RR Charge and Gamma Class

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Recent Progress in 2D GLSM

@ String 2013: I talked about

Exact S^2 partition function of GLSM and its applications

Much more progresses after that : e.g.

- D_2 partition function [Hori,Romo][Honda,Okuda][Sugishita,Terashima]
- RP^2 partition function [Heeyeon Kim,**S.L.**,Piljin Yi]
- Elliptic genera [Benini,Eager,Hori,Tachikawa][Gukov,Gadde]

@ String 2014: I would like to discuss

Exact D_2 and RP^2 partition functions and their applications

Motivation

@ Strings 2013

The Γ-class

This gives some evidence in favor of a proposal by Iritani and Katzarkov–Kontsevich–Pantev to modify the usual identification

$$E \mapsto \mathrm{ch}(E)\sqrt{\mathrm{Td}_X}$$

of K-theory with cohomology, used in describing the integral structure in mirror symmetry (and in specifying D-brane charges), to

$$E \mapsto \mathrm{ch}(E)\hat{\Gamma}_c(X). \quad \textbf{???}$$

This proposal is very CONFUSING for many reasons !

Ramond-Ramond Charge

Does the conventional expression of the RR-charge need modification ?

Wess-Zumino Coupling: minimal coupling to RR gauge fields C

$$S_{\rm WZ} = \int_M C \wedge Q_{\rm RR}$$

- Long history to find the correct form of RR-charge

- Anomaly inflow mechanism provides conditions that Q_{RR} should satisfy

[Green,Harvey,Moore]
[Cheung,Yin]
[Dasgupta,Jatkar,Mukhi]
[Minasian,Moore] and many others

Ramond-Ramond Charge

Ramond-Ramond Charge ($2\pi\alpha' = 1$)

D-branes

$$Q_{\rm RR}^{D} = {\rm ch}(\mathcal{F})\underbrace{\sqrt{\frac{\hat{A}(R_T)}{\hat{A}(R_N)}}}$$

$$\mathcal{F} = F - B$$

$$\hat{A}(x) = \frac{x/2}{\sinh x/2} = 1 - \frac{1}{24}x^2 + \cdots$$

Geometric Witten Effect

O-planes

$$Q_{\rm RR}^{O_p} = \underbrace{\pm 2^{p-4}}\overbrace{\sqrt{\frac{\mathcal{L}(R_T/4)}{\mathcal{L}(R_N/4)}}}$$

Tadpole cancellation

$$\mathcal{L}(x) = \frac{x}{\tanh x} = 1 + \frac{1}{3}x^2 + \cdots$$

- Disk amplitudes with a RR vertex [Craps,Roose] [Morales,Scrucca,Serone]

- Dirac charge quantization condition

New RR-Charge Formula Needed ?

Central Charge (Tension) of D-brane & O-plane in Calabi-Yau space

In large volume limit of CY (semi-classical limit)

$$Z = \int_M e^{-iJ} \wedge Q_{\mathrm{RR}}$$

$$Q_{\mathrm{RR}}^{D} = \mathrm{ch}(\mathcal{F})\sqrt{\frac{\hat{A}(R_T)}{\hat{A}(R_N)}} \qquad \mathcal{F} = F - B$$

$$Q_{\mathrm{RR}}^{O_p} = \pm 2^{p-4}\sqrt{\frac{\mathcal{L}(R_T/4)}{\mathcal{L}(R_N/4)}}$$

Recently, mathematicians point out that this formula needs to be modified

$$Z^D = \int_M e^{-(B+iJ)}\mathrm{ch}(F)\frac{\hat{\Gamma}_c(R_T)}{\hat{\Gamma}_c(-R_N)}$$

[Libgober][Iritani]
[Katzarkov,Kontsevich,Pantev]…

New RR-Charge Formula Needed ?

Puzzle ?

$$Z^D = \int_M e^{-(B+iJ)} \mathrm{ch}(F) \frac{\hat{\Gamma}_c(R_T)}{\hat{\Gamma}_c(-R_N)} \qquad \hat{\Gamma}_c(x) = \Gamma\left(1 + \frac{x}{2\pi i}\right)$$

- Confirmed by recent exact D_2 partition function of GLSMs [Hori,Romo] [Honda,Okuda]

- **New RR-Charge formula ?**

$$Q_{\mathrm{RR}}^{\mathrm{new}} = \mathrm{ch}(\mathcal{F}) \frac{\hat{\Gamma}_c(R_T)}{\hat{\Gamma}_c(-R_N)}$$

[1] Anomaly inflow: OK

[2] Dirac charge quantization: OK

[3] However **S_{WZ}** contains **imaginary terms**, if this is case ???

SUSY Theories on S^2

[Benini,Cremonesi]
[Doroud,Gomis,Le Floch,**S.L.**]

See also Gomis' talk.

GLSM on S^2

Gauged Linear Sigma Model

- 2d N=(2,2) gauge theory with vector + chiral multiplets

- Focus on GLSMs that flow in the IR to NLSM on Calabi-Yau spaces
  - Complexified FI parameters = Kahler moduli of CY
  - Complex parameters in superpotential = Complex structure moduli of CY

GLSM on S^2

- SUSY on S^2: SU(2|1) parametrized by Killing spinors (l: radius of S^2)

$$\nabla_i \epsilon = +\frac{1}{2l}\gamma_i \gamma^3 \epsilon \qquad \nabla_i \bar{\epsilon} = -\frac{1}{2l}\gamma_i \gamma^3 \bar{\epsilon}$$

- SUSY Lagrangian on S^2: add suitable corrections suppressed by 1/l

Exact S^2 Partition Function

Sphere Partition Function: use the localization technique

$$Z_{S^2} = \frac{1}{|W|} \sum_B \int_{\mathfrak{t}} d^r\sigma \; e^{-4\pi i \xi \sigma + i\theta B} \times Z_{\text{1-loop}}(\sigma)$$

$$Z_{\text{v.m.}} = \prod_{\alpha \in \Delta^+} \left[\left(\frac{\alpha \cdot B}{2} \right)^2 + (\alpha \cdot l\sigma)^2 \right]$$

$$Z_{\text{c.m.}} = \prod_{\rho \in \mathbf{R}} \frac{\Gamma\left(\frac{q}{2} - il(\rho \cdot \sigma + m) - \frac{\rho \cdot B}{2}\right)}{\Gamma\left(1 - \frac{q}{2} + il(\rho \cdot \sigma + m) - \frac{\rho \cdot B}{2}\right)}$$

$\boldsymbol{\xi}$: FI parameter

$\boldsymbol{\theta}$: theta angle

W : Weyl group

r : rank of G

B : Flux on S^2

q : U(1) R charge

NB: no dependence on gauge coupling constant and superpotential parameters

What Does It Compute ?

Exact Zamolodchikov's Metric

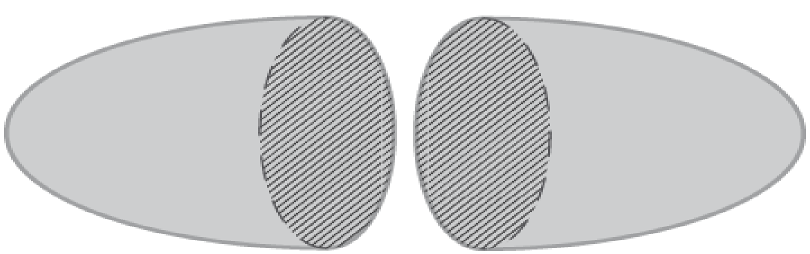
[Jockers,Kumar,Lapan,Morrison,Romo]
[Gomis,**S.L.**][Gerchkovitz,Gomis,Komargodski]

$$Z_{S^2}(\tau,\bar{\tau}) = e^{-K(\tau,\bar{\tau})} \qquad \tau = \frac{\theta}{2\pi} + i\xi$$

- Exact (in α') Kahler potential on the Kahler moduli space of CY

- No dependence on the SUSY squashing parameter implies

Z_{S^2} =



$= {}_{\mathrm{R}}\langle\bar{0}|0\rangle_{\mathrm{R}} = e^{-K(\vec{\tau},\vec{\bar{\tau}})}$

[Cecotti,Vafa]

SUSY Theories on D_2

[Hori,Romo]
[Honda,Okuda]
[Sugishita,Terashima]

Boundary Data

SUSY Theories on D_2 = SUSY Theories on S^2 + Boundary Data

[1] Boundary Condition: Neumann (N) or Dirichlet (D)

[2] Boundary Interaction: to preserve 2 supercharges

- Chan-Paton Vector Space (*V*):
- Tachyon Profile ***Q***

$$\mathcal{Q}^2 = \mathcal{W}\ \mathbf{1}_{\mathcal{V}}$$

Matrix Factorization:
not unique solution

- Boundary Interaction (Tachyon Condensation)

$$\left\{\mathcal{Q}, \mathcal{Q}^\dagger\right\} = V_{bd}(\phi)\ \mathbf{1}_{\mathcal{V}}$$

D-Branes

Tangential: Neumann

$$D_\theta \phi^T(\theta = \pi/2, \varphi) = 0$$

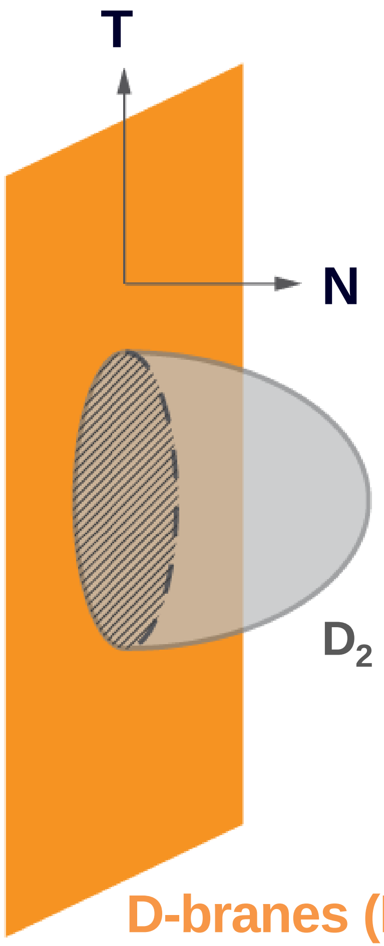
Normal: two equivalent descriptions

[1] Dirichlet

$$\phi^N(\theta = \pi/2, \varphi) = 0$$

[2] Neumann + Tachyon Condensation

$$V_{bd}(\phi^N = 0) = 0$$



D-branes (B-type)

Exact D_2 Partition Function

Hemi-Sphere Partition Function $\left(\tau = \frac{\theta}{2\pi} + i\xi\right)$ [Hori,Romo][Honda,Okuda][Sugishita,Terashima]

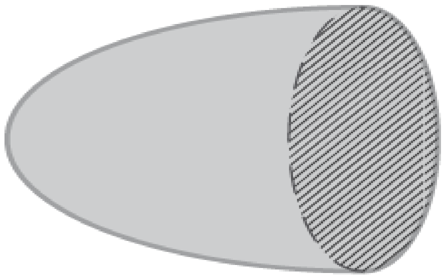
$$Z_{D_2}(\tau, \mathfrak{B}) = \frac{1}{|W|} \int d^r\sigma \; e^{-2\pi\tau \mathrm{tr}\sigma} \mathrm{Tr}_{\mathcal{V}}\left[e^{2\pi\rho_*(\sigma) + i\pi r_*} \right] Z_{1\text{-loop}}(\sigma)$$

$$Z_{1\text{-loop}}(\sigma) = \underbrace{\prod_{\alpha \in \Delta^+} \alpha\cdot\sigma \sinh \alpha\cdot\sigma}_{\text{Vector multiplets}} \times \underbrace{\prod_a \prod_{w_a \in \mathbb{R}_a} \Gamma\left[\frac{q_a}{2} - i w_a \cdot \sigma\right]}_{\text{Chiral matter multiplet}}$$

Vector multiplets

Chiral matter multiplet

What does it compute ?



$= {}_R\langle \bar{0} | \mathfrak{B} \rangle_R =$ central charge of D-branes

[Ooguri,Oz,Yin]

A Simple Example

CY_{N-2} Hypersurface in CP^{N-1}

2d N=(2,2) SUSY gauge theory with G=U(1) gauge group, coupled to

N chiral multiplets X_a (a=1,2,..,N) of electric charge **+1**

a chiral multiplets P of electric charge **- N**

with superpotential **$W = P \cdot G_N(X)$**

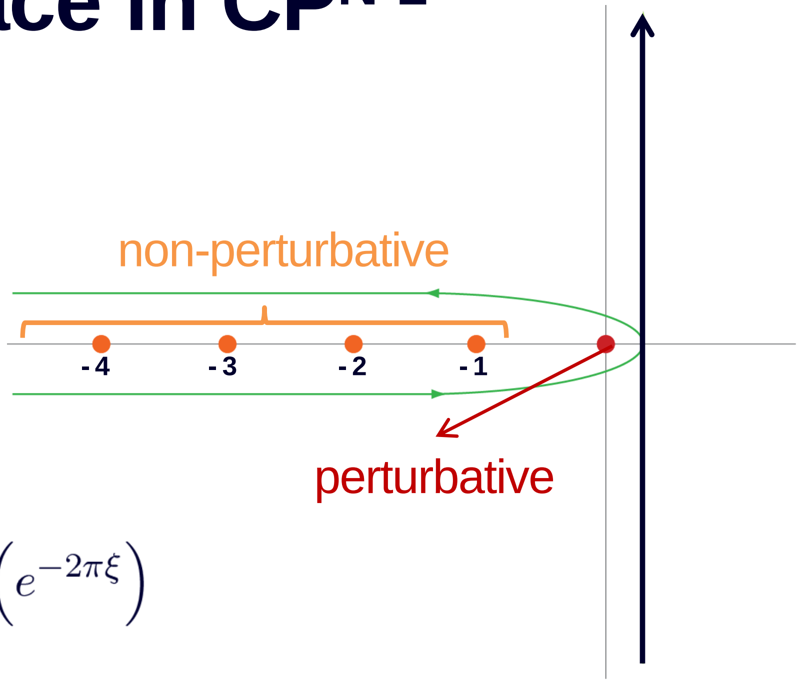
$G_N(x)$: homogeneous polynomial of degree N

e.g. N=5: **Quintic Threefold**

CY_{N-2} Hypersurface in CP^{N-1}

D-brane Wrapped on Entire $X=CY_{N-2}$

$$\mathcal{Z}_D = \int_{0^+-i\infty}^{0^++i\infty} \frac{d\epsilon}{2\pi i}\, e^{2\pi\xi\epsilon - i\theta\epsilon} \cdot \frac{\Gamma^N(\epsilon)}{\Gamma(N\epsilon)}$$



$\xi > 0$

$$= \oint \frac{d\epsilon}{2\pi i}\, e^{+2\pi\xi\epsilon - i\theta\epsilon} \cdot \frac{N}{\epsilon^{N-1}} \cdot \frac{\Gamma(1+\epsilon)^N}{\Gamma(1+N\epsilon)} + \mathcal{O}\left(e^{-2\pi\xi}\right)$$

$\xi \to \infty$

$$\simeq \int_X e^{-B-iJ} \wedge \hat{\Gamma}_c(R_X)$$

$$\int_X H^{N-2} = N$$

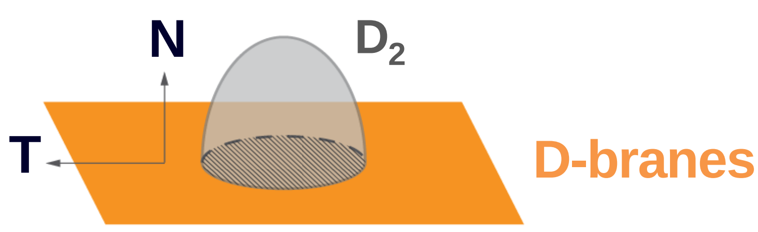
$$J = \xi H$$

$$2\pi B = \theta H$$

H : Toric divisor of CP^{N-1}

Lower-Dimensional D-brane in CY_{N-2}

$$\mathcal{Z}_D \simeq \int_M e^{-B-iJ} \wedge \frac{\hat{\Gamma}_c(R_T)}{\hat{\Gamma}_c(-R_N)}$$



SUSY Theories on RP^2

[Heeyeon Kim,**S.L.**,Piljin Yi]

Parity Projection

SUSY Theories on RP^2 = SUSY Theories on S^2 + Parity Projection

[1] Projection

$$\phi_T(\pi-\theta,\pi+\varphi)=+\phi_T(\theta,\varphi)$$

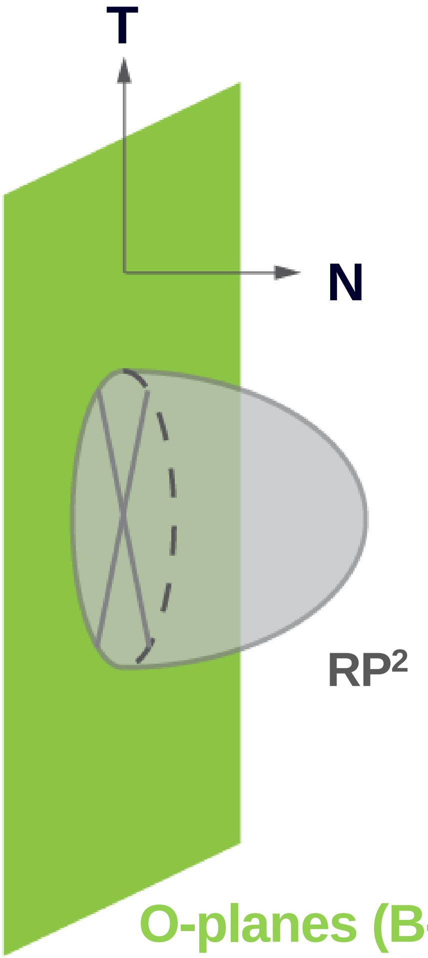
$$\phi_N(\pi-\theta,\pi+\varphi)=-\phi_N(\theta,\varphi)$$

[2] 2 Supercharge: O-plane on holomorphic cycles

[3] Theta angle: $\theta = 0 \text{ or } \pi$

otherwise, the topological term breaks the parity

two values distinguish **O$^+$** and **O$^-$** planes



O-planes (B-type)

Exact RP² Partition Function

RP² Partition Function [H.Kim,**S.L**,P.Yi]

$\theta = 0 \text{ or } \pi$

$$Z_{\mathbf{RP}^2}(\xi) = \frac{1}{|W|}\int d^r\sigma\, e^{-2\pi i \xi \mathrm{tr}\sigma}\left[Z^{\text{even}}_{\text{1-loop}} \pm Z^{\text{odd}}_{\text{1-loop}}\right]$$

$$Z^{\text{even}}_{\text{1-loop}} = \underbrace{\prod_{\alpha\in\Delta^+} \alpha\cdot\sigma \tan\left[\frac{\pi}{2}\alpha\cdot\sigma\right]}_{\text{vector multiplets}} \cdot \underbrace{\prod_{w\in\mathbf{R}} \Gamma\left[\frac{q}{2} - iw\cdot\sigma\right]\cos\left[\frac{\pi}{2}\left(\frac{q}{2} - iw\cdot\sigma\right)\right]}_{\text{chiral matter multiplet}}$$

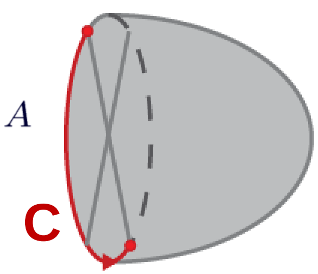
vector multiplets

chiral matter multiplet

$$Z^{\text{odd}}_{\text{1-loop}} = \overbrace{\prod_{\alpha\in\Delta^+} \alpha\cdot\sigma \tan\left[\frac{\pi}{2}\alpha\cdot\sigma - \frac{\alpha\cdot a}{2}\right]} \cdot \overbrace{\prod_{w\in\mathbf{R}} \Gamma\left[\frac{q}{2} - iw\cdot\sigma\right]\cos\left[\frac{\pi}{2}\left(\frac{q}{2} - iw\cdot\sigma\right) - \frac{\omega\cdot a}{2}\right]}$$

NB: discrete holonomy group on **RP²**

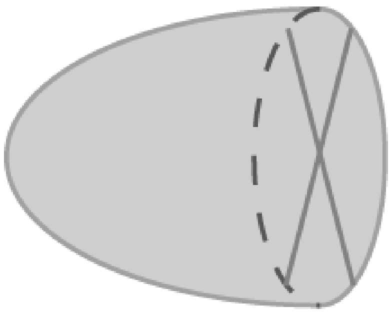
$$e^{ia} = e^{i\oint_C A}$$



RP²

Exact RP^2 Partition Function

What does RP^2 partition function compute ?



$= \; {}_R\langle \bar{0} | \mathfrak{C} \rangle_R \; =$ central charge of O-planes

[Ooguri,Oz,Yin]

[NB] EXACT in the corrections α' including world-sheet instanton effects

CY_{N-2} Hypersurface in CP^{N-1}

O-plane Wrapped on $X=CY_{N-2}$

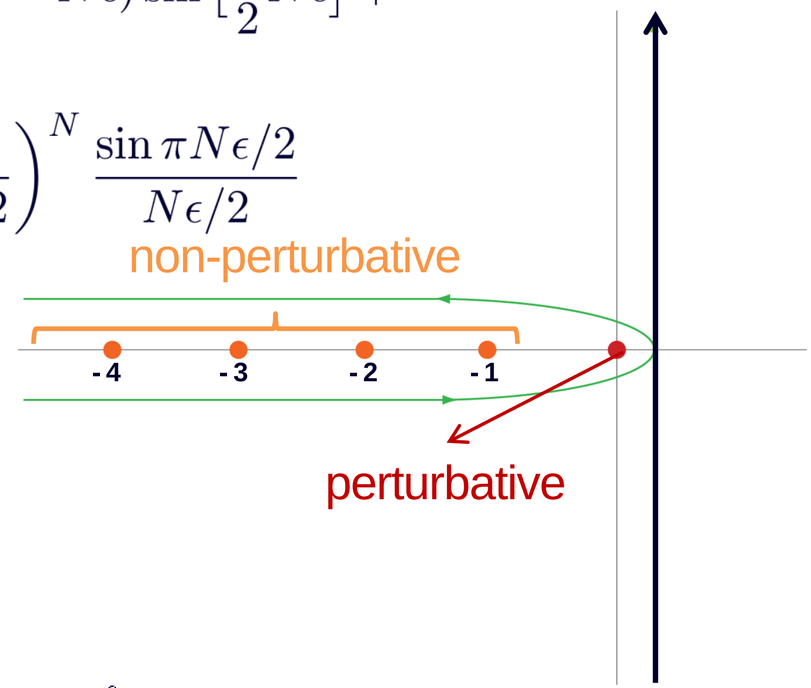
$$\mathcal{Z}_O \simeq \int_{0^+-i\infty}^{0^++i\infty} \frac{d\epsilon}{2\pi i}\, e^{2\pi\xi\epsilon}\cdot\Gamma(\epsilon)^N \cos^N\left[\frac{\pi}{2}\epsilon\right]\cdot\Gamma(1-N\epsilon)\sin\left[\frac{\pi}{2}N\epsilon\right]+\cdots$$

$\xi > 0$

$$\sim \oint \frac{d\epsilon}{2\pi i}\, e^{2\pi\xi\epsilon}\frac{N}{e^{N-1}}\cdot\frac{\Gamma(1+N\epsilon)}{\Gamma(1-\epsilon)^N}\cdot\left(\frac{\epsilon/2}{\sin\pi\epsilon/2}\right)^N\frac{\sin\pi N\epsilon/2}{N\epsilon/2}$$

$\xi \to \infty$

$$= \int_X e^{-iJ}\wedge\frac{\hat{A}(R_X/2)}{\hat{\Gamma}_c(-R_X)}$$



Lower-Dimensional O-plane in CY_{N-2}

$$\mathcal{Z}_{O_p} = \pm 2^{p-4}\int_M e^{-iJ}\wedge\frac{\hat{A}(R_T/2)}{\hat{A}(R_N/2)}\frac{\hat{\Gamma}_c(R_N)}{\hat{\Gamma}_c(-R_T)}$$

$$\int_X H^{N-2} = N \qquad J = \xi H$$

H : Toric divisor of CP^{N-1}

What Does the Gamma Class Correct ?

Gamma Class

D-branes: In the large volume limit,

$$\mathcal{Z}_D = \int_M e^{-B-iJ} \wedge \mathrm{ch}[F] \wedge \frac{\hat{\Gamma}_c(R_T)}{\hat{\Gamma}_c(-R_N)}$$

$$= \int_M e^{-B-iJ} \wedge \mathrm{ch}[F] \wedge \sqrt{\frac{\hat{A}(R_T)}{\hat{A}(R_N)}} \cdot \sqrt{\frac{\hat{\Gamma}_c(R_X)}{\hat{\Gamma}_c(-R_X)}}$$

$$\hat{\Gamma}_c(x)\hat{\Gamma}_c(-x) = \hat{A}(x)$$

(X = Calabi-Yau)

THE SAME FACTOR !

O-planes

$$\mathcal{Z}_{O_p} = \pm 2^{p-4} \int_M e^{-iJ} \wedge \frac{\hat{A}(R_T/2)}{\hat{A}(R_N/2)} \frac{\hat{\Gamma}_c(R_N)}{\hat{\Gamma}_c(-R_T)}$$

$$= \pm 2^{p-4} \int_M e^{-iJ} \wedge \sqrt{\frac{\mathcal{L}(R_T/4)}{\mathcal{L}(R_N/4)}} \cdot \sqrt{\frac{\hat{\Gamma}_c(R_X)}{\hat{\Gamma}_c(-R_X)}}$$

$$\sqrt{\hat{A}(x) \cdot \mathcal{L}(x/4)} = \hat{A}(x/2)$$

Gamma Class

What are these corrections ? (X = Calabi-Yau)

$$\sqrt{\frac{\hat{\Gamma}_c(R_X)}{\hat{\Gamma}_c(-R_X)}} = \mathrm{Exp}\left[\frac{i\gamma}{2\pi}\mathrm{ch}_1(R_X) + i\sum_{k\geq 1}(-1)^k \frac{(2k)!}{(2\pi)^{2k+1}}\zeta(2k+1)\mathrm{ch}_{2k+1}(R_X)\right]$$

[1] Purely imaginary terms, starting from 6-form $\mathbf{ch_3(R_X)}$ terms in CY

[2] Depend on the entire target space **X**, BUT do not care of the sub-manifolds

M that D-branes or O-planes wrap on

[3] Should be identified as the α'**- correction** to the volume of X, not RR-charge

$$e^{-iJ} \;\longrightarrow\; e^{-iJ + i\sum_{k\geq 1}(-1)^k \frac{2k!}{(2\pi)^{2k+1}}\zeta(2k+1)\mathrm{ch}_{2k+1}(R_X)}$$

Gamma Class

Yet another support : in the large-volume limit,

[Halverson,Jocker,Lapan,Morrison]

$$Z_{S^2} = e^{-K(\tau,\bar{\tau})} \simeq \int_X e^{-2(B+iJ)} \cdot \frac{\hat{\Gamma}_c(R_X)}{\hat{\Gamma}_c(-R_X)} + \cdots$$

world-sheet instanton corrections

$$= -\frac{i}{3!} C_{ijk}(\tau - \bar{\tau})^i (\tau - \bar{\tau})^j (\tau - \bar{\tau})^k + \frac{\zeta(3)}{4\pi^3}\chi(X) + \cdots$$

CY_3

[1] Classical volume of CY_3

[2] Four-loop correction in NLSM on CY_3 [Grisaru,van de Ven,Zanon]

[3] Prediction on the perturbative α'**- correction** to the volume of any CY_N

Summary

Exact D_2,RP^2 partition function of GLSM

α'- exact central charge of D/O wrapped on holomorphic (B) cycles

Gamma class & Quantum volume

New and direct method of computing stringy corrections

A factor associated with **Spin^c** D-brane world-volume
[Minasian,Moore][Freed,Witten]

Consistent to the Hori-Vafa mirror symmetry

Characteristic Class

In terms of skew-eigenvalue 2-forms y_i of $\frac{R}{2\pi}$,

$$\hat{A}(R) = \prod_i \frac{y_i/2}{\sinh(y_i/2)}$$

$$\mathcal{L}(R) = \prod_i \frac{y_i}{\tanh y_i}$$

$$\hat{\Gamma}_c(R) = \prod_i \Gamma\left(1 + \frac{y_i}{2\pi i}\right)$$

$$\mathrm{ch}_{s+}(R) = \prod_i e^{y_i/2}$$

Precise Form of Exact S^2 Partition Function

S^2 Partition Function [Doroud,Gomis,Le Floch,**S.L**][Benini,Cremonesi]

$$Z_{S^2}^{\rm GLSM} = \frac{(l\Lambda)^{c/3}}{|W|} \sum_B \int_{\mathfrak{t}} d^r\sigma \;\; e^{-4\pi i \xi_{\rm ren}\sigma + i\theta B} \times Z_{\text{1-loop}}^{\rm reg}(\sigma)$$

$$Z_{\rm v.m.} = \prod_{\alpha\in\Delta^+} \left[\left(\frac{\alpha\cdot B}{2}\right)^2 + (\alpha\cdot l\sigma)^2 \right]$$

$$Z_{\rm c.m.} = \prod_{\rho\in\mathbf{R}} \frac{\Gamma\left(\frac{q}{2} - il(\rho\cdot\sigma + m) - \frac{\rho\cdot B}{2}\right)}{\Gamma\left(1 - \frac{q}{2} + il(\rho\cdot\sigma + m) - \frac{\rho\cdot B}{2}\right)}$$

$\boldsymbol{\xi}$: FI parameter

$\boldsymbol{\theta}$: theta angle

W : Weyl group

r : rank of G

B : Flux on S^2

q : U(1) R charge

- Central charge (scale anomaly)

$$\frac{c}{3} = \underbrace{\sum_i \dim[\mathbf{R}_i](1-q_i)}_{\text{chiral multiplets}} - \underbrace{\dim[G]}_{\text{vector multiplet}} = \mathrm{Tr}_f[R]$$

[Silverstein,Witten]
[Hori,Tong]