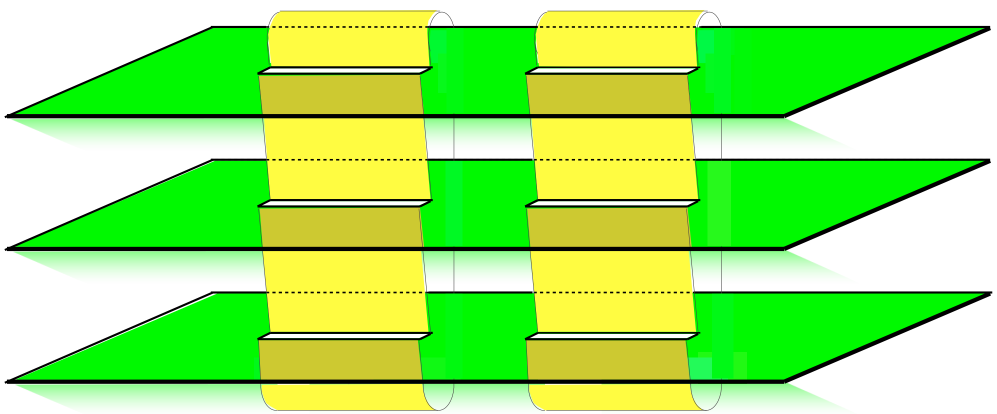

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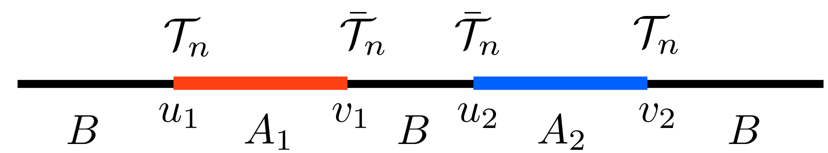
$$\mathrm{Tr}\left(\rho^{T_2}_{A_1 \cup A_2}\right)^n$$



- The partial transposition exchanges $\mathcal{T}_n$ and $\bar{\mathcal{T}}_n$

[Calabrese, Cardy, E.T., (2012)]

Partial transposition: two disjoint intervals

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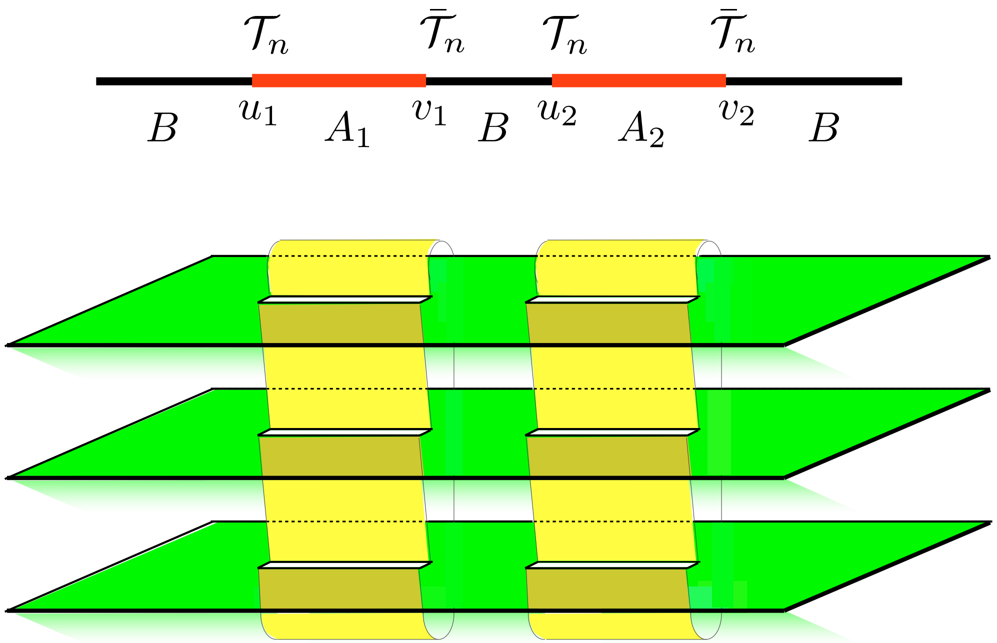

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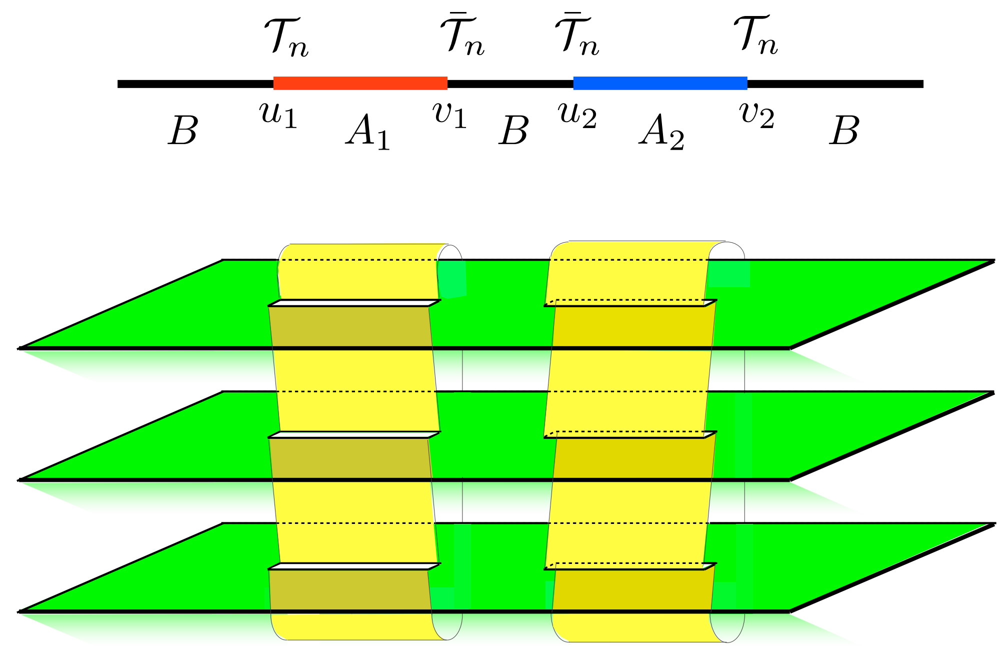

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Renyi entropies vs traces of the Partial Transpose

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Partial Transposition for bipartite systems: pure states

$$\mathcal{H} = \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$$

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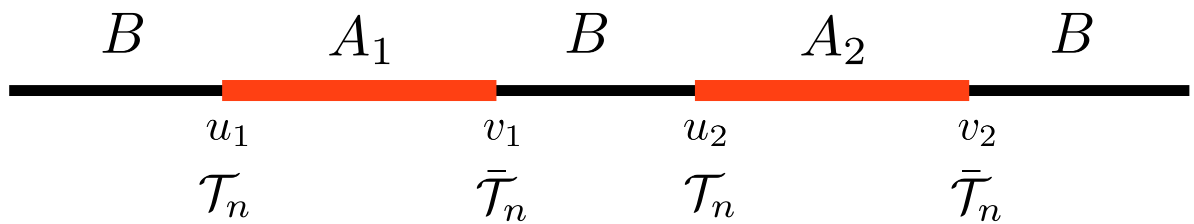

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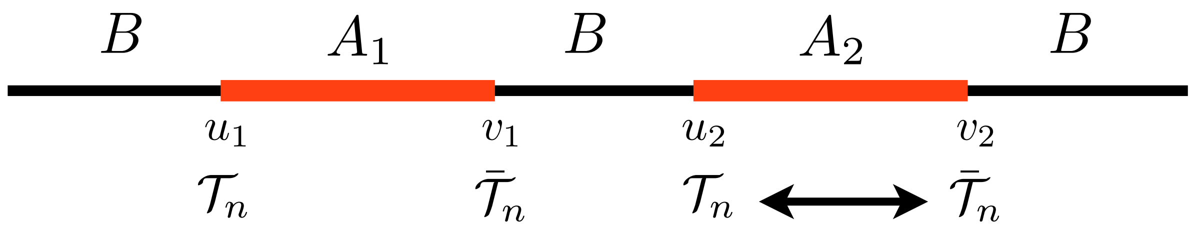

Partial Transposition = exchange two twist fields

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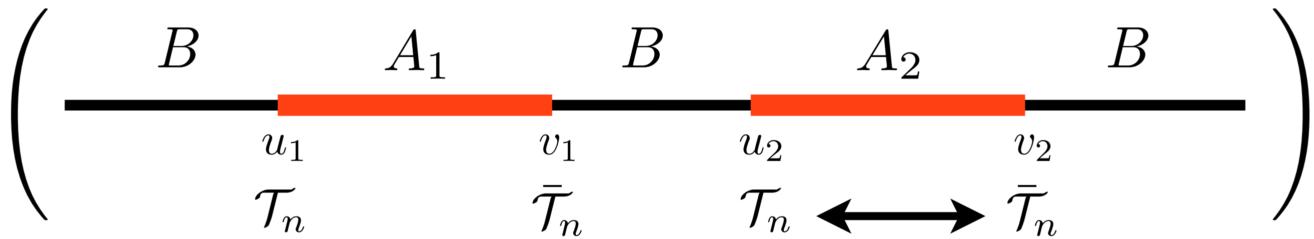

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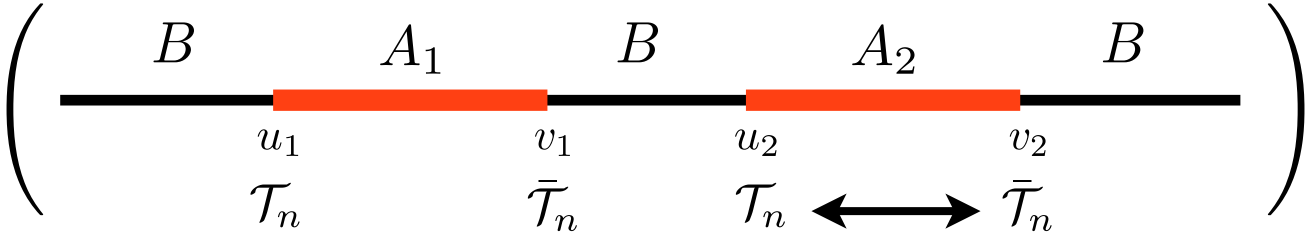

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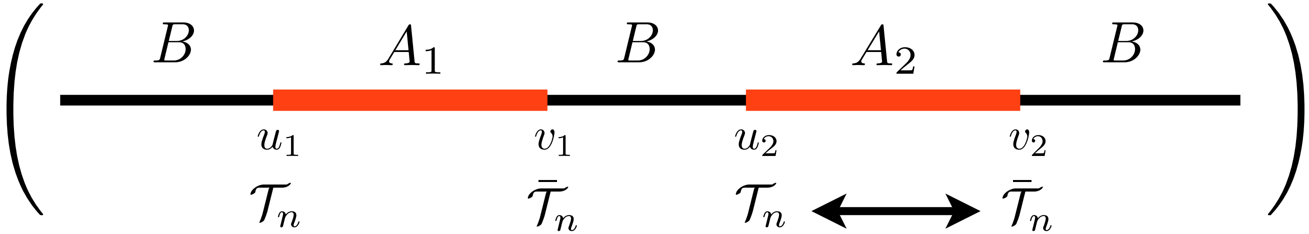

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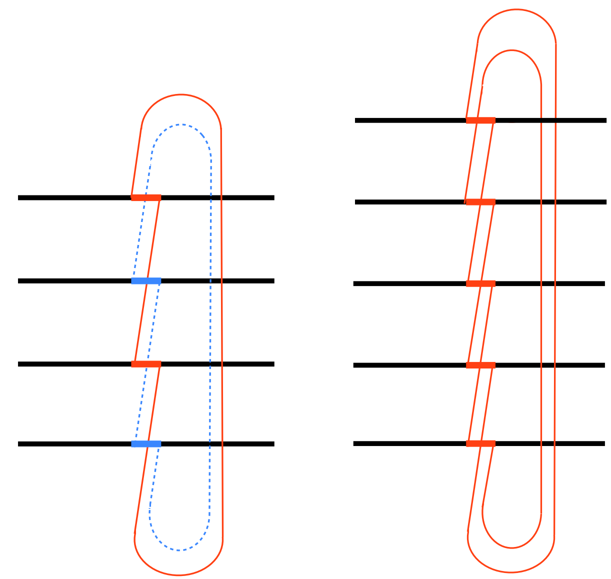

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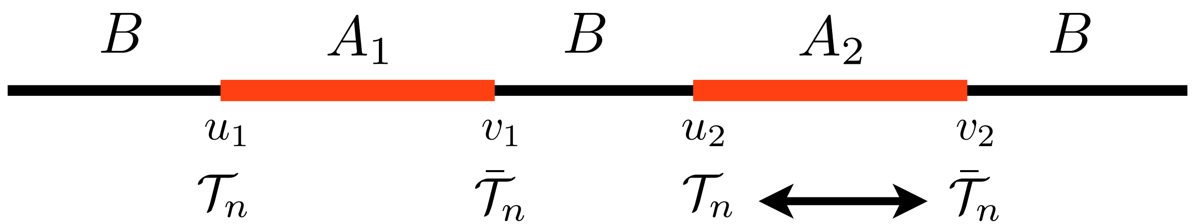
- $\lim_{B \to \emptyset}$ ()

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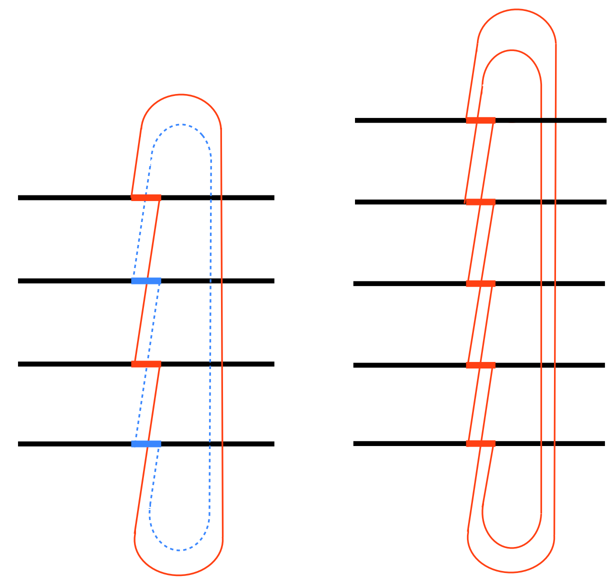

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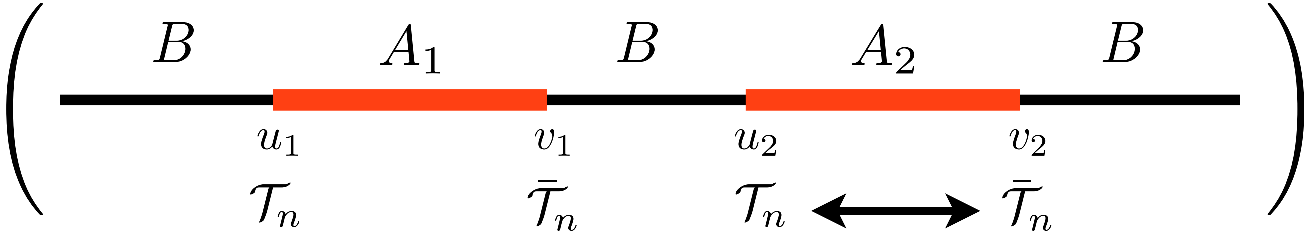
- $\lim_{B \to \emptyset}$ 

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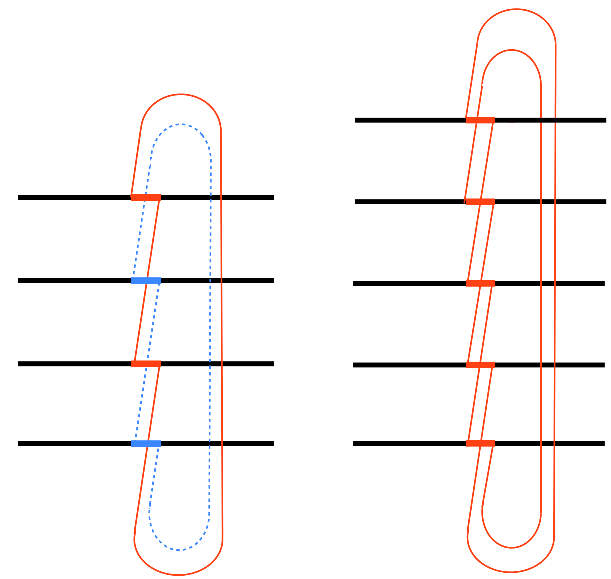

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- Two dimensional CFTs

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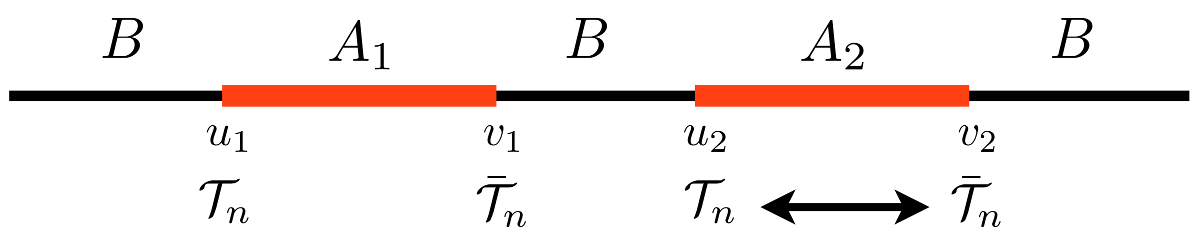

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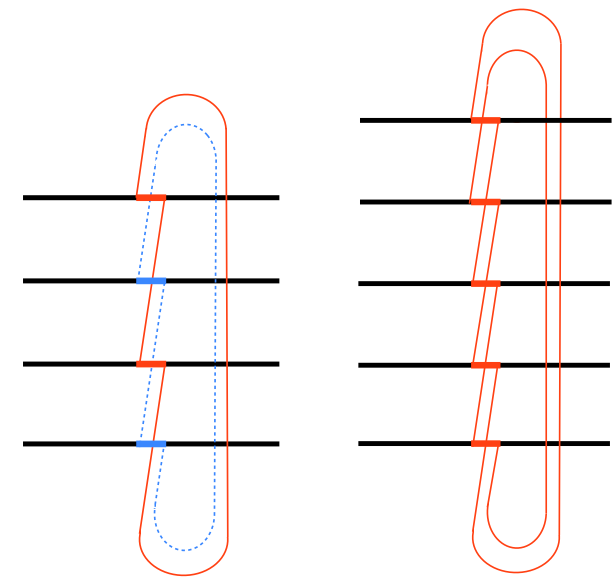

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- Two dimensional CFTs

$$\Delta_{\mathcal{T}_{n_o}^2} = \frac{c}{12}\left(n_o - \frac{1}{n_o}\right) = \Delta_{\mathcal{T}_{n_o}}$$

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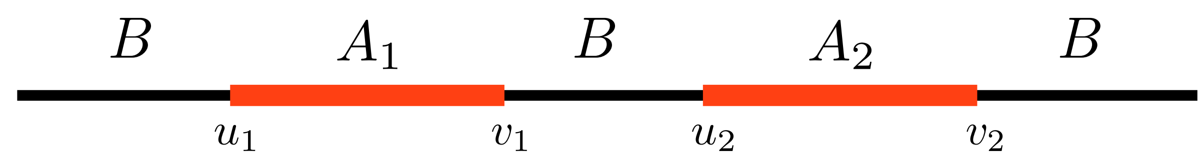

$\mathcal{T}_n$ $\bar{\mathcal{T}}_n$ $\mathcal{T}_n \longleftrightarrow \bar{\mathcal{T}}_n$

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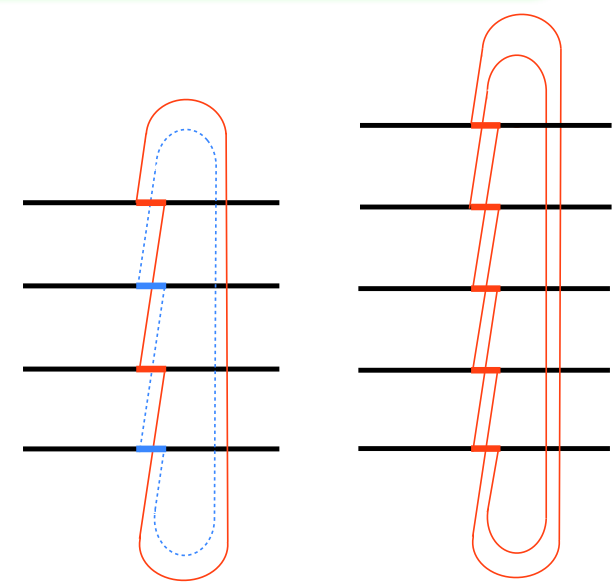

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- Two dimensional CFTs

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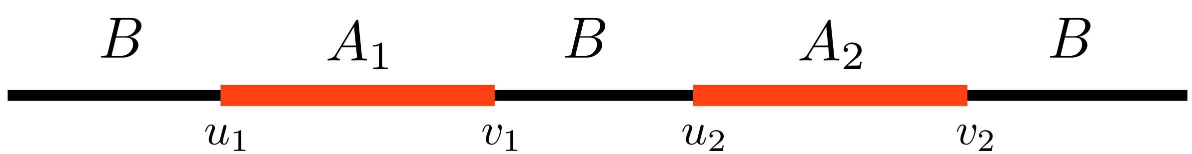
- $\lim_{B \to \emptyset}$ ()

$$\mathcal{T}_n \quad \bar{\mathcal{T}}_n \quad \mathcal{T}_n \longleftrightarrow \bar{\mathcal{T}}_n$$

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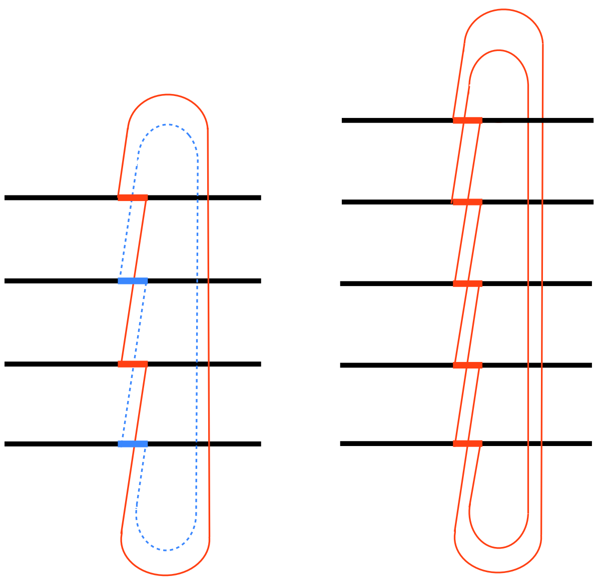

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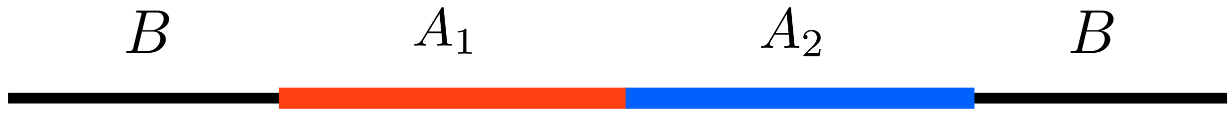
$$\mathcal{E} = \frac{c}{2}\ln \ell + \mathrm{const}$$

Partial Transpose in 2D CFT: two adjacent intervals



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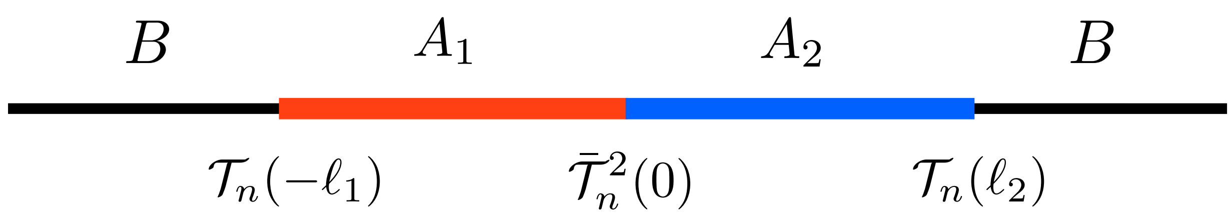

- Three point function

$$\mathrm{Tr}(\rho_A^{T_2})^n = \langle \mathcal{T}_n(-\ell_1)\bar{\mathcal{T}}_n^2(0)\mathcal{T}_n(\ell_2)\rangle$$

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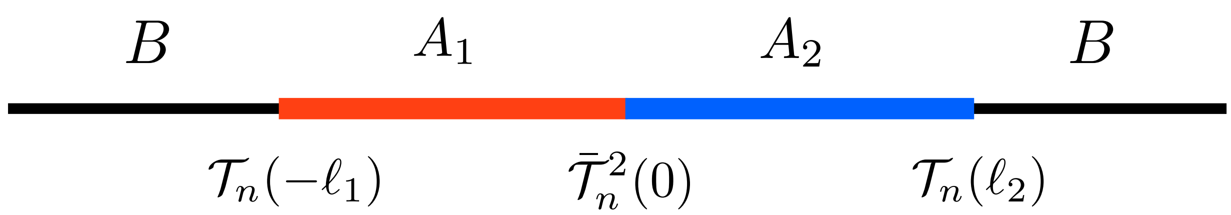

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$$\mathrm{Tr}(\rho_A^{T_2})^{n_o} \propto \big(\ell_1\ell_2(\ell_1+\ell_2)\big)^{-\frac{c}{12}\left(n_o-\frac{1}{n_o}\right)}$$

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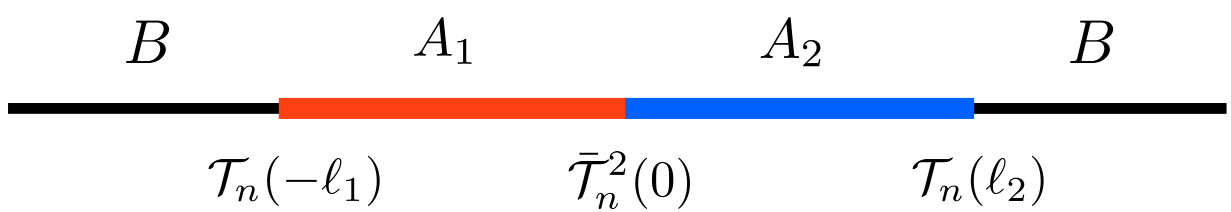

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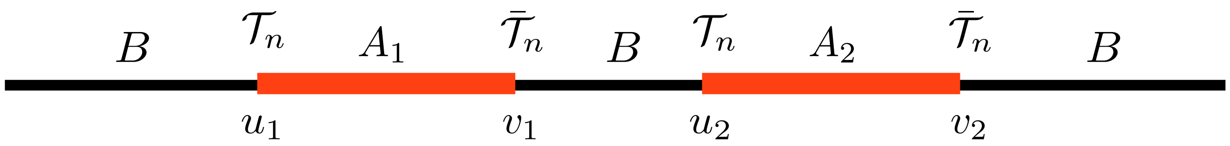
- $$\mathcal{E} = \frac{c}{4}\ln\left(\frac{\ell_1\ell_2}{\ell_1+\ell_2}\right) + \text{const}$$

Partial Transpose in 2D CFT: two disjoint intervals

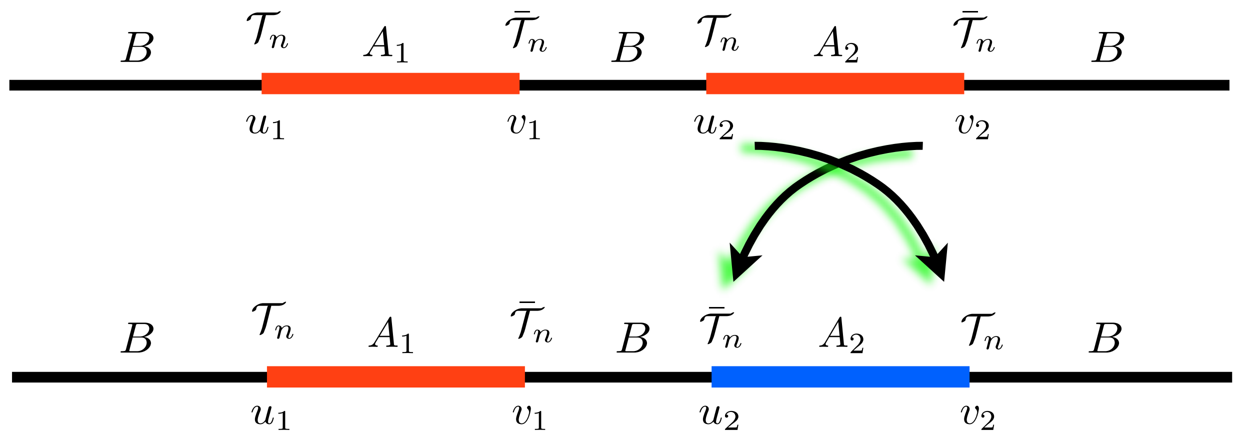
$\mathrm{Tr}\rho^n_{A_1 \cup A_2}$



Partial Transpose in 2D CFT: two disjoint intervals

$\mathrm{Tr}\rho_{A_1 \cup A_2}^n$

$\mathrm{Tr}\left(\rho_{A_1 \cup A_2}^{T_2}\right)^n$



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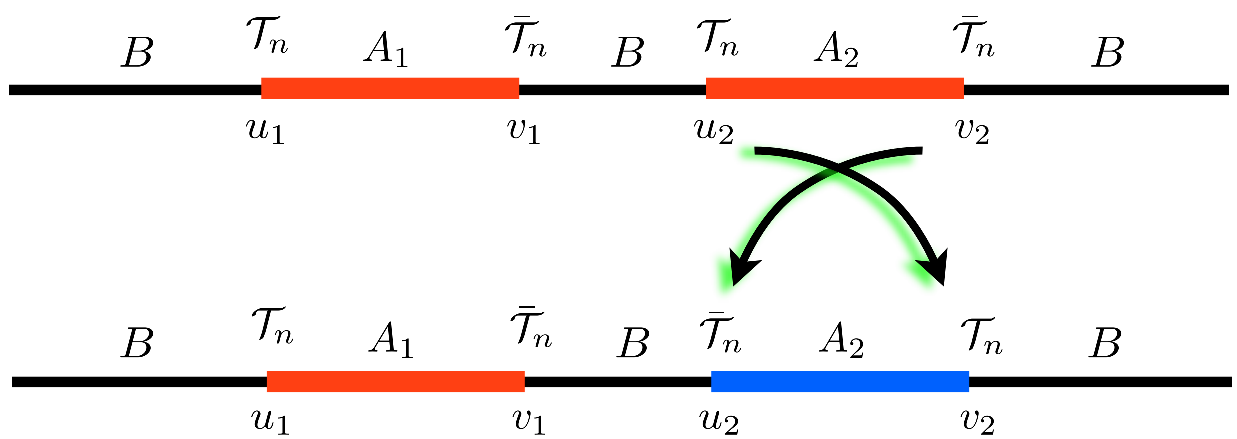

$$\mathrm{Tr}\left(\rho^{T_2}_{A_1\cup A_2}\right)^n = c_n^2\left[\ell_1\ell_2(1-y)\right]^{-\frac{c}{6}\left(n-\frac{1}{n}\right)}\mathcal{G}_n(y)$$

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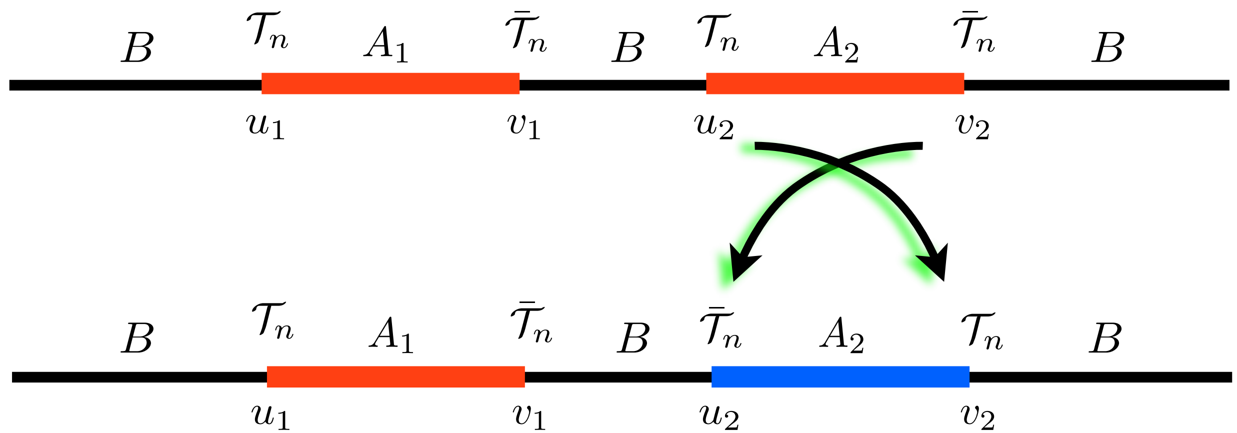

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$$\mathcal{G}_n(y) = (1-y)^{\frac{c}{3}(n-\frac{1}{n})}\,\mathcal{F}_n\left(\frac{y}{y-1}\right)$$

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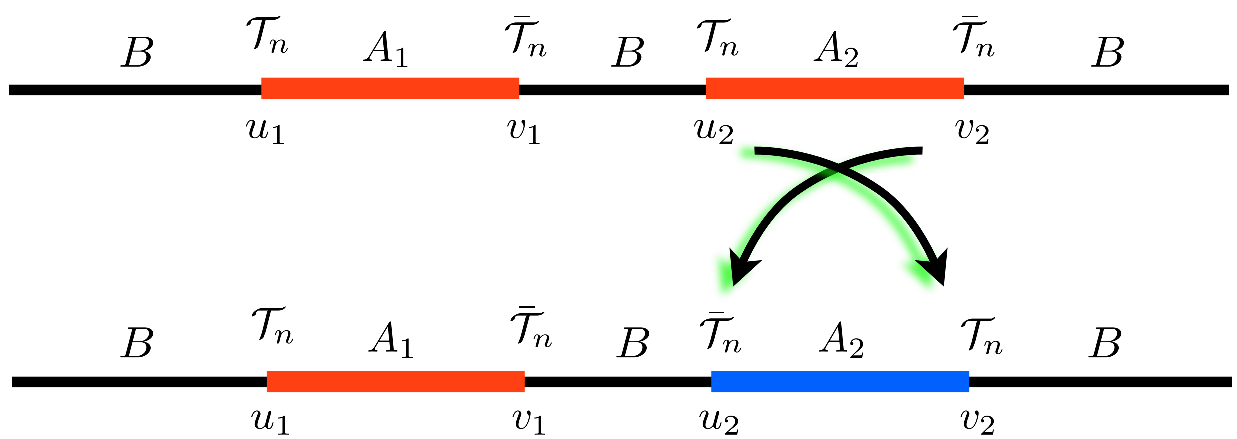

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- $$\mathcal{E}(y) = \lim_{n_e\to 1}\mathcal{G}_{n_e}(y) = \lim_{n_e\to 1}\left[\mathcal{F}_n\left(\frac{y}{y-1}\right)\right]$$

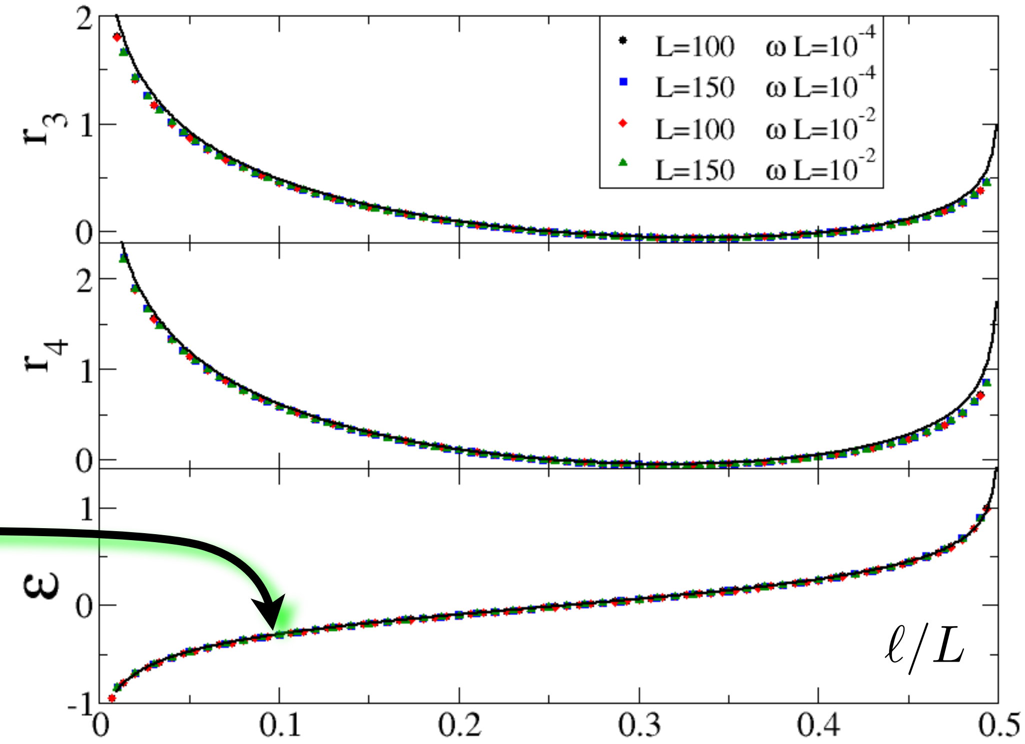
Two adjacent intervals: harmonic chain & Ising model

- Critical periodic harmonic chain

Finite system: $\ell \longrightarrow (L/\pi)\sin(\pi\ell/L)$

$$r_n = \ln \frac{\mathrm{Tr}(\rho_A^{T_{A_2=\ell}})^n}{\mathrm{Tr}(\rho_A^{T_{A_2=L/4}})^n}$$

$$\frac{1}{4}\log\frac{\sin(\pi\ell_1/L)\sin(\pi\ell_2/L)}{\sin(\pi[\ell_1+\ell_2]/L)} + \mathrm{cnst}$$



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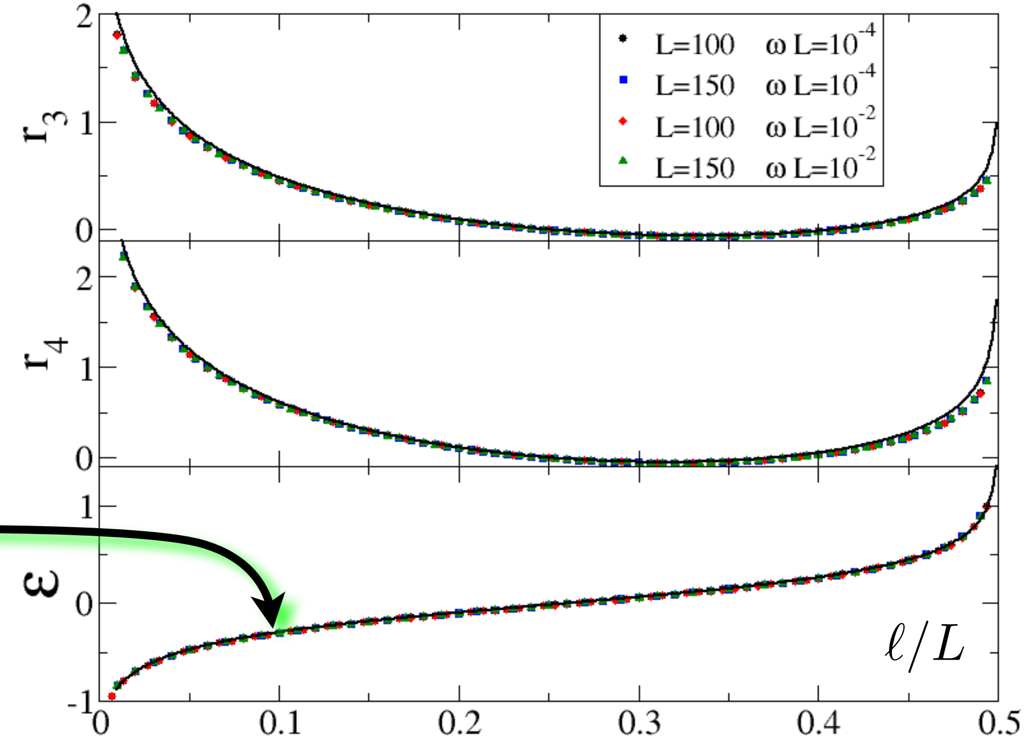
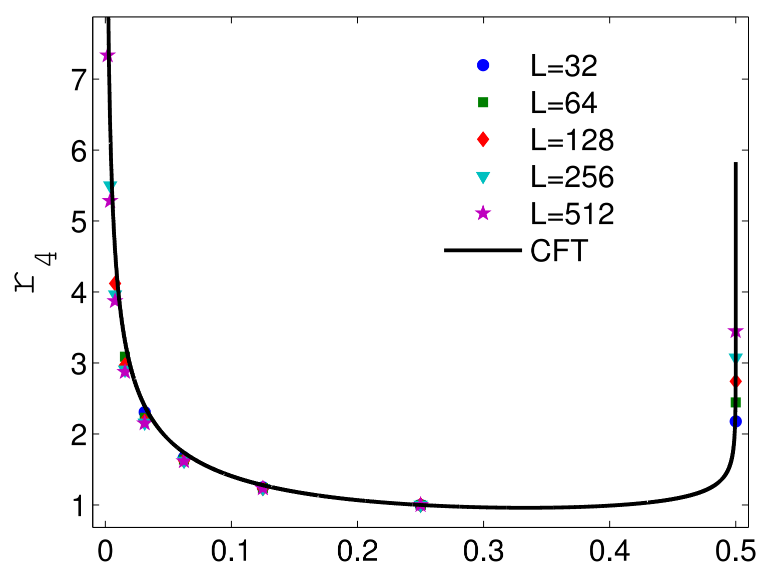
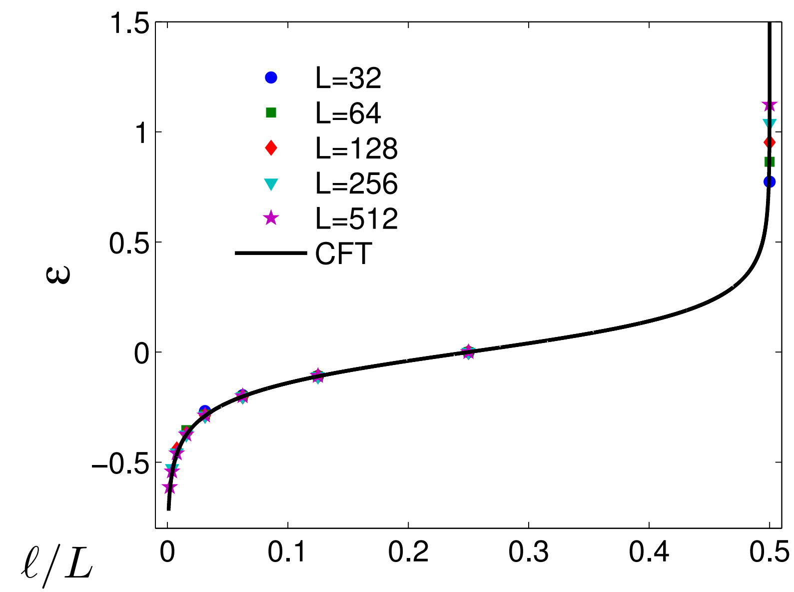


- Ising model:

Monte-Carlo analysis [Alba, (2013)]

Tree Tensor Network [Calabrese, Tagliacozzo, E.T., (2013)]

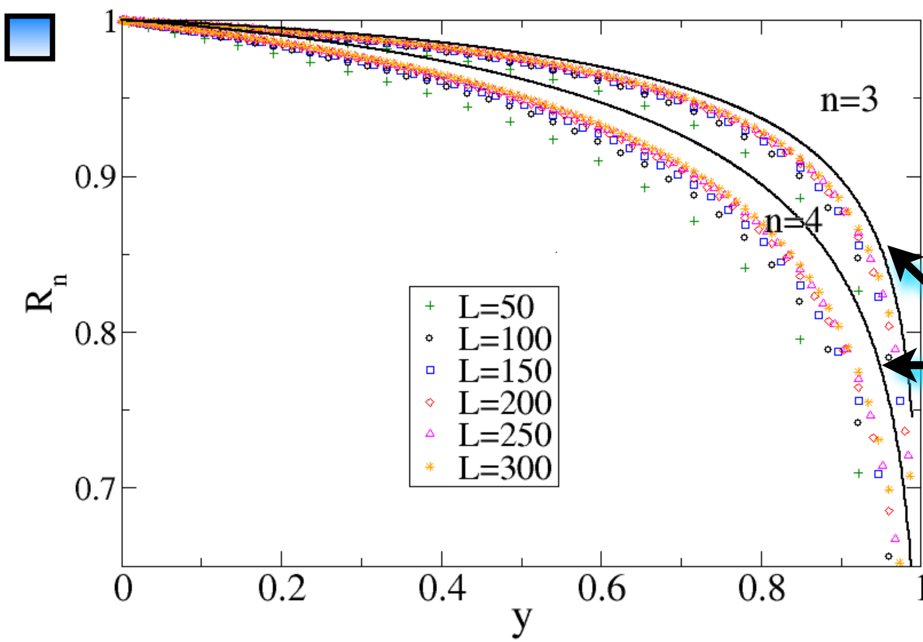




Two disjoint intervals: periodic harmonic chains

- Previous numerical results for $\mathcal{E}$: Ising (DMRG) and harmonic chains

[Wichterich, Molina-Vilaplana, Bose, (2009)]
[Marcovitch, Retzker, Plenio, Reznik, (2009)]

- 

CFT curves

- Two disjoint intervals

[Calabrese, Cardy, E.T., (2012)]

$$R_n = \frac{\mathrm{Tr}(\rho_A^{T_2})^n}{\mathrm{Tr}\,\rho_A^n}$$

$$R_n = \left[\frac{(1-y)^{\frac{2}{3}(n-\frac{1}{n})} \prod_{k=1}^{n-1} F_{\frac{k}{n}}(y)\, F_{\frac{k}{n}}(1-y)}{\prod_{k=1}^{n-1} \mathrm{Re}\big(F_{\frac{k}{n}}(\frac{y}{y-1})\, \bar{F}_{\frac{k}{n}}(\frac{1}{1-y})\big)} \right]^{\frac{1}{2}}$$

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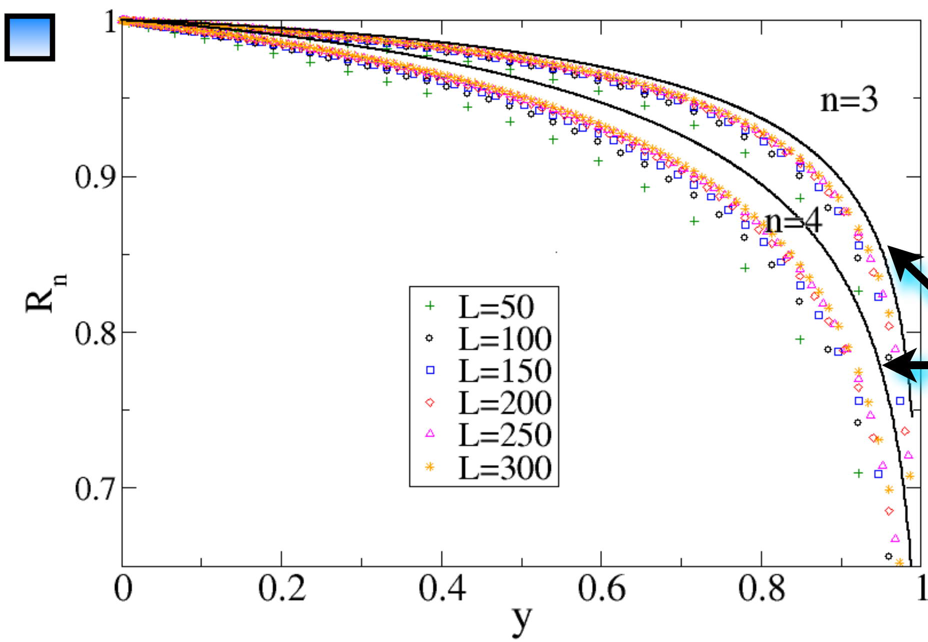

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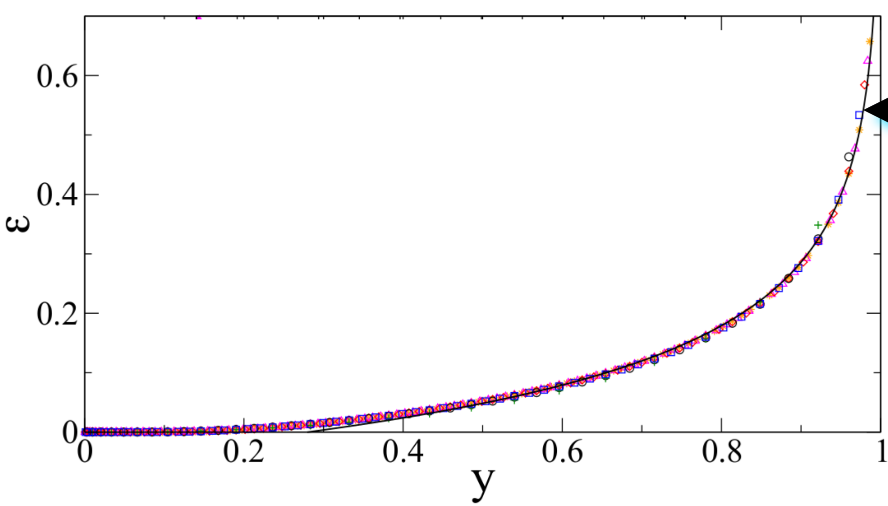

- Analytic continuation for $y \sim 1$

$$\mathcal{E} = -\frac{1}{4}\log(1-y) + \log K(y) + \mathrm{cnst}$$

- Analytic continuation $n_e \to 1$ for $0 < y < 1$ is missing
- It goes to zero faster than any power

Two disjoint intervals: Ising model

[Calabrese, Tagliacozzo, E.T., (2013)]

- CFT

$$\mathcal{G}_n(y) = (1-y)^{(n-1/n)/6} \frac{\sum_{\mathbf{e}} |\Theta[\mathbf{e}](\mathbf{0}|\tau(\frac{y}{y-1}))|}{2^{n-1}\prod_{k=1}^{n-1}|F_{k/n}(\frac{y}{y-1})|^{1/2}} \qquad 0<y<1$$

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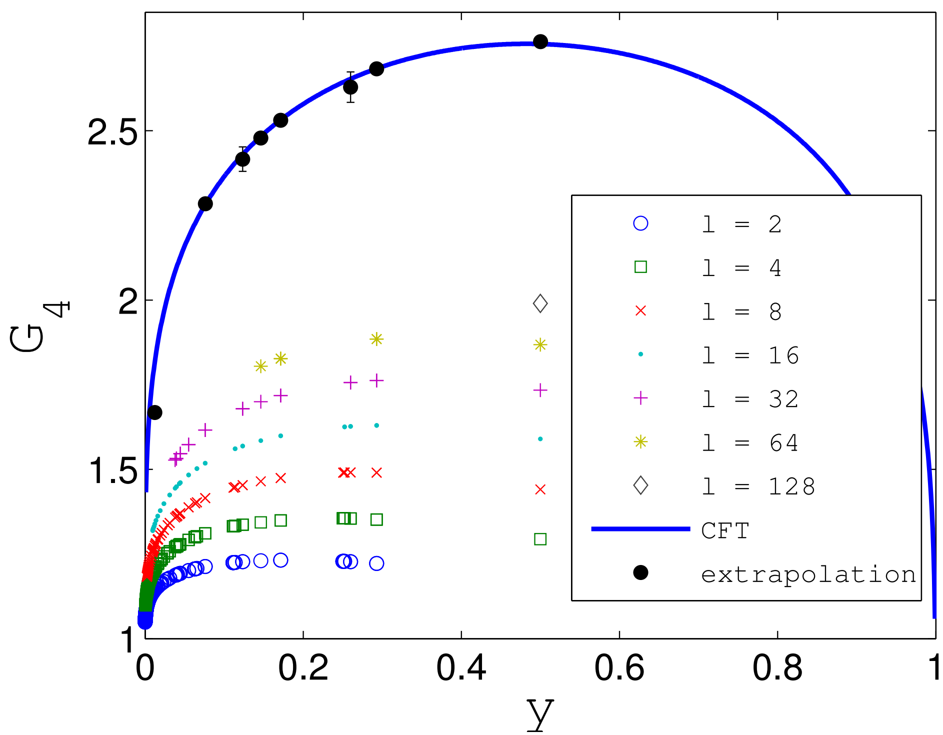

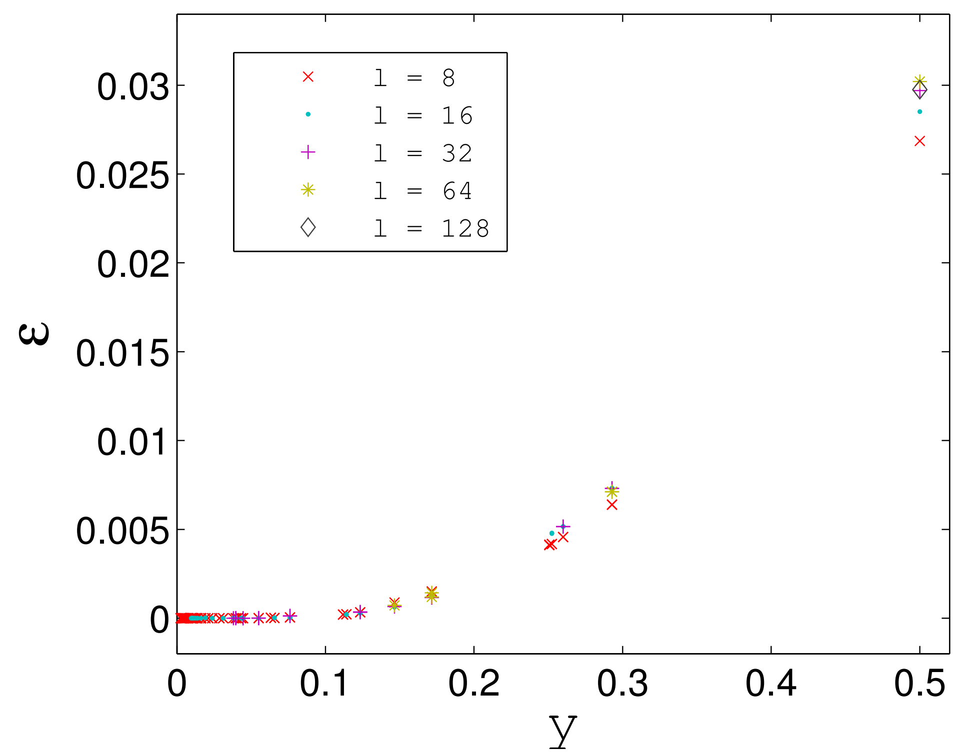

Global quantum quench: CFT evolution

- Global quench:
  - System prepared in the ground state $|\psi_0\rangle$ of H_0
  - At $t = 0$ sudden change of the Hamiltonian $H_0 \to H$

Unitary evolution:

$$|\psi(t)\rangle = e^{-\mathrm{i}Ht}|\psi_0\rangle \qquad \rho(t) = |\psi(t)\rangle\langle\psi(t)|$$

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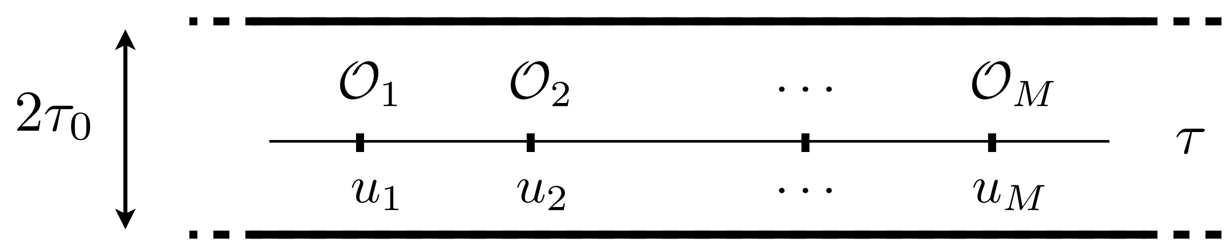
- Path integral formulation and critical H: correlation functions on the strip
  [Calabrese, Cardy, (2005), (2006), (2007)]



Analytic continuation $\tau = \tau_0 + \mathrm{i}t$, then $t \gg \tau_0$ and $|u_i - u_j| \gg \tau_0$

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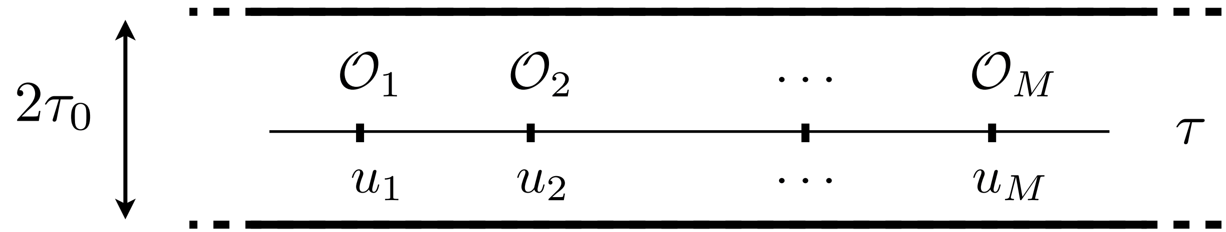

Analytic continuation $\tau = \tau_0 + it$, then $t \gg \tau_0$ and $|u_i - u_j| \gg \tau_0$

- Rényi entropies and traces of the partial transpose:
  - $\mathrm{Tr}\rho_A^n \longrightarrow \langle \prod_{i=1}^{N} \mathcal{T}_n(u_{2i-1})\bar{\mathcal{T}}_n(u_{2i}) \rangle_{\text{strip}}$ [Calabrese, Cardy, (2005)]
  - $\mathrm{Tr}(\rho_A^{T_0})^n \longrightarrow$ proper sequence of $\mathcal{T}_n$, $\bar{\mathcal{T}}_n$, $\mathcal{T}_n^2$ and $\bar{\mathcal{T}}_n^2$ within $\langle \dots \rangle_{\text{strip}}$

[Calabrese, Coser, E.T., 14xx.xxxx]

Negativity after a global quench: bipartition of the system

- Global quench of the mass in the periodic harmonic chain

$$H(\omega) = \frac{1}{2} \sum_{j=1}^{L} \left[p_j^2 + \omega^2 q_j^2 + (q_{j+1} - q_j)^2 \right]$$

$$\omega_0 = 100 \quad \longrightarrow \quad \omega = 10^{-5}$$

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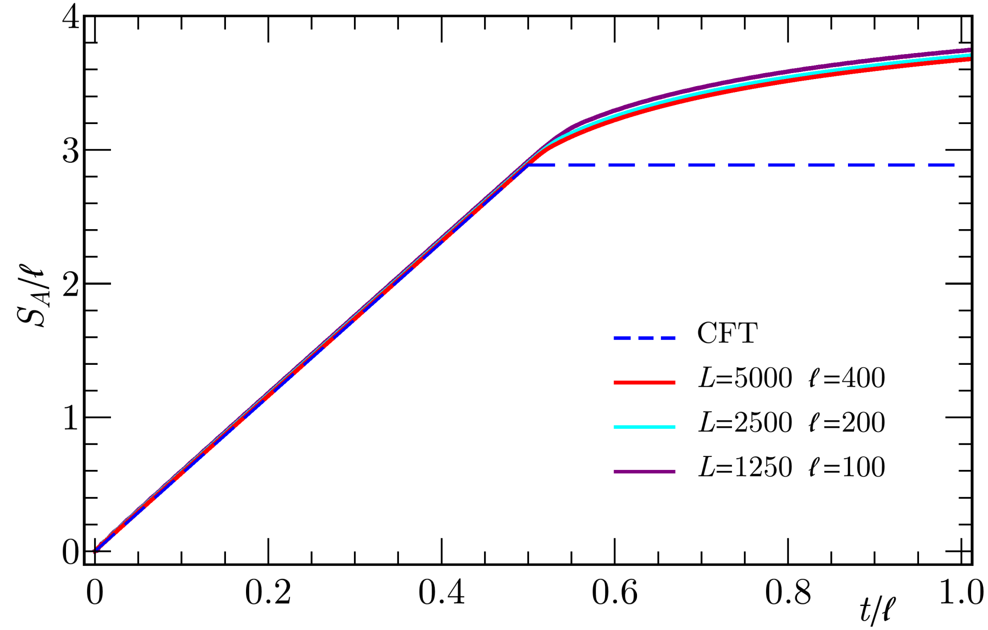

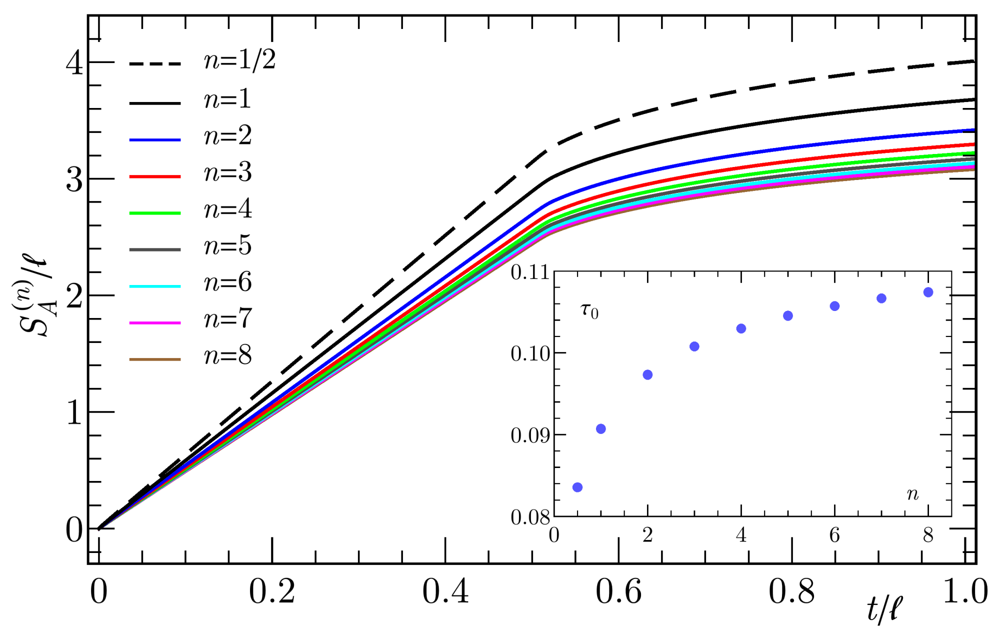

Negativity after a global quench: two adjacent intervals

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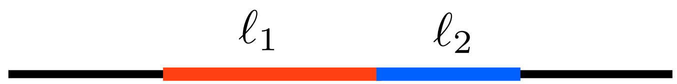

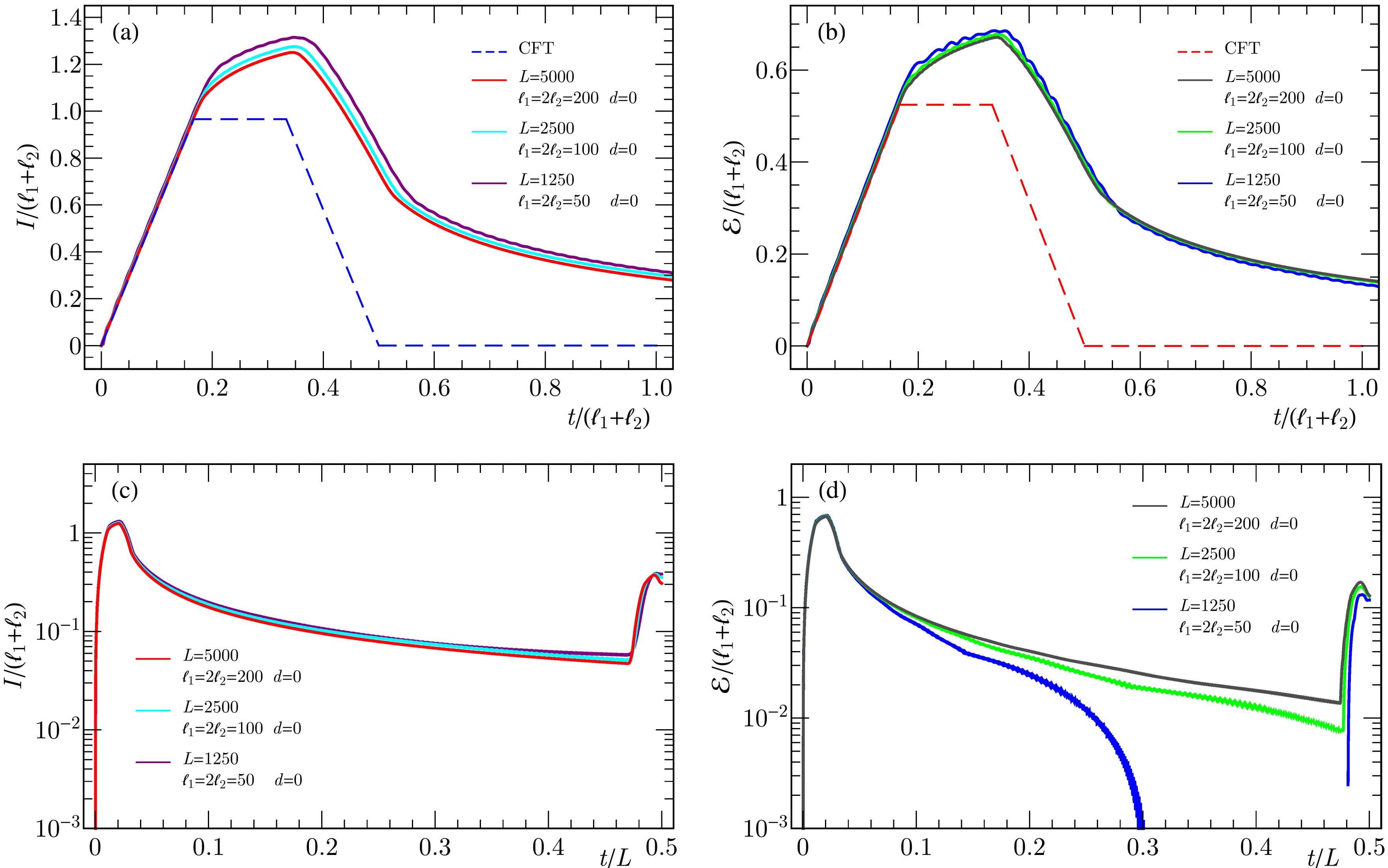

Sudden death of entanglement

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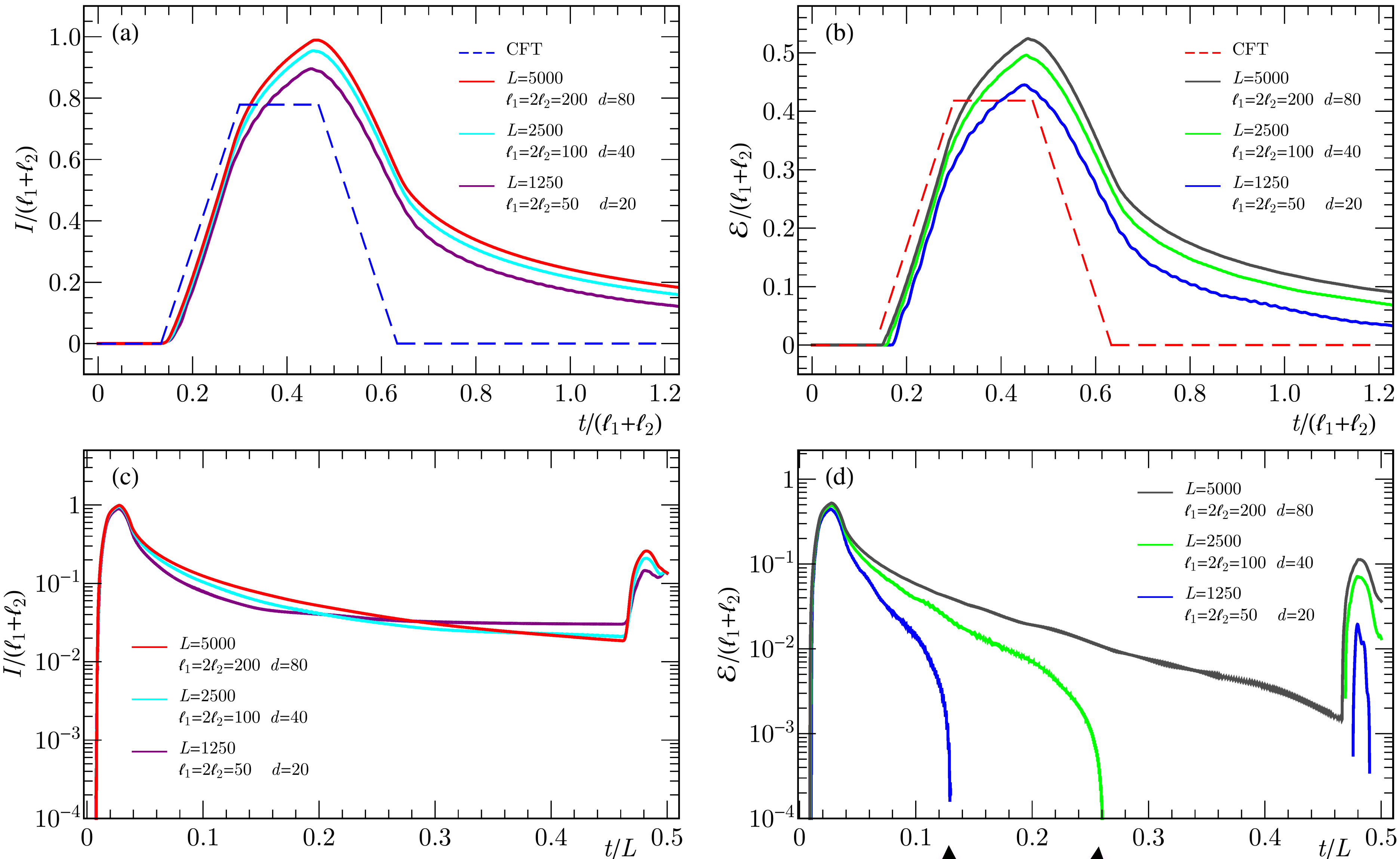

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Conclusions & open issues

- Entanglement for mixed states.

  Entanglement negativity in QFT (1+1 CFTs): $\mathrm{Tr}(\rho^{T_2})^n$ and $\mathcal{E}$

  → free boson on the line and Ising model

- Some generalizations:

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→ Analytic continuations

→ Finite temperature

→ Negativity for fermions

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Thank you!

Free compactified boson & Ising model

- $\mathcal{R}_{N,n}$ is $$y^n = \prod_{\gamma=1}^{N} (z - x_{2\gamma-2}) \Big[\prod_{\gamma=1}^{N-1} (z - x_{2\gamma-1}) \Big]^{n-1}$$ $g = (N-1)(n-1)$ [Enolski, Grava, (2003)]

- Partition function for a generic Riemann surface studied long ago in string theory [Zamolodchikov, (1987)] [Alvarez-Gaume, Moore, Vafa, (1986)] [Dijkgraaf, Verlinde, Verlinde, (1988)]

  Riemann theta function with characteristic $$\Theta[\boldsymbol{e}](\boldsymbol{0}|\Omega) = \sum_{\boldsymbol{m} \in \mathbb{Z}^p} \exp\left[\mathrm{i}\pi (\boldsymbol{m}+\boldsymbol{\varepsilon})^{\mathrm{t}} \cdot \Omega \cdot (\boldsymbol{m}+\boldsymbol{\varepsilon}) + 2\pi\mathrm{i} (\boldsymbol{m}+\boldsymbol{\varepsilon})^{\mathrm{t}} \cdot \boldsymbol{\delta} \right]$$

- Free compactified boson $(\eta \propto R^2)$ [Coser, Tagliacozzo, E.T., (2013)]

$$\mathcal{F}_{N,n}(\boldsymbol{x}) = \frac{\Theta(\boldsymbol{0}|T_\eta)}{|\Theta(\boldsymbol{0}|\tau)|^2} \qquad T_\eta = \begin{pmatrix} \mathrm{i}\,\eta\,\mathcal{I} & \mathcal{R} \\ \mathcal{R} & \mathrm{i}\,\mathcal{I}/\eta \end{pmatrix} \qquad \tau = \mathcal{R} + \mathrm{i}\,\mathcal{I}$$

period matrix

- Ising model $$\mathcal{F}_{N,n}^{\text{Ising}}(\boldsymbol{x}) = \frac{\sum_{\boldsymbol{e}} |\Theta[\boldsymbol{e}](\boldsymbol{0}|\tau)|}{2^g\, |\Theta(\boldsymbol{0}|\tau)|}$$

Nasty n dependence

- Two intervals case: [Caraglio, Gliozzi, (2008)] [Furukawa, Pasquier, Shiraishi, (2009)] [Calabrese, Cardy, E.T., (2009), (2011)] [Fagotti, Calabrese, (2010)] [Alba, Tagliacozzo, Calabrese, (2010), (2011)]