Asymptotic Symmetries for Gauge & Gravitational Theories in Minkowski Space

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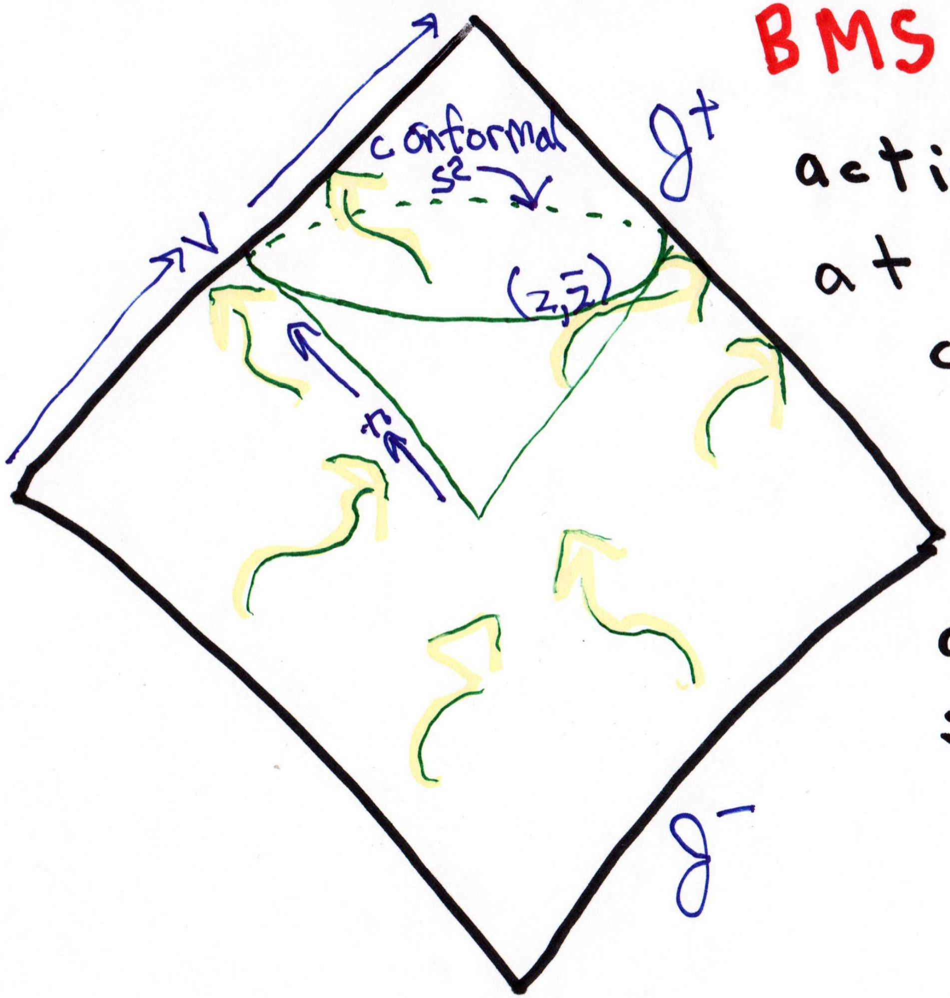
A half-century ago, Bondi, van der Burg, Metzner & Sachs[2] discovered the ∞-dimensional

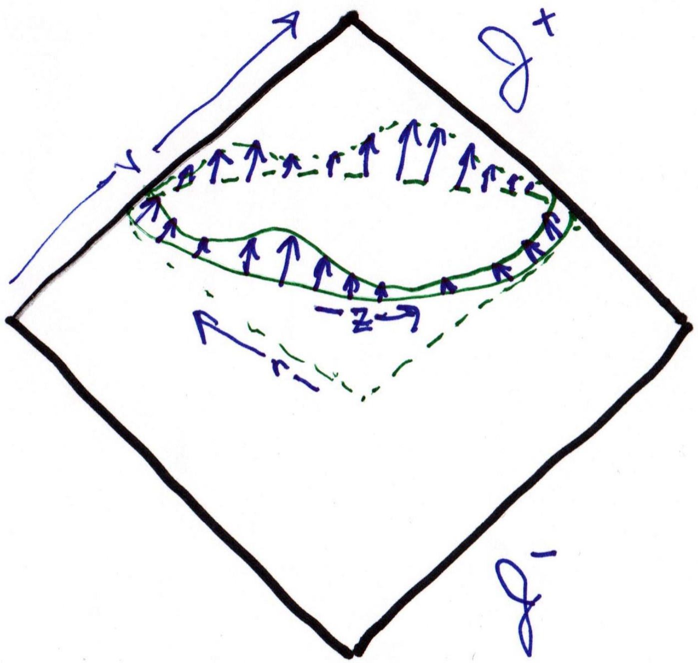
BMS group $= \dfrac{\text{allowed diffeos}}{\text{trivial diffeos}}$

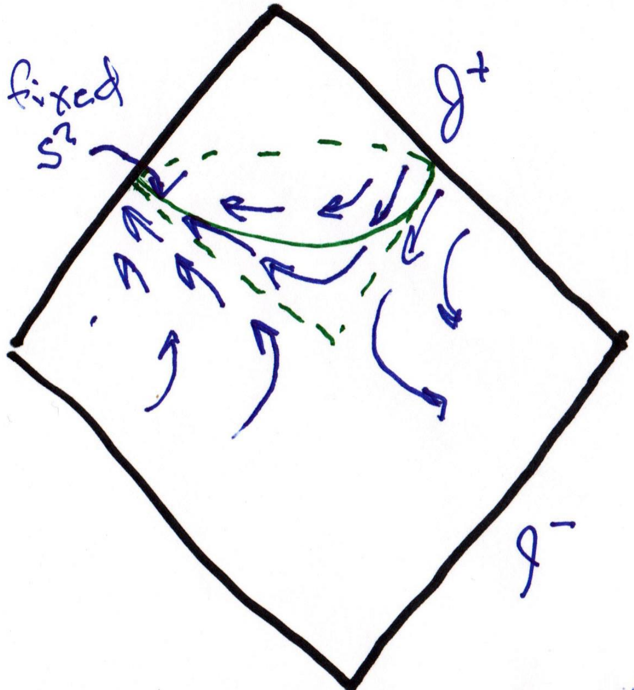
acting on physical data at $\mathscr{I}^+$. Many S^2s, many charges. **BMS** should play a central role in the study of Minkowski scattering.



$$ds^2 \xrightarrow{r\to\infty} -dv^2 - 2\,dr\,dv + 2r^2\gamma_{z\bar{z}}\,dz\,d\bar{z} \qquad \gamma_{z\bar{z}} = \frac{1}{(1+z\bar{z})^2}$$

Two types of BMS generators





∞ **supertranslations**

$$\xi_T = T(z,\bar{z})\partial_v + \mathcal{O}\left(\tfrac{1}{r}\right)$$

any function

6 **SL(2,C) conformal**

odd

$$\xi_{conf} = \xi(z)\partial_z - \tfrac{1}{2}D\cdot\xi\,(r\partial_r - v\partial_v)$$

$\sim$ **Lorentz boost** $+\,\mathcal{O}\left(\tfrac{1}{r}\right)$

$\xi(z) = 1, z, z^2$

= Global conf. killing vectorons?

motivated current investigations

A fascinating proposal

Barnich & Troessaert (2010)

Let

$$\xi_{conf} = \xi(z)\partial_z - \tfrac{1}{2} D\cdot\xi\,(r\partial_r - v\partial_v) + O\left(\tfrac{1}{r}\right)$$

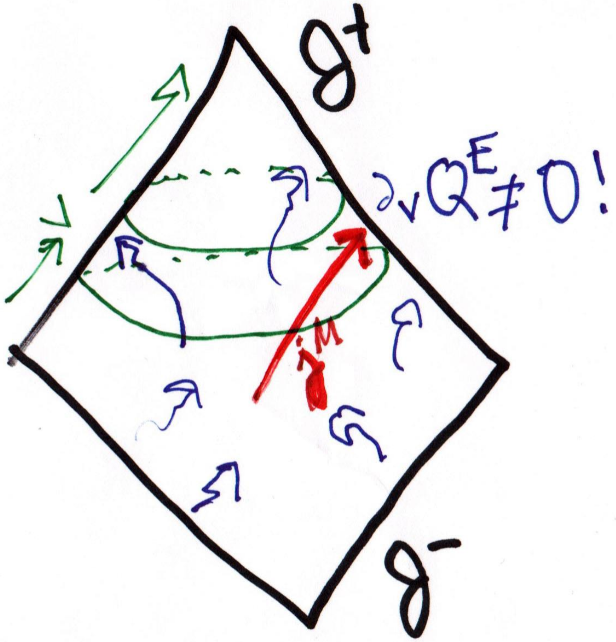
with $\xi(z)\partial_z$ <u>any</u> S^2 conformal killing vector (allow analytic singularities).

Logic similar to BPZ (1982).

Minkowski scattering $\leftrightsquigarrow$ CFT_2?

Will see below possibly implicit in Weinberg (1965) soft graviton theorem.

SIMPLER TOY PROBLEMS



I. What are the asymptotic symmetries at $\mathscr{I}^+$ for electrodynamics with massless charged particles?

II What are the asymptotic symmetries at $\mathscr{I}^+$ for (Coulomb-phase) Yang-Mills theories with group G?

Comments

(i) Of interest in own right.

(ii) No universal def. of "asymp. symmetry", part of problem is to define it. Should be useful.

(iii) Conjecture G-Kac-Moody, possibly level $k = 2/g_{YM}^2$. Coupling-ready to gravity!

(iv) My understanding is incomplete. Comments welcome!

This is how it goes....

Important Clue

Maldacena (unpublished note, 2012)

Weinbergs theorem (1965)

$$\langle \mathcal{O}_1(p_1)\ldots\mathcal{O}_n(p_n)\mathcal{O}^{\gamma}(\varepsilon,q)\ldots\rangle \sim \sum_{k=1}^{n} Q_k \frac{\varepsilon_\mu p_k^\mu}{q_\mu p_k^\mu}\langle \mathcal{O}_1(p_1)\ldots\mathcal{O}_n(p_n)\ldots\rangle$$

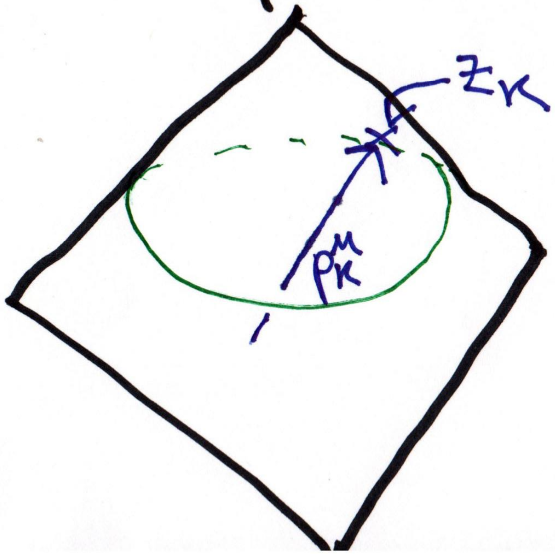
charge of $\mathcal{O}_k$

photon polarization, momentum

can be rewritten in position space on the conformal S^2 at $\mathscr{I}^+$ (massless particles)

$$\langle \mathcal{O}_1(z_1)\ldots\mathcal{O}_n(z_n)\mathcal{O}^{\gamma}(\varepsilon,q)\rangle \sim \sum_{k=1}^{n}\frac{Q_k}{z-z_k}\langle \mathcal{O}_1(p_1)\ldots\mathcal{O}_n(p_n)\ldots\rangle$$

~ Kac-Moody Ward identity. Special thanks to J. Maldacena for explanation.

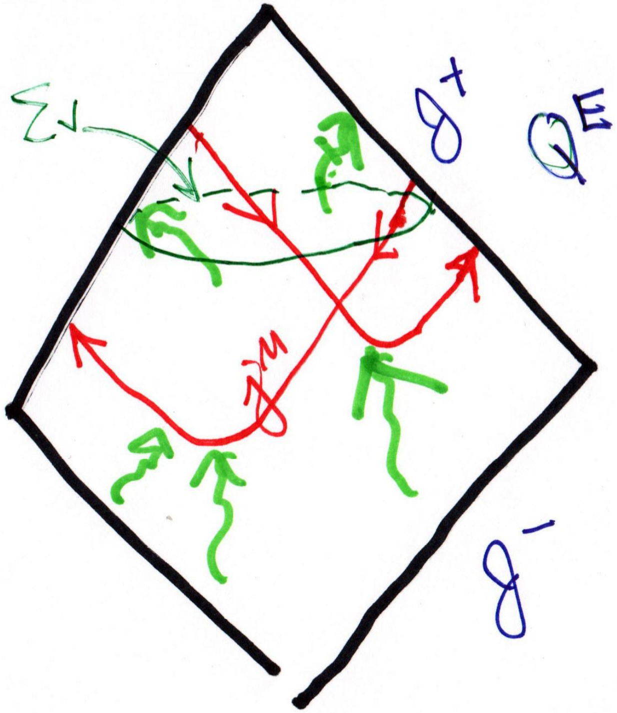


Simplest first case: "massless QED"

no massive particles

classical asymptotic analysis

concentrate on $\mathcal{I}^+$: no incoming j^M_μ

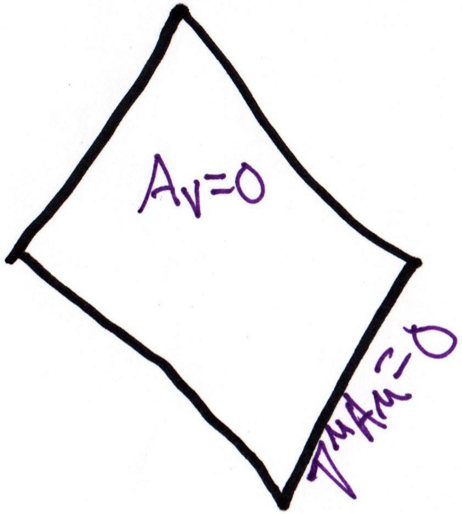


$$Q^E(v) = \frac{1}{4\pi e^2}\int_{\Sigma_v} d^2z\,\gamma_{z\bar z}\, r^2 F_{rv}$$

$$\nabla^\mu F_{\mu v} = j^M_v\, e^2$$

$$\Rightarrow \partial_v Q^E(v) = -\int_{\Sigma_v} d^2z\,\gamma_{z\bar z}\, r^2 j^E_v$$

Radiation Gauge choice & residual symmetry



Temporal gauge

$A_v = 0$ everywhere

$\nabla^\mu A_\mu = 0$ on $\mathcal{I}^-$

(requires $g^{u}|_{\mathcal{I}^-} = 0$)

Residual symmetry

$\delta A_\mu = \partial_\mu \hat{\varepsilon}$

$\partial_v \hat{\varepsilon} = 0, \quad 2\partial_z \partial_{\bar{z}} \hat{\varepsilon} + \gamma_{z\bar{z}} \partial_r (r^2 \partial_r \hat{\varepsilon}) = 0$

locally solved by

$$\boxed{\hat{\varepsilon} = \varepsilon(z) + \cancel{\bar{\varepsilon}(\bar{z})} = \text{large gauge transf.}}$$

ignore

$\delta A_z = \partial_z \varepsilon$ = "boundary photon"

Note: might be better gauge choice for this problem. ($A_r=0$?)

These large gauge transformations are singular at points, like the generic CKV on S^2. Nevertheless we will find they are useful to consider — the Ward identities lead to Weinbergs soft photon theorem.

The Final Data Problem

c.f. Pfefferman-Graham expansion, or BMS asymptotic analysis

$A_r \sim \frac{1}{r^2}$ ensures $Q^{\varepsilon}(v) \sim \int r^2 \gamma_{z\bar{z}} \partial_r A_v$ is finite.

$A_z \sim r^0$ implies $\int r^2 \gamma_{z\bar{z}} T_{vv} = \int \partial_v A_z \partial_v A_{\bar{z}}$

= energy flux is finite and nonzero

allows plane waves

Expand

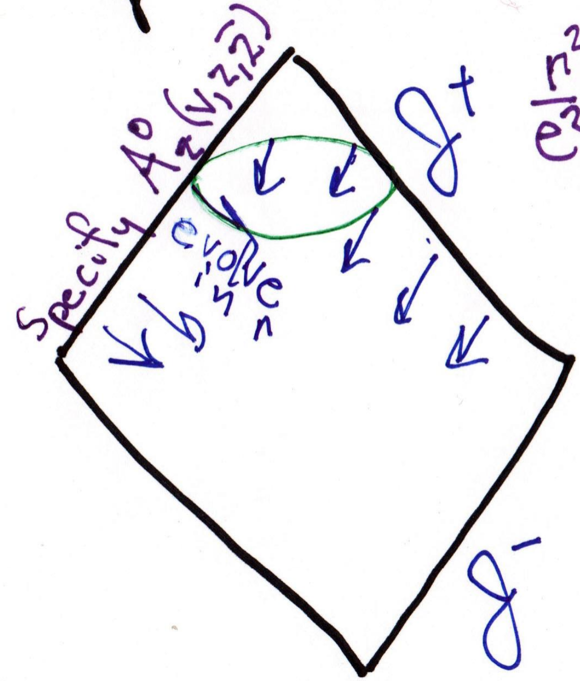
$$A_r(r,v,z,\bar{z}) = \frac{1}{r^2} A_r^0(v,z,\bar{z}) + \frac{1}{r^3} A_r^1(v,z,\bar{z}) + \ldots$$

$$A_z(r,v,z,\bar{z}) = A_z^0(v,z,\bar{z}) + \frac{1}{r} A_z^1(v,z,\bar{z}) + \ldots$$

$$j_v^M(r,v,z,\bar{z}) = \frac{1}{r^2} j_v^{M0}(v,z,\bar{z}) + \ldots$$

Maxwells equation then determine all terms in $A_\mu(r,v,z,\bar{z})$ perturbatively from the "free" final data $A_z^0(v,z,\bar{z})$ (and of course j^M)

For example the leading constraint equation on $\mathcal{I}^+$ is



$$\frac{r^2}{e^2}\nabla^\mu F_{\mu z} \overset{r\to\infty}{=} -\frac{\gamma^{z\bar z}}{e^2}\partial_v\Big[2\partial_{\bar z}A^0_z + F^0_{z\bar z} - \gamma_{z\bar z}F^0_{vr}\Big]$$

$$= j^{M0}_v$$

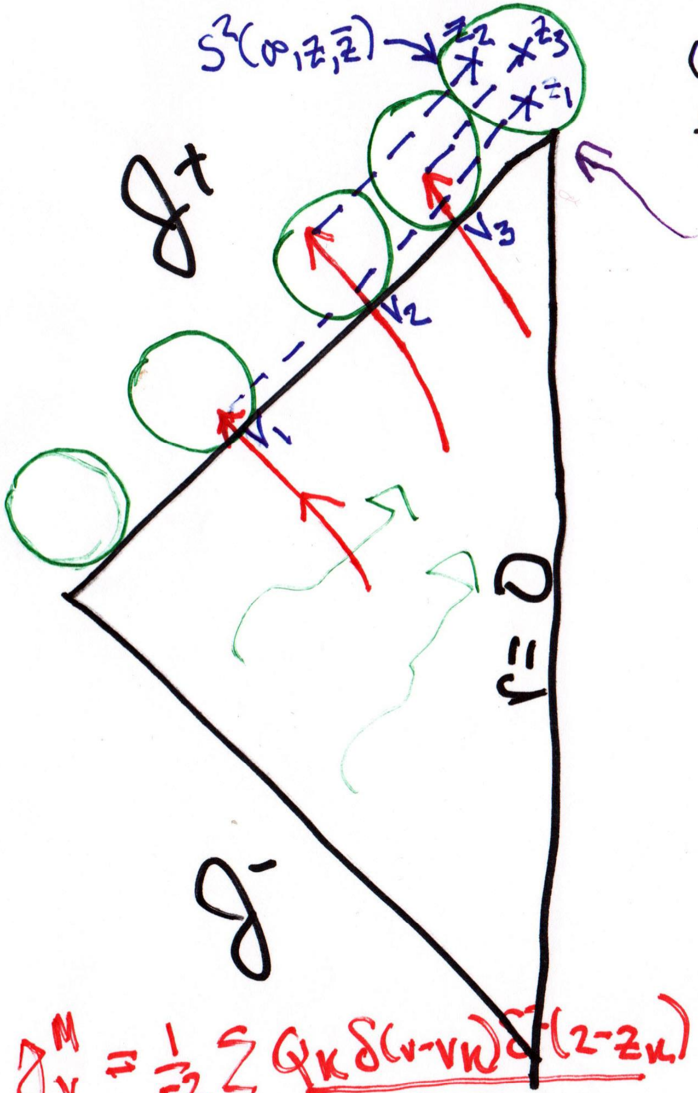
$\Rightarrow$

$$A^0_r = +e^2\int\int^v j^M_v + 2\gamma^{z\bar z}\int^v \gamma^{z\bar z}(\partial_{\bar z}A^0_z + \partial_z A^0_{\bar z})$$

at each $(z,\bar z)$

Subleading terms similarly determined.

The Boundary Current



$$j^M_v = \frac{1}{r^2}\sum_k Q_k \frac{\delta(v-v_k)\,\delta^2(z-z_k)}{\gamma_{z\bar z}}$$

Consider data which reverts to the vacuum at $v \to \infty$ on $\mathcal{I}^+$:

$$F^0_{z\bar z}(\infty,z,\bar z) = F^0_{vr}(\infty,z,\bar z) = 0$$

Define

$$\boxed{J_z(z,\bar z) = -\frac{4\pi}{e^2} A^0_z(\infty,z,\bar z)}$$

obeying

$$\partial_{\bar z} J_z = 2\pi \gamma_{z\bar z} \int_{-\infty}^{\infty} j^{M0}_v \, dz$$

$$= 2\pi \sum_k Q_k \delta^2(z-z_k)$$

$$\Rightarrow$$

$$J_z = \sum_k \frac{Q_k}{z-z_k}$$

The semiclassical operator version of this relation is

$$\langle 0|\mathcal{O}_1(z_1)\dots\mathcal{O}_n(z_n)\,J(z)\,|\,in\rangle$$

$$\approx \sum_{k=1}^{n}\frac{Q_k}{z-z_k}\,\langle 0|\mathcal{O}_1(z_1)\dots\mathcal{O}_n(z_n)\,|\,in\rangle$$

= Weinbergs soft photon theorem (Maldacena)

Moreover J contour integrals generate large gauge transf. on regions of $\mathcal{J}^+$

$$\oint_C dz\,\varepsilon(z)\,\langle 0|\mathcal{O}_1(z_1)\dots J(z)\,|\,in\rangle = \delta_\varepsilon\,\langle 0|\mathcal{O}_1(z_1)\dots|\,in\rangle$$

⇒ **Weinbergs theorem = Ward identity of asymptotic U(1) Kac-Moody symmetry!**

Level?

J_z is itself ~~not~~ not gauge invariant:

$\delta_\varepsilon J_z = -\frac{4\pi}{e^2}\partial_z \varepsilon$ or

$$\langle 0 | J_z J_w | 0 \rangle \sim \frac{k}{(z-w)^2}, \quad k = \frac{2}{e^2}$$

$\Rightarrow$ level k Kac-Moody. However subleading terms contain non-holomorphic corrections I dont understand, so wont discuss further!

Å] Our choice of temporal gauge breaks the conformal SL(2,C) symmetry

Note: Weinberg uses a slightly different gauge-invariant but non-local soft photon operator with no level.

The analysis of the G non Abelian case including a relation of a Ward identity from a single G Kac Moody insertion to a gluonic version of Weinbergs theorem and hints of a level $k = 2/g_{YM}^2$.
We now turn to the much more intricate case of...

GRAVITY

Bondi coordinates

$$ds^2 = -\left(1-\frac{2m_B}{r}\right)dv^2 - 2\,dv\,dr + 2\gamma_{z\bar z}\left(r\,dz + U^z dv + \frac{N^z}{r}dv\right)\left(r\,d\bar z + U^{\bar z} dv + \frac{N^{\bar z}}{r}dv\right)$$

$$+2r\left(C_{zz}\,dz^2 + C_{\bar z\bar z}\,d\bar z^2\right) + \dots$$ irrelevant subleading

m_B, U^z, N^z, C_{zz} depend on $(v, z, \bar z)$ but not r.

Constraints

$$r^2 G_{vv} = 8\pi G_N T^M_{vv} = -\tfrac{1}{4}\partial_v C^2 - \partial_v m_B + \tfrac{1}{2}\partial_v\left(D^z U_z + D^{\bar z} U_{\bar z}\right)$$

matter radiation flux — Bondi news — total derivative

$$r^2 G_{vz} = 8\pi G_N T^M_{vz} = \partial_z m_B + \tfrac{1}{16}\partial_z\partial_v C^2 - U^z\partial_v C_{zz} - \tfrac{1}{2}D^{\bar z}\left(\partial_{\bar z}U_z - \partial_z U_{\bar z}\right)$$

If the system reverts to the vacuum for $v\to\infty$ second equation implies

$$m_B(\infty, z, \bar z) = \partial_{z} U_{\bar z}(\infty, z, \bar z) - \partial_{\bar z} U_z(\infty, \bar z, z) = 0$$

Supertranslation Boundary Current

Define

$$P_z(z,\bar z) = U_z(\infty, z, \bar z)$$

which obeys

$$\partial_{\bar z} P_z = \int_{-\infty}^{\infty} dv \big(\underbrace{r^2 8\pi G_N T^M_{vv} + \tfrac{1}{4}(\partial_v C)^2\big)\gamma_{z\bar z}}_{= \sum_k E_k \delta(v-v_k)\delta^2(z-z_k)}$$

$$\Rightarrow \quad \boxed{P_z = \sum_k \frac{E_k}{z-z_k}}$$

(i) Quantum version is limit of Weinbergs soft graviton theorem.

(ii) Insertions of P contour integrals generate holomorphic supertranslations.

BMS $\longleftrightarrow$ Weinberg SGT

Trying to make similar construction for Virasoro generators of extended BMS.

Conclusion

$\mathscr{I}^+$ of Minkowski space has an interesting structure for both gauge & gravity theories that we do not fully understand. Recent developments in holography, and CFT_2, may reveal new insights into this old problem.