Unity of tree-level superstring amplitudes

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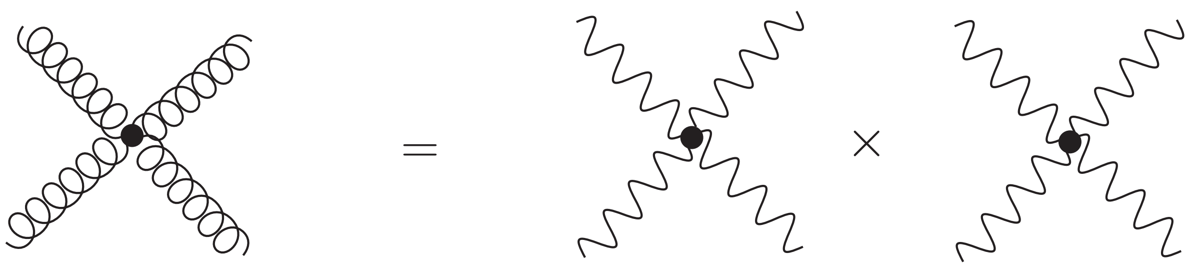
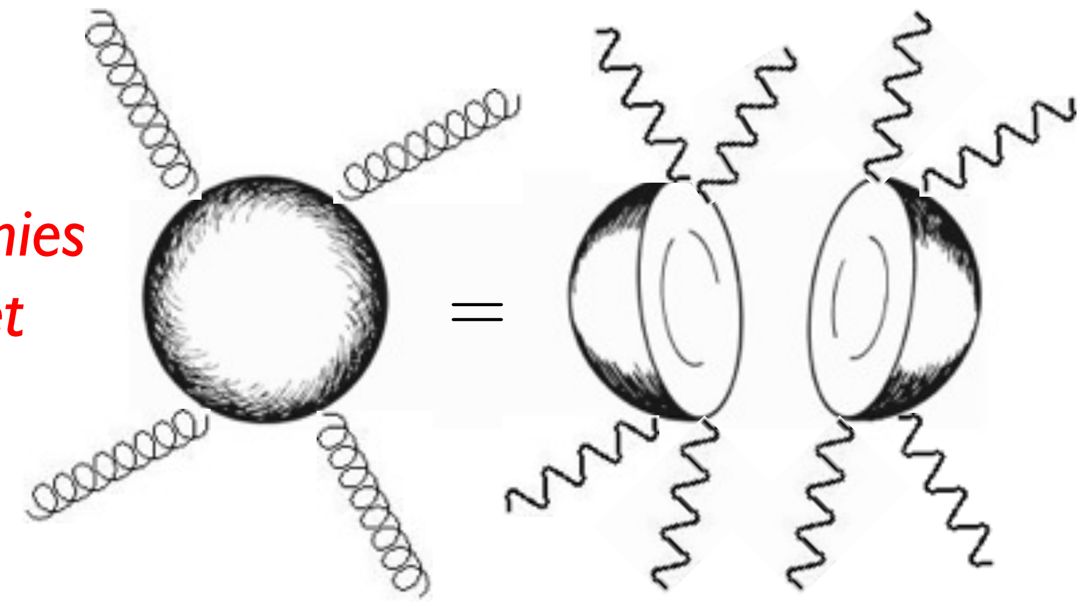
Perturbative String Amplitudes

(a) Structure of string amplitudes has ***deep impact***
on the ***form*** *and* ***organization*** *of quantum field theory amplitudes*



from monodromies of world-sheet



KLT

$$M_{FT}(1,2,3,4) \quad = \quad s_{12}\, A_{FT}(1,2,3,4)\, \tilde{A}_{FT}(1,2,4,3)$$

graviton amplitudes = (gauge amplitudes) × (gauge amplitudes)

many **relations in field-theory** emerge from
properties of string world-sheet:
monodromy on world-sheet yield KLT, BCJ, … relations

(b) (Quite recent) mathematical concepts from/in string amplitudes

elements of **arithmetic algebraic geometry** (e.g. motives, symbols, coproduct, …) underlay the structure of superstring amplitudes

α' - expansion

open, closed superstring amplitude

decomposition of motivic MZVs

Drinfeld, Deligne associators

(c) Amplitudes are key players in establishing string dualities

unexpected relations between open and closed string amplitudes
(beyond KLT)

***new** string **duality** may emerge*
(to all orders in α' i.e. beyond BPS)

based on:

- St.St.: **Closed superstring amplitudes, single-valued multiple zeta values and the Deligne associator,**
J. Phys. A47 (2014) 155401, [arXiv:1310.3259]

- St.St., T.R. Taylor: **Closed string amplitudes as single-valued open string amplitudes,**
Nucl. Phys. B881 (2014) 269-287, [arXiv:1401.1218]

Heterotic gauge amplitudes as single-valued type I gauge amplitudes

Tree-level N-point type I open superstring gauge amplitude:

$$\mathfrak{A}_N^{\rm I} = (g_{YM}^{\rm I})^{N-2} \sum_{\Pi \in S_N/\mathbf{Z}_2} {\rm Tr}(T^{a_{\Pi(1)}} \ldots T^{a_{\Pi(N)}})\ \mathcal{A}^{\rm I}(\Pi(1),\ldots,\Pi(N))$$

Tree-level N-point heterotic closed string gauge amplitude:

$$\mathfrak{A}_N^{\rm HET} = (g_{YM}^{\rm HET})^{N-2} \sum_{\Pi \in S_N/\mathbf{Z}_2} {\rm Tr}(T^{a_{\Pi(1)}} \ldots T^{a_{\Pi(N)}})\ \mathcal{A}^{\rm HET}(\Pi(1),\ldots,\Pi(N)) + \mathcal{O}(1/N_c^2)$$

Result: $$\boxed{\mathcal{A}^{\rm HET}(\Pi) = {\rm sv}\left(\mathcal{A}^{\rm I}(\Pi)\right)}$$

sv= single-valued projection

This will be generalized to ***any*** *closed string amplitude:*
closed string amplitudes as
single-valued open string amplitudes

at the level of world-sheet integrals:

e.g. N=4:
$$\int_{\mathbf{C}} d^2z \, \frac{|z|^{2s} \, |1-z|^{2u}}{z \, (1-z) \, \overline{z}} = \mathrm{sv} \left(\int_0^1 dx \; x^{s-1} \; (1-x)^u \right)$$

$$\frac{1}{s} \, \frac{\Gamma(s) \, \Gamma(u) \, \Gamma(t)}{\Gamma(-s) \, \Gamma(-u) \, \Gamma(-t)} = \mathrm{sv} \left(\frac{\Gamma(s) \, \Gamma(1+u)}{\Gamma(1+s+u)} \right)$$

$$s = \alpha'(k_1+k_2)^2$$
$$t = \alpha'(k_1+k_3)^2$$
$$u = \alpha'(k_1+k_4)^2$$

No KLT relations necessary !

KLT: $$\int_{\mathbf{C}} d^2z \, \frac{|z|^{2s} \, |1-z|^{2u}}{z \, (1-z) \, \overline{z}} = \sin(\pi u) \left(\int_0^1 x^{s-1} \, (1-x)^{u-1} \right) \left(\int_1^\infty x^{t-1} \, (1-x)^u \right)$$

Type I gauge amplitude:

$\pi, \rho \in S_{N-3}$

$$\mathcal{A}^{\mathrm{I}}(\pi) = (-1)^{N-3} \sum_{\sigma \in S_{N-3}} \sum_{\rho \in S_{N-3}} Z_\pi(\rho)\; S[\rho|\sigma]\; A_{YM}(\sigma)$$

Mafra, Schlotterer, St.St. (2011)
Broedel, Schlotterer, St.St. (2013)

fundamental world-sheet **disk** integrals:
iterated real integral on $\mathbf{RP}^1 \backslash \{0, 1, \infty\}$

Heterotic gauge amplitude:

$$\mathcal{A}^{\mathrm{HET}}(\rho) = (-1)^{N-3} \sum_{\sigma \in S_{N-3}} \sum_{\overline{\rho} \in S_{N-3}} J[\rho\,|\,\overline{\rho}]\; S[\overline{\rho}|\sigma]\; A_{YM}(\sigma)$$

Taylor, St.St. (2014)

$$\boxed{J = \mathrm{sv}\,(Z)}$$

fundamental world-sheet **sphere** integrals:
integral on $\mathbf{P}^1 \backslash \{0, 1, \infty\}$

Supergravity N-graviton amplitude:

$$\mathcal{M}_{FT}(1, \ldots, N) = (-1)^{N-3}\, \kappa^{N-2} \sum_{\sigma \in S_{N-3}} \sum_{\rho \in S_{N-3}} \tilde{A}_{YM}(\rho)\; S[\rho|\sigma]\; A_{YM}(\sigma)$$

S = KLT kernel

S = KLT kernel

$$S[\rho|\sigma] \quad := \quad S[\,\rho(2,\dots,N-2)\,|\,\sigma(2,\dots,N-2)\,]$$

$$= \quad \prod_{j=2}^{N-2}\left(s_{1,j_\rho} \;+\; \sum_{k=2}^{j-1}\theta(j_\rho,k_\rho)\, s_{j_\rho,k_\rho} \right)$$

$$s_{ij} = \alpha'(k_i+k_j)^2$$

Bern, Dixon, Perelstein, Rozowsky (1998)
Bjerrum-Bohr, Damgaard, Sondergaard, Vanhove (2010)

side note: in super-twistor or dual super-twistor space:

$$\tilde{A}_{YM}(\rho) \simeq Z_\pi(\rho)$$

subject to Mellin transformation

gives rise to superstring/supergravity Mellin correspondence

Taylor, St.St. (2013)

Multiple zeta-values in superstring theory

Disk integrals: iterated real integral on $\mathbf{RP}^1 \backslash \{0,1,\infty\}$

Expand w.r.t. α':

$$Z_{23}(23) \equiv V_{\rm CKG}^{-1} \int_{z_i<z_{i+1}} \left(\prod_{j=1}^{5} dz_j\right) \prod_{1\leq i<j\leq 5} \frac{|z_{ij}|^{s_{ij}}}{z_{12}z_{23}z_{35}z_{54}z_{41}}$$

$$= \frac{1}{s_{12}s_{45}} + \frac{1}{s_{23}s_{45}} + \zeta(2)\left(1 - \frac{s_{34}}{s_{12}} - \frac{s_{12}}{s_{45}} - \frac{s_{23}}{s_{45}} - \frac{s_{51}}{s_{23}}\right) + \mathcal{O}(\alpha')$$

Terasoma & Brown: the coefficients of the Taylor expansion of the Selberg integrals w.r.t. the variables s_{ij} can be expressed as linear combinations of MZVs over $\mathbf{Q}$

$$\zeta_{n_1,\ldots,n_r} := \zeta(n_1,\ldots,n_r) = \sum_{0<k_1<\ldots<k_r} \prod_{l=1}^{r} k_l^{-n_l} \quad , \quad n_l \in \mathbf{N}^+ \ , \ n_r \geq 2 \ ,$$

Commutative graded $\mathbf{Q}$ - algebra: $\mathcal{Z} = \bigoplus_{k\geq 0} \mathcal{Z}_k \ , \quad \dim_{\mathbf{Q}}(\mathcal{Z}_N) = d_N$

with: $d_N = d_{N-2} + d_{N-3}, \quad d_0 = 1,\ d_1 = 0,\ d_2 = 1, \ldots$ (Zagier)

Recall: $\mathcal{A}^{\mathrm{I}}(\pi) = (-1)^{N-3}\ Z\ S\ A_{YM}(\sigma)$ $\qquad \pi, \sigma \in S_{N-3}$

Define: $F_{\pi\sigma} := (-1)^{N-3} \sum_{\rho \in S_{N-3}} Z_\pi(\rho)\ S[\rho|\sigma]$ $\qquad$ F = period matrix

$$F(\alpha') = P\ Q\ \exp\left\{ \sum_{n\geq 1} \zeta_{2n+1}\ M_{2n+1} \right\}$$

organization according to zeta values

$$P = 1 + \sum_{n\geq 1} \zeta_2^n\ P_{2n}, \quad P_{2n} = F(\alpha')|_{\zeta_2^n}$$

$$M_{2n+1} = F(\alpha')|_{\zeta_{2n+1}}$$

$$Q = 1 + \tfrac{1}{5}\ \zeta_{3,5}\ [M_5, M_3] + \left\{ \tfrac{3}{14}\ \zeta_5^2 + \tfrac{1}{14}\ \zeta_{3,7} \right\} [M_7, M_3]$$

$$+ \left\{ 9\ \zeta_2\ \zeta_9 + \tfrac{6}{25}\ \zeta_2^2\ \zeta_7 - \tfrac{4}{35}\ \zeta_2^3\ \zeta_5 + \tfrac{1}{5}\ \zeta_{3,3,5} \right\} [M_3, [M_5, M_3]] + \ldots$$

This form exactly appears in F. Brown's decomposition of motivic multiple zeta values !

Schlotterer, Stieberger, arXiv:1205.1516

To explicitly describe the structure of the algebra $\mathcal{Z}$
MZVs are replaced by their symbols (or motivic MZVs) $\zeta^m \in \mathcal{H}$

Goncharov, Brown:

$$\zeta^m \longrightarrow \phi(\zeta^m)$$

$$\mathcal{H} \longrightarrow \mathcal{U}$$

normalized by:

$$\phi(\zeta_n^m) = f_n, \quad n \geq 2$$

$$\mathcal{U} = \mathbf{Q}\langle f_3, f_5, \ldots \rangle \otimes_{\mathbf{Q}} \mathbf{Q}[f_2]$$

ϕ = map into non-commutative Hopf-algebra

f_{2n+1} = generators of Hopf-algebra

Map ϕ sends every motivic MZV $\xi \in \mathcal{H}$ to a non-commutative polynomial in $f_i's$

E.g.:

$$\phi\left(\zeta_3^m \zeta_5^m\right) = f_3 \sqcup\!\sqcup f_5 \equiv f_3 f_5 + f_5 f_3$$

$$\phi\left(\zeta_{3,5}^m\right) = -5 \; f_5 f_3$$

F. Brown (2012)

Motivic open superstring amplitude:

$$\phi(\mathcal{A}^m) = \left(\sum_{k=0}^{\infty} f_2^k \; P_{2k}\right) \left(1 - \sum_{k=1}^{\infty} f_{2k+1} \; M_{2k+1}\right)^{-1} A_{YM}$$

Schlotterer, Stieberger, arXiv:1205.1516

There is a natural homomorphism:

F. Brown (2013): $$\mathrm{sv}:\ w \mapsto \mathrm{sv}(w) := \sum_{uv=w} u ш \tilde{v}$$

$$\mathrm{sv}(f_a) = 2\ f_a$$

$$\mathrm{sv}(f_a f_b) = 2\ f_a ш f_b \qquad \mathrm{sv}(f_2) = 0$$

Apply sv at open superstring amplitude:

$$\mathrm{sv}(\phi(\mathcal{A}^{\mathrm{I}})) = \left(1 - 2\sum_{k=1}^{\infty} f_{2k+1}\ M_{2k+1}\right)^{-1} A_{YM} = \phi(\mathcal{A}^{\mathrm{HET}})$$

apply ϕ^{-1}: $$\mathrm{sv}:\ \begin{cases} \mathcal{H} \longrightarrow \mathcal{H}^{\mathrm{sv}} \\ \zeta^m \longrightarrow \zeta^m_{\mathrm{sv}} \end{cases}$$

$$\begin{aligned} \zeta^m_{\mathrm{sv}}(2) &= 0 \\ \zeta^m_{\mathrm{sv}}(2n+1) &= 2\ \zeta^m_{2n+1} \\ \zeta^m_{\mathrm{sv}}(3,5) &= -10\ \zeta^m_3\ \zeta^m_5 \end{aligned}$$

$\Rightarrow$ $$\mathrm{sv}(A^{\mathrm{I}}(\pi)) = A^{\mathrm{HET}}(\pi)$$ $\Rightarrow$ $$J = \mathrm{sv}\,(Z)$$

Complex vs. iterated integrals

N=5:
$$\begin{pmatrix} \displaystyle\int_{z_2,z_3\in\mathbf{C}} d^2z_2\, d^2z_3\, \frac{\prod_{i<j}^{4}|z_{ij}|^{2s_{ij}}}{z_{12}z_{23}\ \overline{z}_{12}\overline{z}_{23}\overline{z}_{34}} & \displaystyle\int_{z_2,z_3\in\mathbf{C}} d^2z_2\, d^2z_3\, \frac{\prod_{i<j}^{4}|z_{ij}|^{2s_{ij}}}{z_{13}z_{32}\ \overline{z}_{12}\overline{z}_{23}\overline{z}_{34}} \\ \displaystyle\int_{z_2,z_3\in\mathbf{C}} d^2z_2\, d^2z_3\, \frac{\prod_{i<j}^{4}|z_{ij}|^{2s_{ij}}}{z_{12}z_{23}\ \overline{z}_{13}\overline{z}_{32}\overline{z}_{24}} & \displaystyle\int_{z_2,z_3\in\mathbf{C}} d^2z_2\, d^2z_3\, \frac{\prod_{i<j}^{4}|z_{ij}|^{2s_{ij}}}{z_{13}z_{32}\ \overline{z}_{13}\overline{z}_{32}\overline{z}_{24}} \end{pmatrix}$$

$$= \mathrm{sv} \begin{pmatrix} \displaystyle\int_{0<z_2<z_3<1} dz_2\, dz_3\, \frac{\prod_{i<j}^{4}|z_{ij}|^{s_{ij}}}{z_{12}z_{23}} & \displaystyle\int_{0<z_2<z_3<1} dz_2\, dz_3\, \frac{\prod_{i<j}^{4}|z_{ij}|^{s_{ij}}}{z_{13}z_{32}} \\ \displaystyle\int_{0<z_3<z_2<1} dz_2\, dz_3\, \frac{\prod_{i<j}^{4}|z_{ij}|^{s_{ij}}}{z_{12}z_{23}} & \displaystyle\int_{0<z_3<z_2<1} dz_2\, dz_3\, \frac{\prod_{i<j}^{4}|z_{ij}|^{s_{ij}}}{z_{13}z_{32}} \end{pmatrix}$$

Single-valued MZVs

$$\zeta_{\rm sv}(n_1,\ldots,n_r)\in\mathbf{R}$$

- special class of MZVs, which occurs as the values **at unity** of SVMPs

$$polylogarithms: \quad \ln(z),\ Li_1(z)=-ln(1-z), Li_a(z), Li_{a_1,\ldots,a_r}(1,\ldots,1,z)$$

SVMPs: **multiple polylogarithms** can be combined
with their complex conjugates
to **remove monodromy** at $z=0,1,\infty$
rendering the function **single-valued** on $\mathbf{P}^1\backslash\{0,1,\infty\}$.

$$\mathcal{L}_2(z)=D(z)=Im\left\{Li_2(z)+\ln|z|\ \ln(1-z)\right\}$$

(Bloch-Wigner dilogarithm)

$$\mathcal{L}_n(z)=Re_n\left\{\sum_{k=1}^{n}\frac{(-\ln(|z|)^{n-k}}{(n-k)!}\ Li_k(z)+\frac{\ln^n|z|}{(2n)!}\right\}\ \text{with:}\quad Re_n=\begin{cases}Im, & n\text{ even}\\ Re, & n\text{ odd}\end{cases}$$

(Zagier)

$$\mathcal{L}_n(1)=Re_n\left\{Li_n(1)\right\}=\begin{cases}0, & n\text{ even}\\ \zeta_n, & n\text{ odd}\end{cases}$$

- coefficients of the Deligne associator W:

(reduced) KZ equation: $$\frac{d}{dz}\, L_{e_0,e_1}(z) = L_{e_0,e_1}(z)\ \left(\frac{e_0}{z} + \frac{e_1}{1-z}\right)$$ with generators e_0 and e_1 of the free Lie algebra g

Drinfeld associator Z (generating series of MZVs):

$$Z(e_0,e_1) := L_{e_0,e_1}(1) = \sum_{w\in\{e_0,e_1\}^\times} \zeta(w)\ w = 1 + \zeta_2\ [e_0,e_1] + \zeta_3\ (\ [e_0,[e_0,e_1]] - [e_1,[e_0,e_1]\ ) + \ldots$$

$$\zeta(e_1 e_0^{n_1-1}\ldots e_1 e_0^{n_r-1}) = \zeta_{n_1,\ldots,n_r}$$

with the symbol $w \in \{e_0,e_1\}^\times$ denoting a non-commutative word $w_1 w_2\ldots$ in the letters $w_i \in \{e_0,e_1\}$

$$\zeta(w_1)\zeta(w_2) = \zeta(w_1 ⧢ w_2) \text{ and } \zeta(e_0) = 0 = \zeta(e_1)$$

Deligne associator W (generating series of SVMZVs):

Deligne introduced associator W formally as: $$W \circ {}^\sigma Z = Z$$

with Ihara action $\circ$ providing formal multiplication rule on group-like formal power series in e_0 and e_1

$$F(e_0,e_1)\circ G(e_0,e_1) = G(e_0, F(e_0,e_1)e_1F(e_0,e_1)^{-1})\ F(e_0,e_1)$$

$$\Longrightarrow\quad W(e_0,e_1) = {}^\sigma Z(e_0, We_1W^{-1})^{-1}\ Z(e_0,e_1)$$ (definition only uses Ihara action)

$$W(e_0,e_1) = \sum_{w\in\{e_0,e_1\}^\times} \zeta_{\rm sv}(w)\ w = 1 + 2\ \zeta_3\ ([e_0,[e_0,e_1]] - [e_1,[e_0,e_1]]) + \ldots$$

F. Brown (2013)

side note: explicit representation of associators in limit $\mathrm{mod}\ (\mathrm{g}')^2$

(corresponds to a commutative realization of the Ihara bracket) $(g')^2 = [g,g]^2$

$$u = -\mathrm{ad}_{e_1},\ \ v = \mathrm{ad}_{e_0}$$

$$\mathrm{ad}_x y = [x,y]$$

$$Z(e_0,e_1) = 1 - (uv)^{-1} \left(\frac{\Gamma(1-u)\,\Gamma(1-v)}{\Gamma(1-u-v)} - 1 \right) [e_0,e_1]$$

relates to open superstring amplitude — Drummond, Ragoucy (2013)

$$W(e_0,e_1) = 1 + (uv)^{-1} \left(\frac{\Gamma(-u)\,\Gamma(-v)\,\Gamma(u+v)}{\Gamma(u)\,\Gamma(v)\,\Gamma(-u-v)} + 1 \right) [e_0,e_1]$$

relates to closed superstring amplitude — St.St. (2013)

$W \circ {}^{\sigma} Z = Z$ — KLT relation for associators

Gravitational amplitudes in superstring theory

Type II, type I graviton N-point amplitude:

$$\mathcal{M} = (-1)^{N-3}\ \kappa^{N-2}\ A^t_{YM}\ S_0\ \mathrm{sv}(\mathcal{A}^{\mathrm{I}})$$

only one full-fledged superstring amplitude necessary

with intersection matrix S_0

$$\mathcal{M}_{FT}(1,\ldots,N) = (-1)^{N-3}\ \kappa^{N-2} \sum_{\sigma\in S_{N-3}} \sum_{\overline{\sigma}\in S_{N-3}} A_{YM}(\sigma)\ S_0[\sigma|\overline{\sigma}]\ A_{YM}(\overline{\sigma})$$

Consequence:

$$\mathcal{M} = (-1)^{N-3}\ \kappa^{N-2}\ A^t_{YM}\ S_0\ \mathcal{A}^{\mathrm{HET}}$$

relates gauge and gravitational amplitudes from two different string vacua !

Understand single-valued map sv at the level of string world-sheets

Scalar amplitudes in D=4:

$$\mathcal{A}^{\rm II}(\Phi_1^{i_1j_1},\Phi_2^{i_2j_2},\Phi_3^{i_3j_3},\Phi_4^{i_4j_4}) = \frac{u}{st}\left(t\ \delta_1+s\ \delta_2+\frac{st}{u}\ \delta_3\right)$$

$$\times\ \left(t\ \bar\delta_1+s\ \bar\delta_2+\frac{st}{u}\ \bar\delta_3\right)\ \frac{\Gamma(s)\ \Gamma(u)\ \Gamma(t)}{\Gamma(-s)\ \Gamma(-u)\ \Gamma(-t)}$$

scalars denote geometric moduli fields describing the internal six-dimensional toroidal g^{ij}

$$\delta_1 = g^{i_1i_2}\ g^{i_3i_4}\ ,\quad \delta_2 = g^{i_1i_3}\ g^{i_2i_4}\ ,\quad \delta_3 = g^{i_1i_4}\ g^{i_2i_3},$$
$$\bar\delta_1 = g^{j_1j_2}\ g^{j_3j_4}\ ,\quad \bar\delta_2 = g^{j_1j_3}\ g^{j_2j_4}\ ,\quad \bar\delta_3 = g^{j_1j_4}\ g^{j_2j_3}$$

$$\mathcal{A}^{\rm I}(\Phi_1^{j_1},\Phi_2^{j_2},\Phi_3^{j_3},\Phi_4^{j_4}) = \left(t\ \bar\delta_1+s\ \bar\delta_2+\frac{st}{u}\ \bar\delta_3\right)\ \frac{\Gamma(s)\ \Gamma(1+u)}{\Gamma(1+s+u)}$$

w.r.t. canonical color ordering (1,2,3,4)

scalars denote open string moduli fields (D-brane positions or Wilson lines)

$$\mathcal{A}^{\rm II}(\Phi_1^{i_1j_1},\Phi_2^{i_2j_2},\Phi_3^{i_3j_3},\Phi_4^{i_4j_4}) = -\frac{u}{t}\left(t\ \delta_1+s\ \delta_2+\frac{st}{u}\ \delta_3\right)$$

$$\times\ \ {\rm sv}\left(\mathcal{A}^{\rm I}(\Phi_1^{j_1},\Phi_2^{j_2},\Phi_3^{j_3},\Phi_4^{j_4})\right)$$

Similar relations can also be derived for amplitudes involving more than four scalar fields

- By applying **naively KLT** relations we would **not** have arrived at **these relations**
- **Much deeper connection** between open and closed string amplitudes **than** what is implied by **KLT relations**
- **Full** α'- dependence of **closed string** amplitude is **entirely** encapsulated by **open string** amplitude
- **Any** closed string amplitude can be written as single-valued image of open string amplitude
- **Various connections** between different amplitudes of different vacua can be established

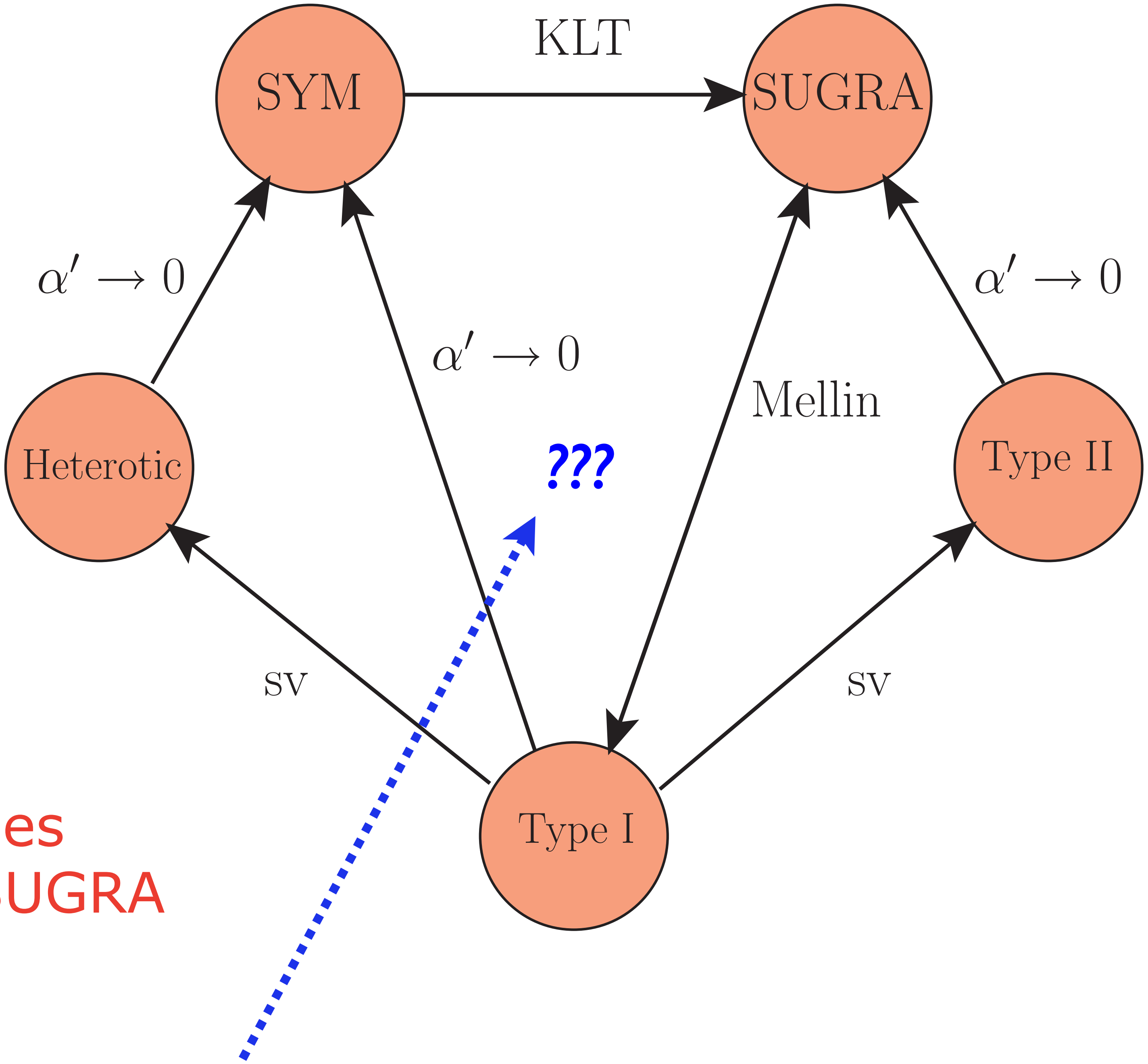
⇒ New kind of *duality* relating amplitudes involving **full tower of massive string excitations** (not just BPS states as in most examples of string dualities)

Unity of tree-level field-theory and superstring couplings

Amplitude space =
space of physical observables

*Unexpected connections
between field-theory,
open and closed string amplitudes !*

Universal α'- dependence
of all tree-level string amplitudes
and their connection to SYM and SUGRA



*Amplitudes in non-trivial background,
e.g.: warped geometries, AdS5,...*

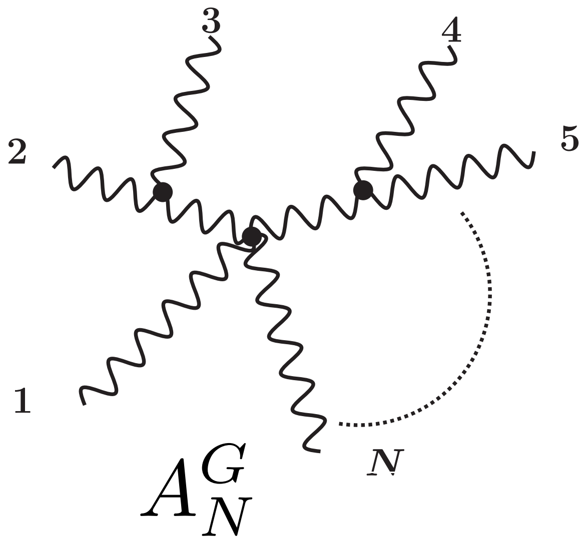
Concluding remarks

- growing set of **<u>interconnections</u>** between open & closed amplitudes with gauge theory and supergravity amplitudes

- **<u>new kind of duality</u>** working beyond usual BPS protected couplings

by combining field and string theory structures obtain information on a possible alternative or ***dual description of perturbative string amplitudes***: obtain amplitudes from ***first principles***

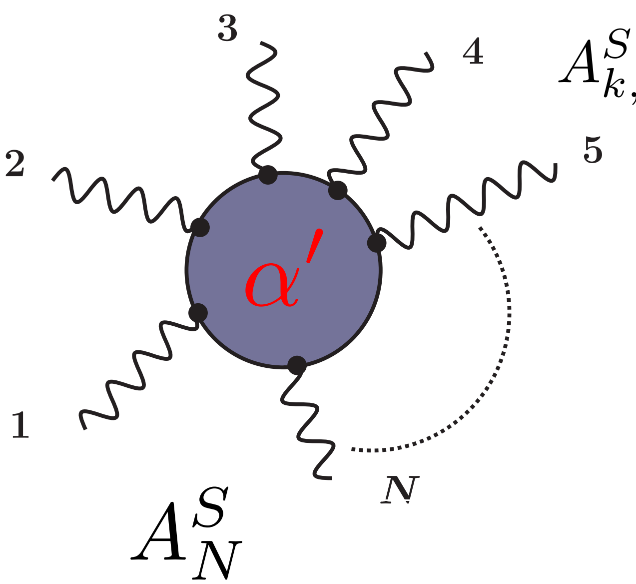
Unified description of superstring and supergravity amplitudes



$$A^G_{k,N} = \frac{1}{\mathrm{vol}(\mathrm{GL}(2))\mathrm{vol}(\widetilde{\mathrm{GL}}(2))} \int d^2\sigma_1 \cdots d^2\sigma_N \int d^2\tilde{\sigma}_1 \cdots d^2\tilde{\sigma}_N \, H_N(\sigma,\tilde{\sigma})$$

$$\times \prod_{\alpha=1}^{k} \delta^{4|4}\Big(\sum_{i=1}^{N} C^V_{\alpha i}[\sigma]\mathcal{W}_i(\eta)\Big) \prod_{\tilde{\alpha}=1}^{k} \delta^{4|4}\Big(\sum_{i=1}^{N} C^V_{\tilde{\alpha} i}[\tilde{\sigma}]\mathcal{W}_i(\tilde{\eta})\Big)$$

generalized Hodges' determinant
(KLT kernel)



$$A^S_{k,N} = \frac{1}{\mathrm{vol}(\mathrm{SL}(2))\mathrm{vol}(\widetilde{\mathrm{GL}}(2))} \int_{D\subset(\mathbb{P}^1)^N} d^2\sigma_1 \cdots d^2\sigma_N \int d^2\tilde{\sigma}_1 \cdots d^2\tilde{\sigma}_N \, H_N(\sigma,\tilde{\sigma})$$

$$\times \prod_{1\leq i<j\leq N} |(ij)|^{s_{ij}} \prod_{\tilde{\alpha}=1}^{k} \delta^{4|4}\Big(\sum_{i=1}^{N} C^V_{\tilde{\alpha} i}[\tilde{\sigma}]\mathcal{W}_i(\tilde{\eta})\Big)$$

multiple Mellin transform

- Striking match between supergravity and open superstring tree-level amplitude communicated by Hodges' determinant (KLT kernel)
- KLT kernel seems to be the key ingredient in our description

Taylor, St.St. (2013)

Generic N-point string form factor (disk integral):

$$B_N(\{s_{k,l}\},\{n_{k,l}\}) = \left(\prod_{i,j\in P} \int_0^\infty du_{i,j}\ u_{i,j}^{s_{i,j}-1+n_{i,j}}\ \theta(1-u_{i,j}) \right)\ \delta(\{u_{p,q})$$

with product of $\frac{(N-2)(N-3)}{2}$ *delta-functions:* $\delta(\{u_{p,q}\}) = \prod_P{}' \delta\left(u_P - 1 + \prod_{\widetilde{P}} u_{\widetilde{P}}\right)$

assembled by rules according to a Pascal triangle

and $\frac{1}{2}N(N-3)$ variables $u_{i,j}$: $0 \leq u_{i,j} \leq 1$ $\begin{cases} i=2,\ j=3,\ldots,N-1\ , \\ i=3,\ldots,N-1\ <\ j=4,\ldots,N \end{cases}$

- Multi-dimensional Mellin transforms from string world-sheet boundary D to the Mellin space of Mandelstam invariants
- Multiple (inverse) Mellin transforms trivialize tree-level string amplitudes
- Mellin transform from string world-sheet into dual space of kinematic invariants thus bypassing space-time, cf. quantum mechanical Fourier transform: $x^{\alpha' s} \leftrightarrow e^{ikx/\hbar}$
- Correlation functions from delta-functions and residua integrals