Scattering via Riemann spheres

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based on works with Freddy Cachazo & Ellis Yuan

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with Yong Zhang **1607.02843**

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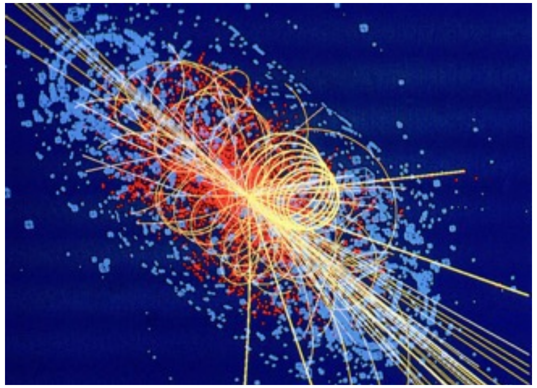
S-matrix in QFT

- Colliders at high energies need amplitudes of e.g. many gluons (tree & loop level)



$gg \rightarrow gg \ldots g$

- Fundamental level: understanding of QFT incomplete; tensions with gravity
  new structures & simplicity seen in (perturbative) scattering amplitudes

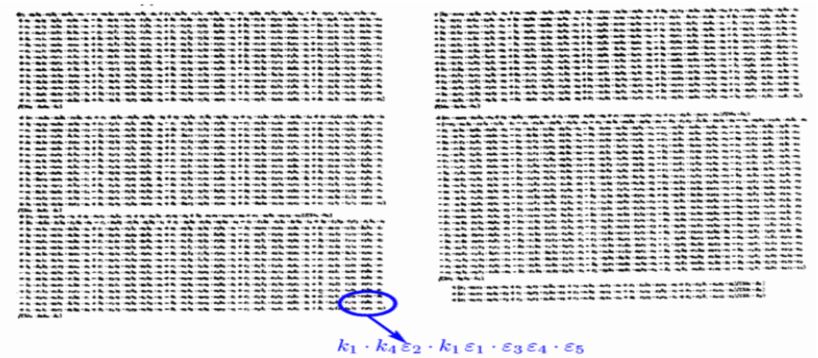
- Goal: deeper understanding of QFT & gravity from studying the S-matrix

Surprising simplicity

- theoretical challenges: many diagrams, many many terms, gauge (non-)invariance

n-gluon scattering (tree)

n	4	5	6	7	8	9	10
# diagrams	4	25	220	2485	34300	559405	10525900



- Why? redundancies in textbook formulation of QFT, but unexpected simplicity & structures emerge for on-shell S-matrix, e.g. MHV gluon amplitudes [Parke, Taylor, 86]

$$M_n(i^-, j^-) = \frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle\, \langle 2\, 3\rangle \cdots \langle n\, 1\rangle},$$

$$k^\mu = (\sigma^\mu)_{\alpha,\dot\alpha} \lambda^\alpha \tilde\lambda^{\dot\alpha}, \quad \epsilon^\mu = \dots$$

$$\langle a\, b\rangle := \varepsilon_{\alpha,\beta}\, \lambda_a^\alpha\, \lambda_b^\beta, \quad [a\, b] := \varepsilon_{\dot\alpha,\dot\beta}\, \tilde\lambda_a^{\dot\alpha}\, \tilde\lambda_b^{\dot\beta}$$

[Xu, Zhang, Chang, 84…]

- Led to 30 years of enormous progress on computing & understanding S-matrix!

Twistor-string revolution

- Witten's twistor string theory → worldsheet model for gluon tree amplitudes

  amps = string correlators with a map from $\mathbb{CP}^1$ to $\mathbb{CP}^{3|4}$(twistor space) [Witten, 2003]

- Key observation: [Nair, 88] Parke-Taylor MHV amps = correlator on $\mathbb{CP}^1$

$$\lambda_i^\alpha \sim (z_i, 1)\,, \qquad PT_n := \frac{1}{(z_1 - z_2)\,(z_2 - z_3)\cdots(z_n - z_1)}\,. \qquad j_A(z) j_B(z') = \frac{f_{AB}^C\, j_C}{z - z'} + \text{double poles} + \ldots$$

- $N^k MHV$ amplitude is the image of PT_n under polynomial map of deg. (k+1);

  polarization dependence naturally encoded by maximal supersymmetry.

- Inspired CSW & BCFW, progress on unitarity method, Grassmannian, etc. etc.

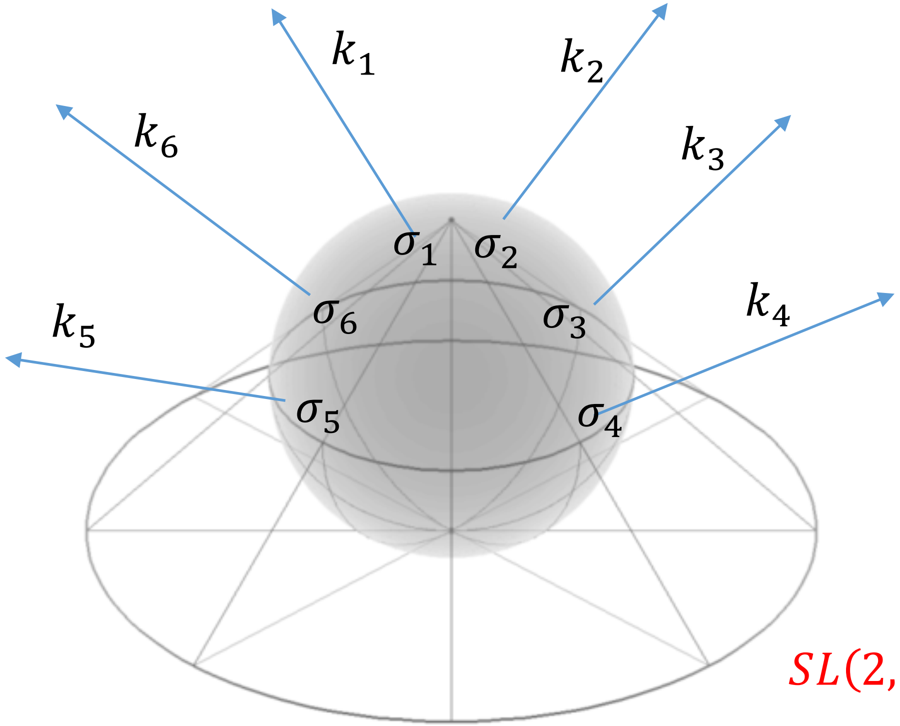
Cachazo-He-Yuan formulation

- Witten's twistor string very special: d=4 N=4 super Yang-Mills theory
  - no supersymmetry? any spacetime dimension?
  - general theories: gravity, Yang-Mills, standard model, effective field theories?
  - generalizations to loop level?
- CHY formulation: scattering of massless particles in any dimension [CHY 2013]
  - *compact formulas* for amplitudes of gluons, gravitons, fermions, scalars, etc.
  - *manifest* gauge (diff) invariance, double-copy relations, soft theorems, etc.
  - *string-theory origin*: "ambitwsitor"/ "chiral" strings [Mason, Skinner; Adamo et al; Berkovits; Siegel] [c.f. Yu-tin's talk]

Scattering equations

$$E_a := \sum_{b=1, b\neq a}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0\,, \qquad a = 1, 2, \ldots, n$$

[CHY 2013] [appeared in Farlie, Roberts, 72; Gross, Mende, 88;…]



- universal, independent of theories; determine locations of punctures in terms of kinematics
- physical singularities ↔ boundary of moduli space for n-punctured Riemann spheres
- simplest "derivation": saddle point eqs in tensionless limit of string amps [Gross, Mende]

$$E_a = \frac{\partial\left[\sum_{i<j} s_{i,j} \ln(\sigma_{i,j})\right]}{\partial \sigma_a}$$

$SL(2, \mathbb{C})$ symmetry

CHY representation of tree amps

$$M_n = \underbrace{\int \frac{d^n\sigma}{\text{vol}\,\mathrm{SL}(2,\mathbb{C})} \prod_a{}' \delta(E_a)}_{d\mu_n} \mathcal{I}(\{k,\epsilon,\sigma\}) = \sum_{\{\sigma\}\in\text{solns.}} \frac{\mathcal{I}(\{k,\epsilon,\sigma\})}{J(\{\sigma\})}$$

- S-matrix = localized integral = sum over solutions, of certain CHY integrand

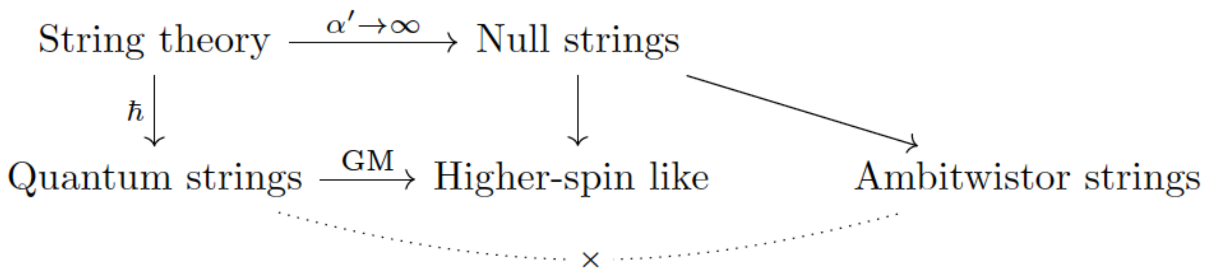
  n-3 integrals/delta functions, (n-3)! solutions [CHY; Dolan, Goddard] → (n-3)! “virtual amps”

- Key: a worldsheet picture of massless scattering via n-punctured Riemann spheres. Feynman diagrams and Lagrangians become emergent!

- Task: find “dynamic part”, i.e. CHY integrands for different massless QFT’s

Further developments [not covered in this talk…]

- Origins from worldsheet: ambitwistor strings, chiral strings & null strings



T. Adamo, N. Berkovits, E. Casali, Y. Geyer, Y.t. Huang, A. Lipstein, L. Mason, R. Monteiro, K. Ohmori, D. Skinner, W. Siegel, P. Tourkine, E. Yuan ……

- Mathematical aspects: solving scattering eqs, analytical evaluations etc.
  J. Bourjaily, C. Baadsgaard, N.E. Bjerrum-Bohr, C. Cardona, P.H. Damgaard, L. Dolan, Y. J. Du, B. Feng, P. Goddard, H. Gomez, Y-H. He, R. Huang, C. Kalousios, C.S. Lam, M.x. Luo, Y. Peng, J. Rao, M. Sogaard, F. Teng, Y.S. Wu, Y. Zhang, C.J. Zhu, M. Zlotnikov, ……

- Extensions & applications: more QFT & string amps, soft theorems, amp relations
  N. Afkhami-Jeddi, N.E. Bjerrum-Bohr, P. Cha, L. Cruz, P.H. Damgaard, A. Kniss, A. Lipstein, Z.W. Liu, S. Mizera, R. Monteiro, D. Nandan, S. Naculich, D. O'Connel, J. Plefka, K. Roehrig, O. Schlotterer, B. Schwab, S. Stieberger, T. Taylor, J.B. Wu, P. Vanhove, A. Volovich, C. Wen, S. Weinzierl, C. White, M. Zlotnikov, ……

CHY formulas: ϕ^3, YM & GR

- All tree amplitudes in bi-adjoint ϕ^3 scalar, Yang-Mills and gravity in any dim [CHY 13]

$$m[\pi|\rho] := \int d\mu_n \; PT[\pi] \; PT[\rho] \,,$$

$$\mathcal{L}_{\phi^3} = -\frac{1}{2}(\partial\phi)^2 + \frac{\lambda}{3!} f^{IJK} f^{I'J'K'} \, \phi^{II'} \phi^{JJ'} \phi^{KK'}$$

$$M_n^{\phi^3} = \sum_{\pi,\rho} \mathrm{Tr}(T^{I_{\pi(1)}} \cdots T^{I_{\pi(n)}}) \; \mathrm{Tr}(T^{I_{\rho(1)}} \cdots T^{I_{\rho(n)}}) \; m[\pi|\rho]$$

$$M_n^{\mathrm{YM}}[\pi] = \int d\mu_n \, \mathrm{PT}[\pi] \, \mathrm{Pf}' \, \Psi$$

$$M_n^{h+B+\phi} = \int d\mu_n \, \mathrm{Pf}'\Psi(\epsilon) \, \mathrm{Pf}'\Psi(\epsilon') \longrightarrow M_n^{\mathrm{GR}} = \int d\mu_n \, \det{}' \, \Psi(\epsilon)$$

- Two ingredients: Parke-Taylor factor (color) & a Pfaffian (polarization): $PT^{2-s} \times Pf^{s}$

$$\mathrm{PT}[\pi] := \frac{1}{(\sigma_{\pi(1)} - \sigma_{\pi(2)})\,(\sigma_{\pi(2)} - \sigma_{\pi(3)}) \cdots (\sigma_{\pi(n)} - \sigma_{\pi(1)})} \qquad \mathrm{Pf}'\Psi(\sigma, k, \epsilon)$$

The Pfaffian

- The (reduced) Pfaffian of a $2n \times 2n$ skew matrix Ψ, with four blocks

$$\mathrm{Pf}'\Psi := \frac{\mathrm{Pf}|\Psi|^{i,j}_{i,j}}{\sigma_{i,j}} \qquad A_{a,b} := \begin{cases} \frac{k_a \cdot k_b}{\sigma_{a,b}} & a \neq b \\ 0 & a = b \end{cases}, \quad B_{a,b} := \begin{cases} \frac{\epsilon_a \cdot \epsilon_b}{\sigma_{a,b}} & a \neq b \\ 0 & a = b \end{cases},$$

$$\Psi := \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}, \quad C_{a,b} := \begin{cases} \frac{\epsilon_a \cdot k_b}{\sigma_{a,b}} & a \neq b \\ -\sum_{c \neq a} C_{a,c} & a = b \end{cases}$$

- Simplified open superstring correlator: $\mathrm{Pf}'\Psi \sim \langle V^{(0)}(\sigma_1) \ldots V^{(-1)}(\sigma_i) \ldots V^{(-1)}(\sigma_j) \ldots V^{(0)}(\sigma_n) \rangle$

- The Pfaffian is permutation invariant, multi-linear in polarizations …
  most importantly gauge invariant on the support of scattering equations!

Gauge & diffeomorphism invariance

- Pf' (det') Ψ as the simplest gauge (diffeo.) invariant object: $\epsilon_a^\mu \sim \epsilon_a^\mu + \alpha k_a^\mu$

$$
\left(\begin{array}{cc|cc}
0 & \cdots & \sum_{b=2}^{n} \frac{k_1\cdot k_b}{\sigma_{1,b}} & \cdots \\
\frac{k_2\cdot k_1}{\sigma_{2,1}} & \cdots & \frac{k_2\cdot k_1}{\sigma_{2,1}} & \cdots \\
\vdots & & \vdots & \\
\frac{k_n\cdot k_1}{\sigma_{2,1}} & \cdots & \frac{k_n\cdot k_1}{\sigma_{2,1}} & \cdots \\
-\sum_{b=2}^{n} \frac{k_1\cdot k_b}{\sigma_{1,b}} & \cdots & 0 & \cdots \\
\frac{\epsilon_2\cdot k_1}{\sigma_{2,1}} & \cdots & \frac{\epsilon_2\cdot k_1}{\sigma_{2,1}} & \cdots \\
\vdots & & \vdots & \\
\frac{\epsilon_n\cdot k_1}{\sigma_{2,1}} & \cdots & \frac{\epsilon_n\cdot k_1}{\sigma_{2,1}} & \cdots
\end{array}\right)
$$

- It vanishes by scattering equations
- invariant for (n-3)! virtual amplitudes
- closed-string=(open-string)^2
- CHY integrand: $GR = YM^2/\phi^3$

Double-copy relations

- First such relations discovered as (FT limit of) Kawai-Lewellen-Tye relations:

$$M_n^{\text{closed}} = \sum_{\alpha,\beta} M_n^{\text{open}}[\alpha]\; \mathcal{S}^{\text{string}}[\alpha|\beta]\; M_n^{\text{open}}[\beta] \quad \Longrightarrow \quad M_n^{\text{GR}} = \sum_{\alpha,\beta} M_n^{\text{YM}}[\alpha]\; S[\alpha|\beta]\; M_n^{\text{YM}}[\beta]\,.$$

- How about $GR = YM^2/\phi^3$? KLT derived from inserting two PT's in CHY: $\boldsymbol{S} = \boldsymbol{m}^{-1}$

$$M_n = \int d\mu_n\, \mathcal{I}_L\, \mathcal{I}_R \quad \Longrightarrow \quad M_n = \sum_{\alpha,\beta} M_L[\alpha]\; m^{-1}[\alpha|\beta]\; M_R[\beta], \quad \text{for } M_{L(R)} := \int d\mu_n\, \text{PT}\; \mathcal{I}_{L(R)}\,.$$

- A general way of seeing double-copy relations: splitting a CHY formula into two.

More theories

- Generate CHY formulas of new theories from old ones, e.g. dim reduction

  GR → Einstein-Maxwell (EM), YM → YM-scalar (YMs), with Pfaffian factorizes:

$$M_{n\,\gamma}^{\rm EM} = \int d\mu_n\, {\rm Pf}'A\ {\rm Pf}X\ {\rm Pf}'\Psi\,, \quad M_{n\,s}^{\rm YMs} = \int d\mu_n\, {\rm Pf}'A\ {\rm Pf}X\ {\rm PT}\,; \quad X_{a\,b} = \frac{\delta^{I_a\,I_b}}{\sigma_{a,b}}(1-\delta_{a,b})\,.$$

- A new operation to add non-abelian interactions leads to "direct sum" of theories
  Formulas in Einstein ⊕ Yang-Mills and YM ⊕ bi-adjoint scalar theories [CHY 14]

- A new class: "exceptional" effective field theories (EEFT) with Goldstone scalars

  very special theories: amplitudes have enhanced "Adler's zero"! [Cheung et al 14] [CHY 14]

More theories

- $M_n = \int d\mu_n \; (\mathrm{Pf}'A)^2 \; \mathrm{PT}\,,$ adjoint scalars with two derivative coupling?

  U(N) NLSM (the chiral Lagrangian) $\mathcal{L} = \mathrm{Tr}(\partial_\mu U^+ \partial^\mu U)$

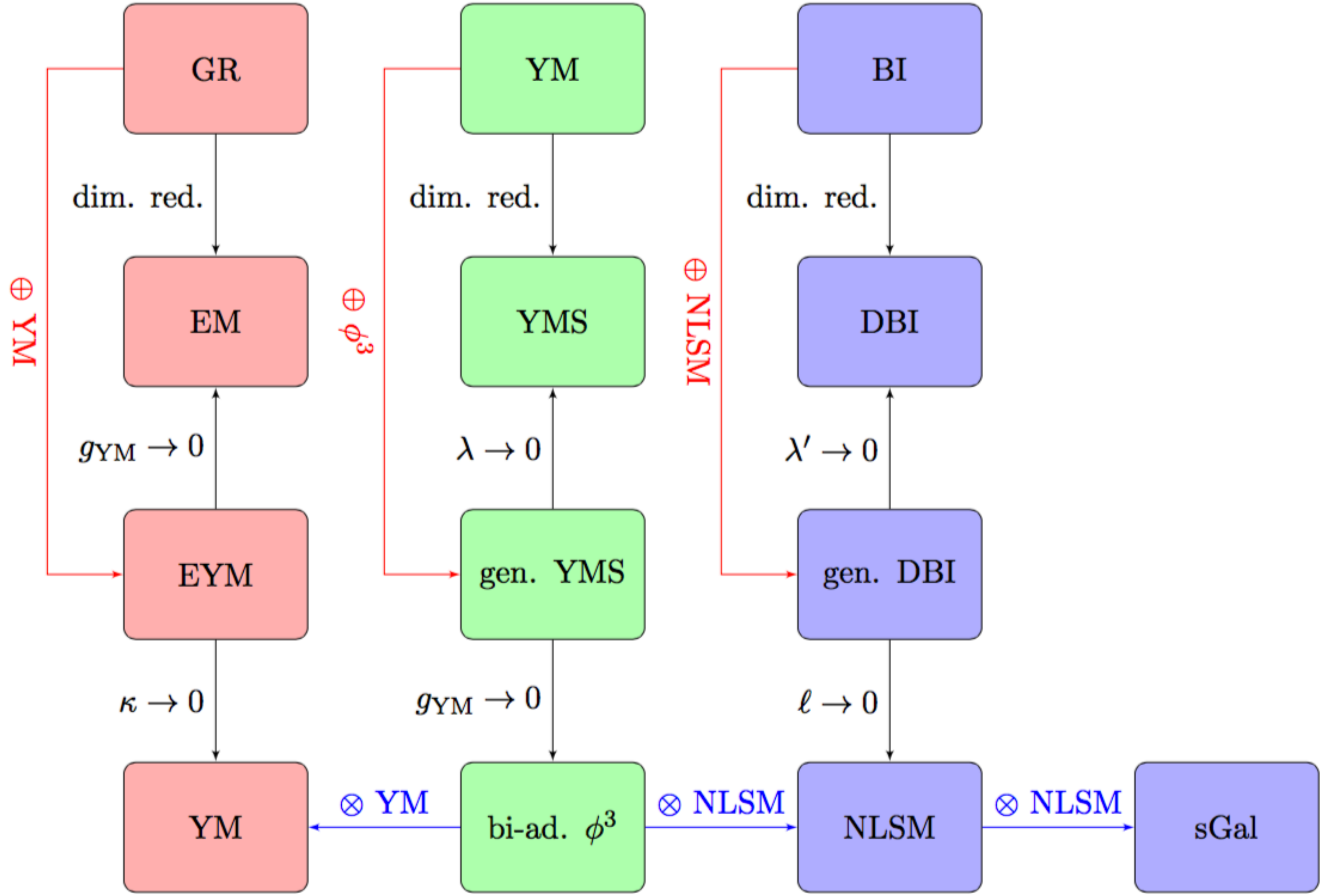
- $M_n = \int d\mu_n \; (\mathrm{Pf}'A)^2 \; \mathrm{Pf}'\Psi\,,$ higher-derivative-coupled photons?

  Born-Infeld theory (BI) & DBI by dim reduction $\mathcal{L} = \sqrt{-\det(\eta_{\mu\nu} - \ell F_{\mu\nu} - \ell^2 \partial_\mu \phi \partial_\nu \phi)}$

- a special Galileon (single scalar with many derivatives) $M_n^{\mathrm{sGal}} = \int d\mu_n \; (\mathrm{Pf}'A)^4$

- double-copy relations: $BI \sim YM \otimes NLSM\,,\; DBI \sim YMs \otimes NLSM\,,\; sGal \sim NLSM^2$

A landscape of massless theories



Soft theorems in CHY

- CHY makes manifest old & new soft theorems; connections to BMS etc. [Strominger,...]

$$M_n^{\text{gauge}} = (\mathcal{S}^{(0)}+\mathcal{S}^{(1)})\, M_{n-1}^{\text{gauge}} + O(\tau)\,, \quad M_n^{\text{GR}} = (\mathcal{S}^{(0)}+\mathcal{S}^{(1)}+\mathcal{S}^{(2)}) M_{n-1}^{\text{GR}} + O(\tau^2)\,,$$

- EEFT's are the only scalar EFT's with vanishing soft behavior as O(τ^p) for p=1,2,3
  Non-linearly realized symmetry: coset from double-soft-scalar theorems [CHY 15]

$$M_n^{\text{NLSM}} = (S^{(0)} + S^{(1)})\, M_{n-2}^{\text{NLSM}} + O(\tau^2)\,, \quad M_n^{\text{DBI}} = (S^{(0)} + S^{(1)} + S^{(2)}) M_{n-2}^{\text{DBI}} + O(\tau^4)\,,$$

(also for sGal). Striking similarities with gauge/gravity soft theorems. Why?

- 4d: manifest double soft theorems in N=4 SYM, N=8 SUGRA & DBI-Volkov-Akulov

  e.g. double fermions: non-linearly realized SUSY [Huang et al 14] $\leftrightarrow$ 16+16 [Bergshoeff et al 14]

Loops from trees

- Feynman's loop-tree theorem → loop amps from (generally divergent) forward limits of trees; need off-shell momenta ($l^2 \neq 0$) & regularizations; e.g.

$$M_n^{1-\text{loop}} \sim \int \frac{d^D\ell}{\ell^2} \sum_{I_+=I_-,\epsilon_+=(\epsilon_-)^*} M_{n+2}^{\text{tree}}(\,\{(k_i;0)\},\ \pm(\ell,|\ell|)\,)\,,$$

- Both resolved in CHY: loop-level eqs & formulas on a sphere [Geyer et al 15][HY, CHY 15]

$$M_n^{(1)} = \int d^D\ell \frac{1}{\ell^2} \int d\mu_n^{(1)}\, \mathcal{I}_n(\{\sigma,k,\epsilon\};\ell)\,, \qquad \mathcal{E}_a = \sum_{b\neq a} \frac{k_a\cdot k_b}{\sigma_a-\sigma_b} + \frac{k_a\cdot\ell}{\sigma_a}\,, \quad \text{for } a=1,\ldots,n\,.$$

- Agree with ambitwistor results [Geyer et al 15]; two-loop progress [Geyer et al 16; Feng 16].
- Seems to give wrong propagators $1/(l+P)^2 - l^2$, but equivalent to standard rep (difference integrates to zero) → a new rep of loop integrands [c.f. Baadsgaard et al 15]

Loops from trees

- Suming over colors/polarizations gives 1-loop "PT" & "Pfaffians" [Geyer et al 15][HY, CHY 15]

$$\mathrm{PT}_n^{(1)}[1,2,\ldots,n] := \sum_{i=1}^{n} \mathrm{PT}_{n+2}[1,\ldots,i,+,-,i{+}1,\ldots,n]\,.$$

$$\mathrm{Pf}_{\mathbf{s}}^{(1)} = \frac{1}{\sigma_{+,-}^2}\mathrm{Pf}\Psi_n(\ell), \quad \mathrm{Pf}_{\mathbf{g}}^{(1)} = \sum_{\epsilon_+=(\epsilon_-)^*} \mathrm{Pf}'\Psi_{n+2}(\ell), \quad \mathrm{Pf}_{\mathbf{f}}^{(1)} = \ldots$$

- One-loop formula for ϕ^3, YM & GR, also susy-theories by adding fermions [Geyer et al 15]

$$\mathcal{I}_n^{\phi^3} = (\mathrm{PT}_n^{(1)})^2, \quad \mathcal{I}_n^{\mathrm{YM}} = \mathrm{PT}_n^{(1)}\,\mathrm{Pf}_{\mathbf{g}}^{(1)}, \quad \mathcal{I}_n^{\mathrm{GR}} = (\mathrm{Pf}_{\mathbf{g}}^{(1)})^2 - c_d(\mathrm{Pf}_{\mathbf{f}}^{(1)})^2\,,$$

$$\mathcal{I}_n^{\mathrm{SYM}} = \mathrm{PT}_n^{(1)}\,(\mathrm{Pf}_{\mathbf{g}}^{(1)} - c_d\mathrm{Pf}_{\mathbf{f}}^{(1)})\,, \quad \mathcal{I}_n^{\mathrm{SUGRA}} = (\mathrm{Pf}_{\mathbf{g}}^{(1)} - c_d\mathrm{Pf}_{\mathbf{f}}^{(1)})^2\,.$$

- Gauge invariance, soft theorems, unitarity cuts, SUSY etc. manifest, but how to systematically understand loop amplitudes/integrals in this new rep?

Back to four dimensions

- CHY rep. simplifies in 4d → old & new "connected" formulas [RSV 04; Cachazo, Skinner 12; SH et al 16]
- Key: scattering eqs split into RSV-Witten eqs for sectors, k=2,3…,n-2 [CHY 13; SH et al 16]

  A prior nothing to do with helicities, any amp decomposes into n-3 sectors:

$$M_n = \sum_{k=2}^{n-2} \sum_{\text{soln. } k} \frac{I_n^{\text{CHY}}}{J_n^{\text{CHY}}} := \sum_{k=2}^{n-2} M_{n,k}\,, \quad \Rightarrow \quad M_{n,k} = \int d\mu_{n,k}^{\text{4d}}\; I_{n,k}^{\text{4d}}\,.$$

- YM & GR: for k neg. hel., Pf'Ψ=0 for any soln. sector $k' \neq k$; Pf' A=0 for $k' \neq \frac{n}{2}$

  only right sector needed; 4d formulas simplify a lot & natural to have SUSY!

  e.g. 4d integrands in SYM, DBIVA & SUGRA (w. susy-measures): [Geyer et al 14; SH et al 16]

$$\mathcal{I}_{n,k}^{\text{SYM,4d}} = \text{PT}_n\,, \quad \mathcal{I}_{n,\frac{n}{2}}^{\text{DBIVA,4d}} = \det{}' A_n\,, \quad \mathcal{I}_{n,k}^{\text{SUGRA,4d}} = \det{}' H_k \det{}' \tilde{H}_{n-k}\,.$$

Fermions, Higgs & off-shell quantities

- Easy to include fermions in 4d, e.g. gluon-quark amplitudes in massless QCD

$$M_{n,k}(g;q\bar{q}) = \int d\mu_{n,k}\,\mathrm{PT}_n\,\mathcal{J}_{\rm ferm}\,,$$

Jac. depends on helicities & flavors of quarks only!

follow from gluon-gluino ones in SYM → new rep for all QCD tree amplitudes

- Higgs mechanism in CHY? very different from massive CHY from KK reduction

1st step: Higgs + n gluons as (n+2) on-shell legs $p_\phi = \lambda\tilde{\lambda} + \mu\tilde{\mu}$

$$M_{n+1,k}(\phi;n_g) = m_H^4 \int d\mu'_{n+2,k}\,\mathrm{PT}_n\,, \quad \sigma_\lambda,\sigma_\mu \text{ fixed; } \lambda,\mu \text{ eqs removed.}$$

- Opens up CHY for SM amps and off-shell quantities (form factors, correlators…)

Summary & outlook

- New picture: gluons (massless particles) scattering via punctures on a sphere.
  Suggest a weak-weak duality of QFT and string theory for S-matrix?
- Complimentary to FD's: (n-3)! virtual amplitudes with all symmetries manifest!
- Web of theories connected by operations e.g. $\oplus$ (interaction) & $\otimes$ (double-copy)
- Loops: higher-genus vs. higher-punctures; integrands vs. integrated amps
- Massive theories, off-shell quantities etc. Scope of QFT's with natural CHY formula?
- S-matrix as representation theory, of (Poincare $\subset$ BMS $\subset$) some group?

Thank You!