Subtle Points About Saddle Points

in the S-Matrix Theory

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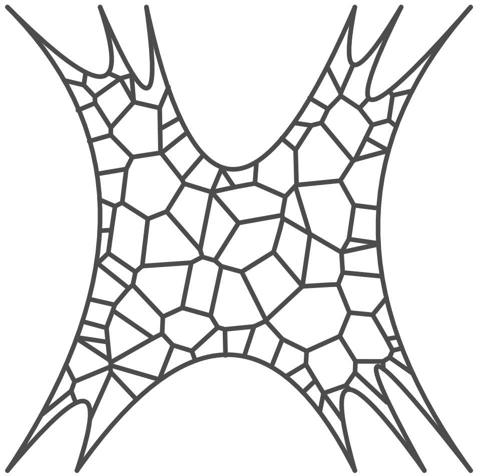
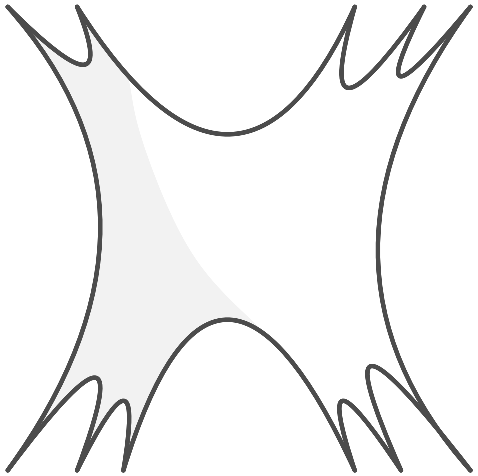
We’ll look at aspects of saddle points in Lorentzian scattering processes without counterparts in other signatures

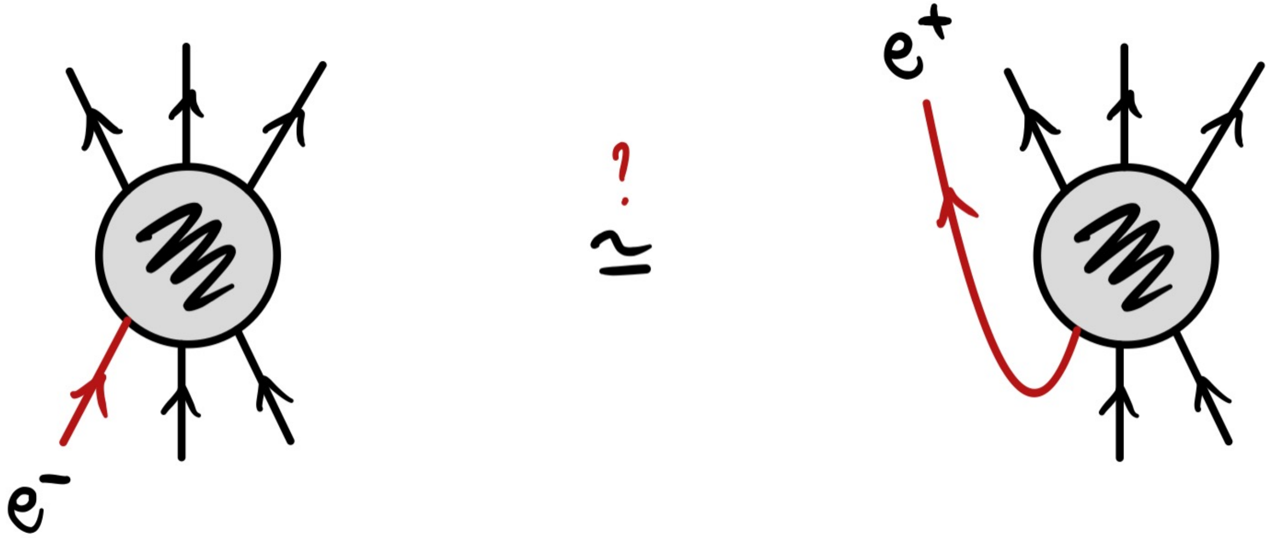


I) worldlines



II) worldsheets

One of the oldest open problems in QFT: *crossing symmetry*



$p_i^2 = M_i^2$

(on-shell)

$\sum_i p_i^\mu = 0$

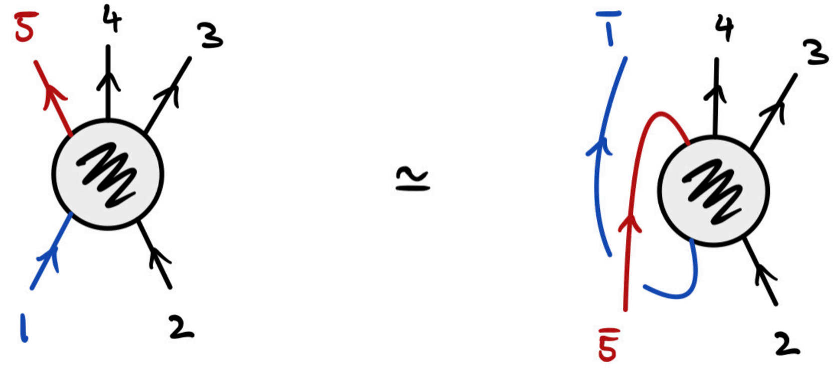
(mom. cons.)

Are the two scattering processes related by analytic continuation?

[Stückelberg, Gell-Mann, Goldberger, Thirring, …]

Partial progress in axiomatic QFT with a mass gap:

[Bros, Epstein, Glaser ‘65]



[Bros ‘86]

Some extensions to string field theory [De Lacroix, Erbin, Sen ‘18]

Technical challenges:

- Massless particles

- Different number of in/out states
(e.g., exchanging a particle for an antiparticle)

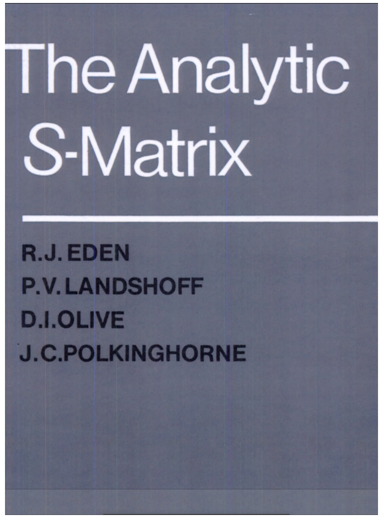
- More than five external states
(where constraints from the space-time dimension first start to matter)

Conceptual challenge:

- Little to no physics understanding

(previous proofs hinge on the use of analytic completion theorems)

Historically abandoned; replaced with a much stronger *mathematical* assumption



crossing symmetry

$\subset$ maximal analyticity

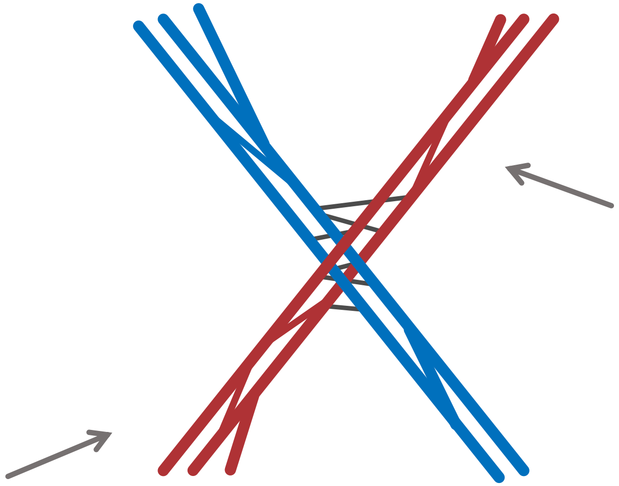
contrast with physical assumptions:
causality, locality, unitarity

The S-matrix theory should *not* be based on assumptions whose physical content we don't understand!

What’s the new strategy?

Revisit the question of crossing symmetry in perturbation theory

Crossing symmetry $\Leftarrow$ absence of a certain class of singularities



[Witten]

Simplification coming with perturbation theory:

singularities $\Leftrightarrow$ wordline saddle points

algebraic problem

Contribution from a single worldline Feynman diagram:

Schwinger parameters $\alpha_{e=1,2,\ldots,\mathrm{E}}$

worldline action

$$\int \frac{\mathrm{d}^{\mathrm{E}}\alpha}{\mathcal{U}^{\mathrm{D}/2}}\, \mathrm{N}\, e^{i\mathcal{V}}$$

determinant of the Laplacian

polynomial numerator (from the OPE's)

The worldline action takes the form

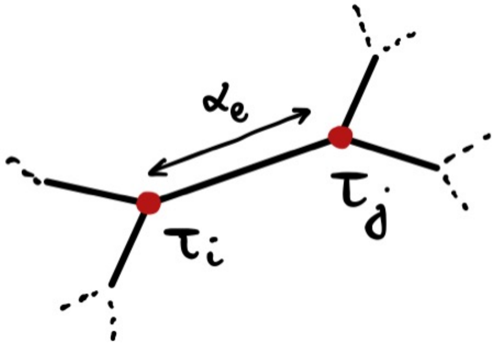
$$\mathcal{V} = -\sum_{i<j} p_i \cdot p_j \, \mathcal{G}_{ij} - \sum_e m_e^2 \, \alpha_e$$

Green's function　　internal masses

Before applying saddle-point analysis (really, stratified Morse theory) we need to make the Green's functions *holomorphic*

$$\mathcal{G}_{ij} = |\tau_i - \tau_j| + \ldots$$

$$\rightsquigarrow \alpha_e + \ldots$$



Saddle point equations:

$$\frac{\partial \mathcal{V}}{\partial \alpha_e} = 0, \qquad e = 1, 2, \ldots, \mathrm{E}$$

(in this talk we ignore boundary saddles)

Wordline action is special because of homogeneity under dilations:

$$\mathcal{V}(\lambda \alpha_e) = \lambda \mathcal{V}(\alpha_e)$$

Three important consequences:

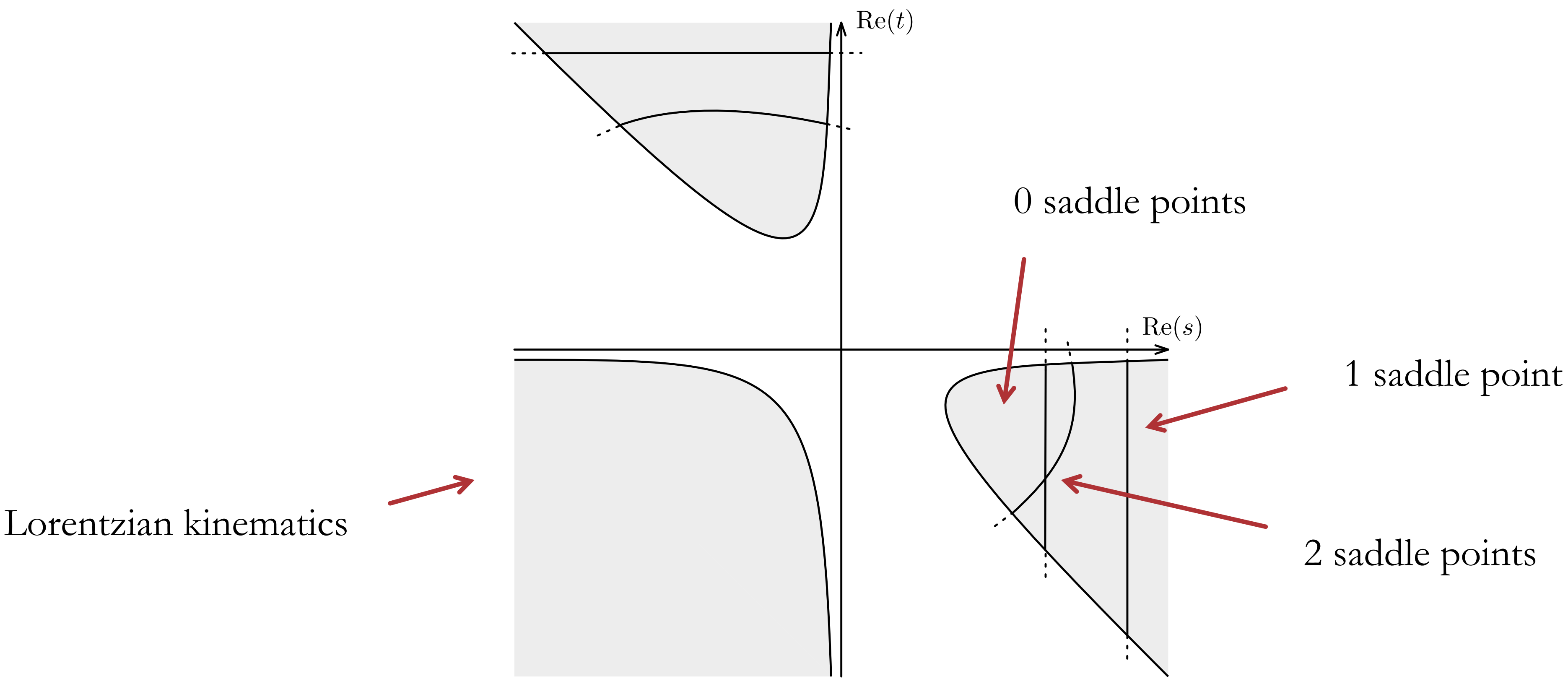
- The action vanishes on the saddle points, $\mathcal{V} = \sum_e \alpha_e \frac{\partial \mathcal{V}}{\partial \alpha_e} = 0$

- Integrating out the overall scale gives $\int \frac{\mathrm{d}\lambda}{\lambda^{1-\gamma}} e^{i\lambda\mathcal{V}} \propto \frac{1}{\mathcal{V}^\gamma}$

  degree of divergence

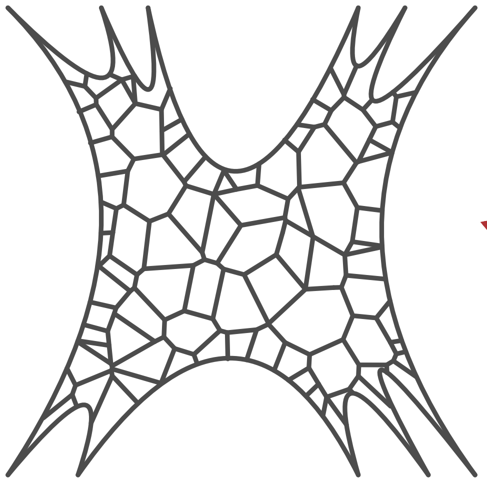
- E equations on E−1 independent Schwinger parameters, leaving at least one constraint on the *external kinematics*

For instance at 4-pt, in terms of the Mandelstam invariants s and t



(while Euclidean scattering doesn't have any saddles at all)

Physical meaning of intermediate particles becoming classical on-shell states (also known as *anomalous thresholds*)



internal propagators on-shell

$$q_e^2 - m_e^2 = 0$$

Saddle point conditions equivalent to *Landau equations*

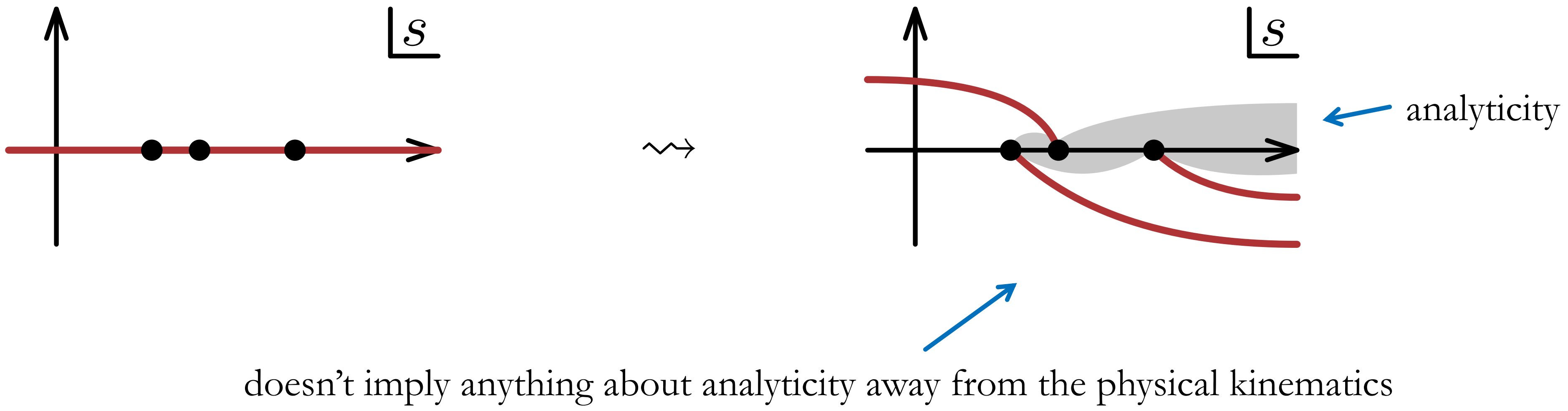
Causality/convergence at infinity is imposed by rotating the worldlines by an infinitesimal amount in complex directions:

$$\alpha_e \to \alpha_e \exp\left(i\varepsilon \frac{\partial \mathcal{V}}{\partial \alpha_e} \right)$$

The action acquires a small positive imaginary part

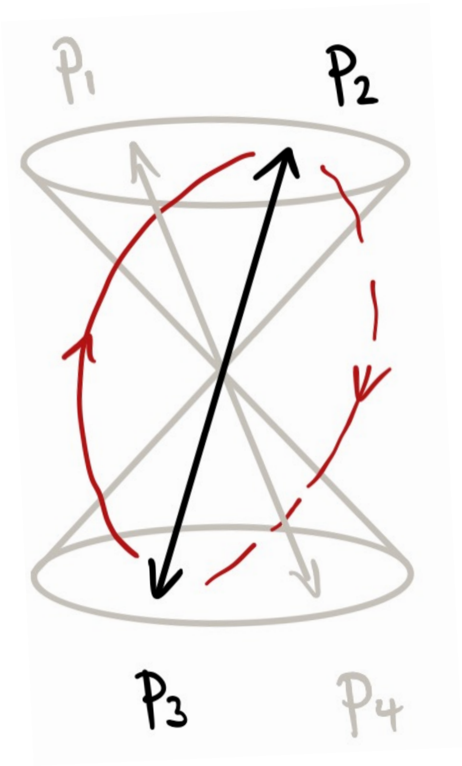
$$\mathcal{V} \to \mathcal{V} + i\varepsilon \underbrace{\sum_e \alpha_e \left(\frac{\partial \mathcal{V}}{\partial \alpha_e} \right)^2}_{>0 \quad \text{except at saddle points}} + \ldots$$

Resolves the branch cuts



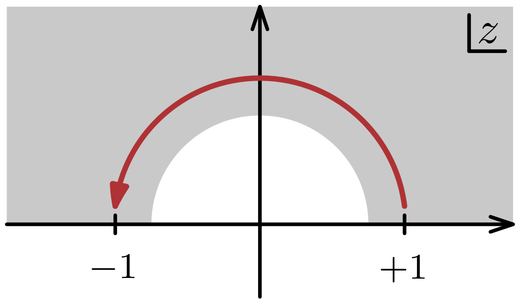
Starting point for general dispersion relations beyond EFT?

Analytic continuation of *external* energies within the complexified lightcone (say at 4-pt):



lightcone coordinates

$$
\begin{aligned}
p_1^\mu &= (\quad p_1^+, \quad p_1^-, \ \vec{p}_1) \\
p_2^\mu &= (\ \ zp_2^+, \ \tfrac{1}{z}p_2^-, \ \vec{p}_2) \\
p_3^\mu &= (-zp_2^+, -\tfrac{1}{z}p_2^-, \ \vec{p}_3) \\
p_4^\mu &= (\,-\,p_1^+, \ -\,p_1^-, \ \vec{p}_4)
\end{aligned}
$$

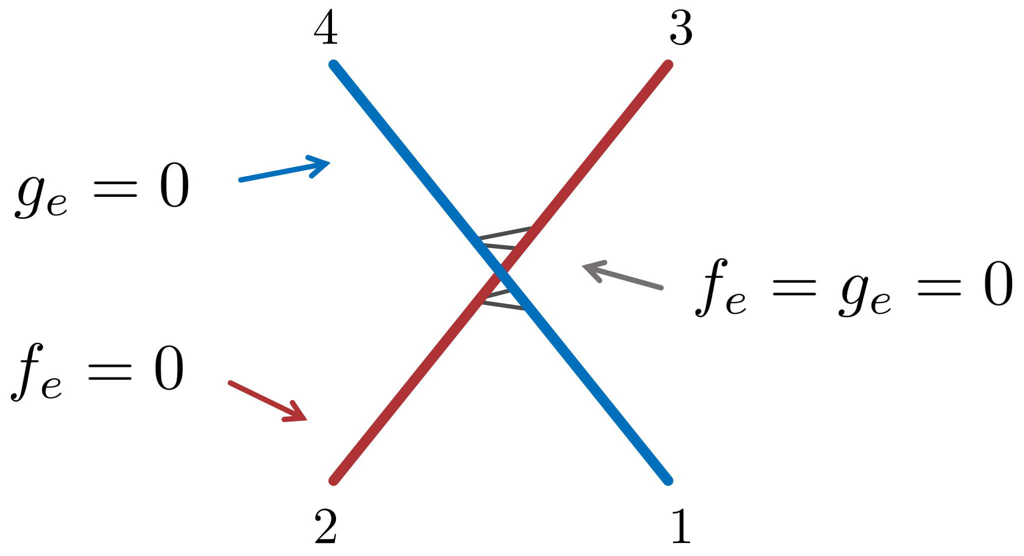


(preserves on-shell conditions and mom. cons.)

Every *internal* momentum q_e^μ can be decomposed as

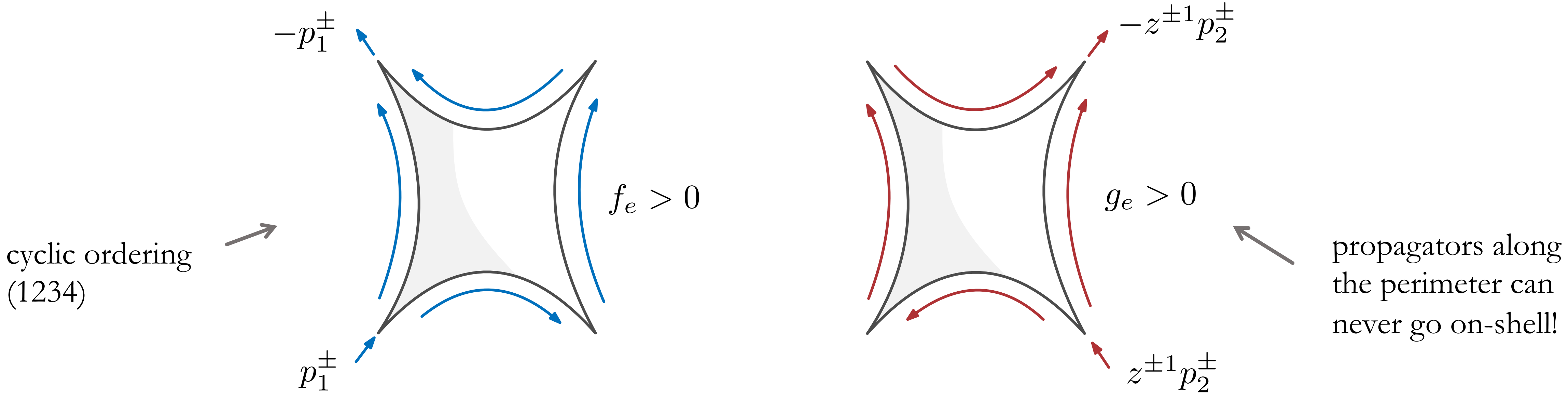
$$q_e^\pm = p_1^\pm f_e + z^{\pm 1} p_2^\pm g_e$$

Putting them on-shell implies $0 = \mathrm{Im}(q_e^2 - m_e^2) \propto f_e g_e$



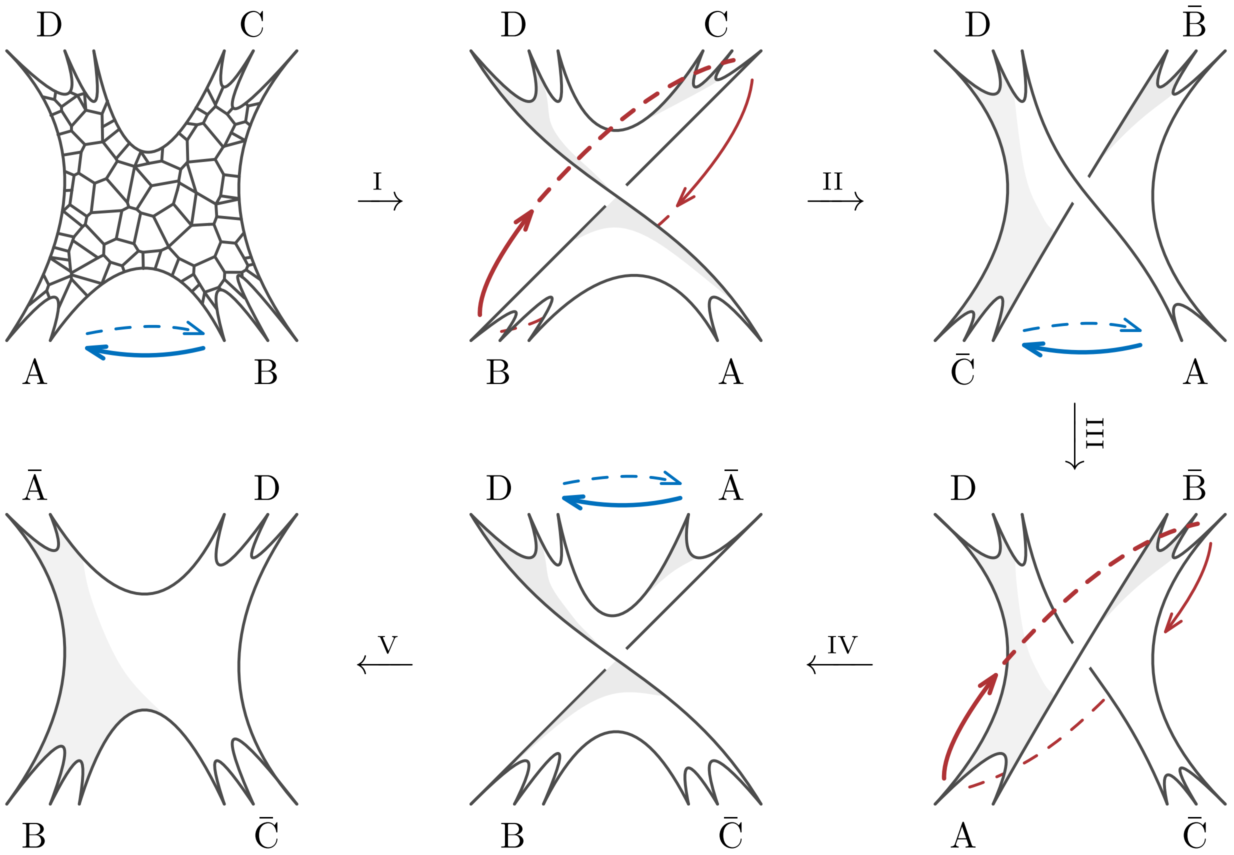
Unknown if such anomalous thresholds exist in an arbitrary theory

For *planar* scattering amplitudes such singularities never appear:



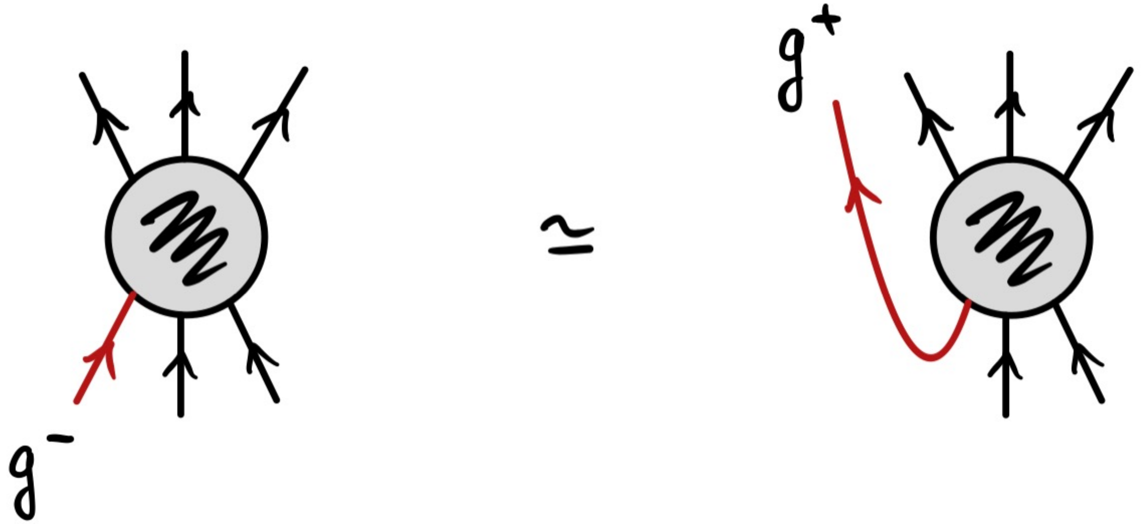
$$\mathrm{Im}(q_e^2 - m_e^2) \neq 0 \quad \Rightarrow \quad \text{analyticity when rotating the energies}$$

Sequence of rotating the energies between crossing channels:



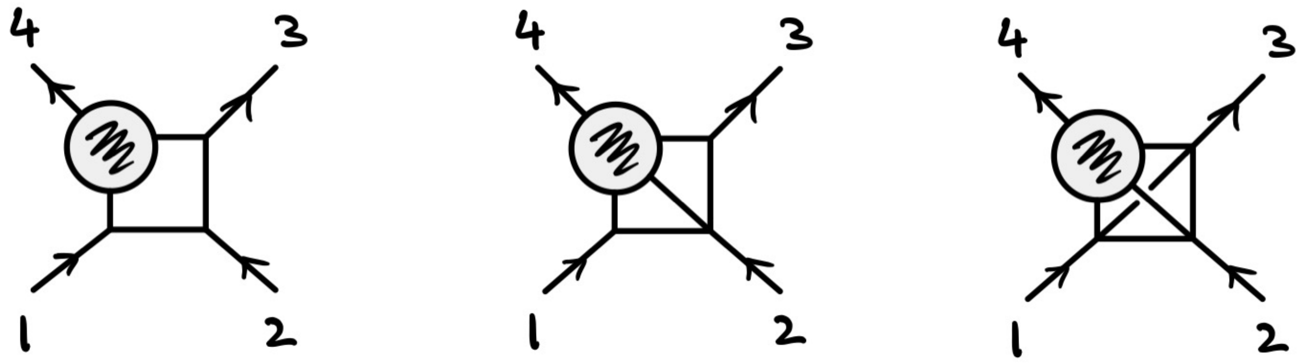
[details in hep-th/2104.12776]

Composition of such moves proves crossing symmetry
for planar amplitudes at every order in perturbation theory
with any masses, spins, multiplicity, …



(for $n \geqslant 5$, processes with consecutive in/out states in CPT-invariant theories)

Similar arguments extend to many non-planar Feynman diagrams, e.g.,

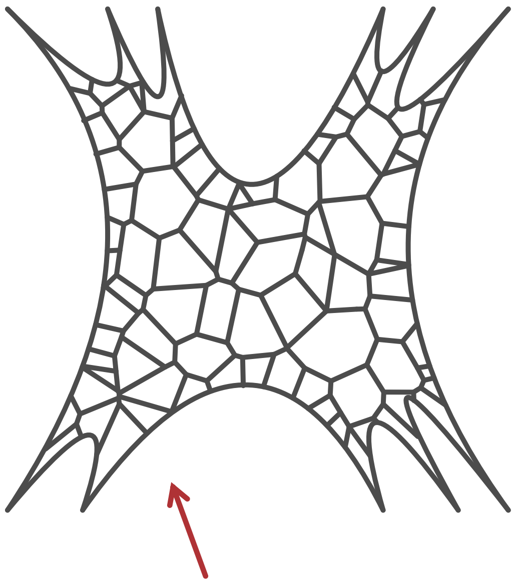


But, more generally, crossing symmetry remains an open problem!

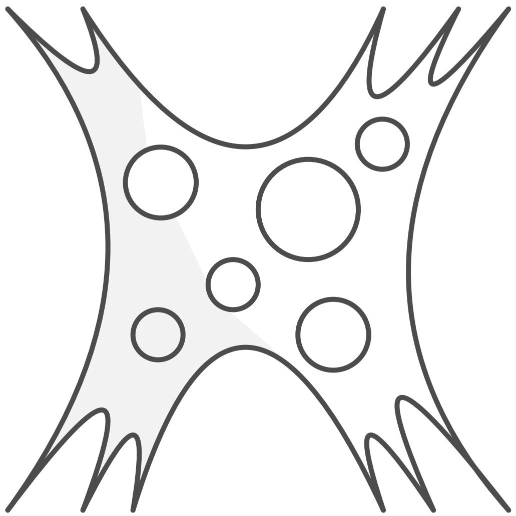
How does it generalize to strings?

(Spoiler: we don't know)

All the QFT anomalous thresholds should be present



$\subset$



for every mass level $m_e^2 \in \mathbb{N}/\alpha'$

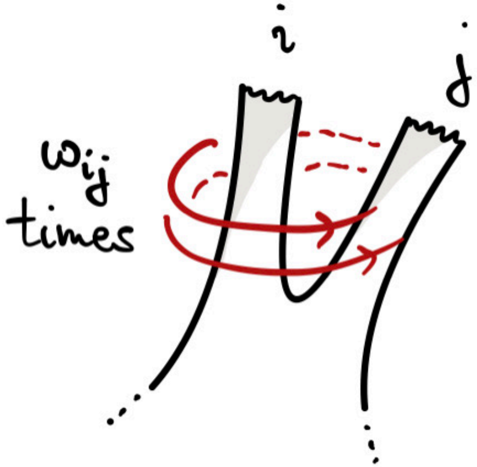
Therefore, Lorentzian string scattering at any genus must involve an *infinite* number of saddle points!

Before applying the saddle-point analysis, one needs to make the Green's functions *holomorphic* (say, for open strings)

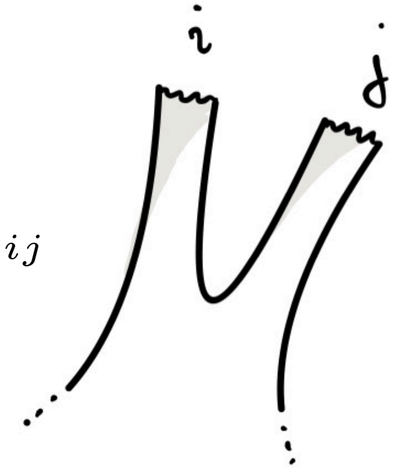
$$\mathcal{G}_{ij} = \log|z_i - z_j| + \ldots$$

$$\rightsquigarrow \log(z_i - z_j) + \ldots$$

Allows for "winding" of strings around each other



$$= \quad e^{2\pi i\alpha' p_i \cdot p_j w_{ij}}$$



Tricky to interpret physically

[cf. Grinberg, Maldacena '20 in black hole physics]

Already non-trivial at genus zero

On each of the sheets counted by $\pi_1^{\mathrm{ab}}(\mathcal{M}_{0,n}) = \mathbb{Z}^{n(n-3)/2}$,
the number of saddle points is $|\chi(\mathcal{M}_{0,n})| = (n-3)!$

worldsheet action (finite)

$$\sum_{a=1}^{(n-3)!} e^{-\mathcal{S}_a} \sum_{\vec{w}_{ij} \in \mathbb{Z}^{n(n-3)/2}} \mathrm{N}_{a,\vec{w}_{ij}} \prod_{ij} e^{2\pi i \alpha' p_i \cdot p_j w_{ij}}$$

integers depending on the crossing channel and cyclic ordering

For example, 4-pt with Chan–Paton ordering (1234)
in the (2,2) signature gives a single saddle

$$e^{-\mathcal{S}}$$

[Fairlie, Roberts '72]

[Gross, Mende '87]

In the Lorentzian signature we have an infinite lattice of saddles

$$
\begin{array}{c|ccccc}
 & \mathrm{N}_{-2,2} & \mathrm{N}_{-1,2} & \mathrm{N}_{0,2} & \mathrm{N}_{1,2} & \mathrm{N}_{2,2} \\
\uparrow & \mathrm{N}_{-2,1} & \mathrm{N}_{-1,1} & \mathrm{N}_{0,1} & \mathrm{N}_{1,1} & \mathrm{N}_{2,1} \\
w_{23} & \mathrm{N}_{-2,0} & \mathrm{N}_{-1,0} & \mathrm{N}_{0,0} & \mathrm{N}_{1,0} & \mathrm{N}_{2,0} \\
 & \mathrm{N}_{-2,-1} & \mathrm{N}_{-1,-1} & \mathrm{N}_{0,-1} & \mathrm{N}_{1,-1} & \mathrm{N}_{2,-1} \\
 & \mathrm{N}_{-2,-2} & \mathrm{N}_{-1,-2} & \mathrm{N}_{0,-2} & \mathrm{N}_{1,-2} & \mathrm{N}_{2,-2} \\
 & & & \xrightarrow[w_{12}]{\hspace{4em}} & &
\end{array}
$$

t-channel

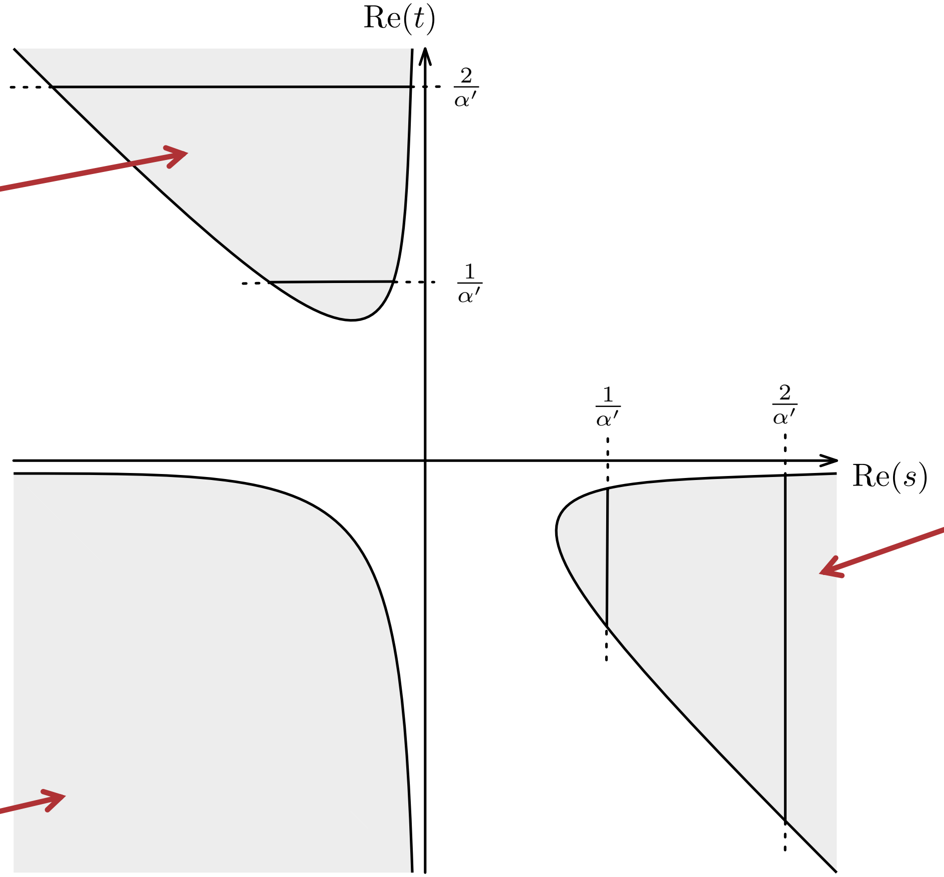
$$w_{23}\uparrow\quad\begin{array}{ccccc} & \vdots & \vdots & & \\ 0 & +1 & -1 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

$$\xrightarrow{\quad w_{12}\quad}$$

u-channel

$$w_{23}\uparrow\quad\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

$$\xrightarrow{\quad w_{12}\quad}$$



s-channel

$$w_{23}\uparrow\quad\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & -1 & -1 & \cdots \\ 0 & 0 & +1 & +1 & +1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \end{array}$$

$$\xrightarrow{\quad w_{12}\quad}$$

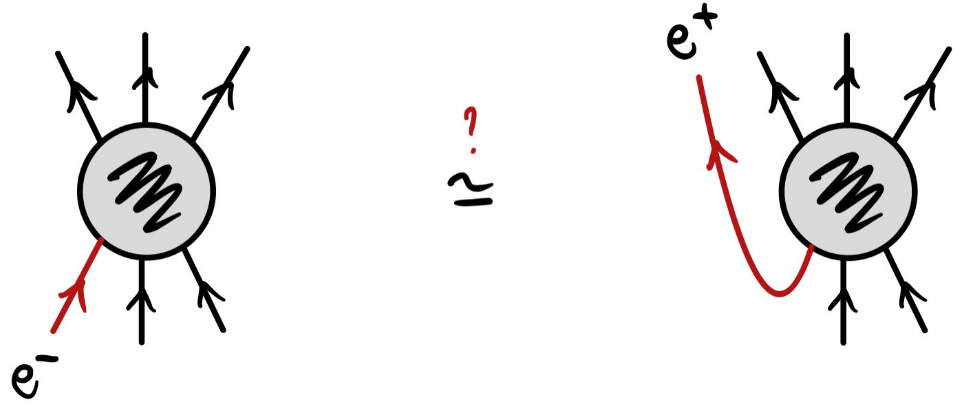
All the saddles resum to

$$\frac{e^{-2\pi i\alpha' t} - e^{2\pi i\alpha' s}}{1 - e^{2\pi i\alpha' s}}\, e^{-\mathcal{S}}$$

resonances

[details in hep-th/1910.11852 with Pokraka]

Summary



- Singularities as worldline saddle points
- Crossing symmetry for planar amplitudes
- Challenges in generalizations to string theory

Thank you!