Strings 2021
ICTP-SAIFR, Sao Paulo
Review Talk

Symmetries and Their Generalizations in Topological Phases of Matter

Shu-Heng Shao
Institute for Advanced Study

Global symmetry

- Global symmetry is one of the few universally applicable tools in analyzing strongly coupled quantum systems.
- Global symmetry can have ‘t Hooft anomalies --- obstructions to gauging it.
- In recent years, the notion of global symmetry has been generalized in different directions.
- This has led to new constraints on renormalization group flows, new organizing principles of topological phases, and new dualities.
- Applications in high energy physics, condensed matter physics, and mathematics.

Three generalizations

- I will discuss three generalizations of ordinary global symmetries:

1. Higher-form symmetries
2. Subsystem symmetries
3. Non-invertible topological operators

- Disclaimer: This is an enormous subject and it's impossible to cover every topic in one talk. I apologize in advance for the variety of fascinating papers that are not discussed below.
- Many other generalizations of global symmetries not discussed here, e.g. dipole symmetry [Pretko 2018, Seiberg 2019, Son's talk…], asymptotic symmetry [Strominger's book 2017, Pasterski's review talk, Strominger and Taylor's discussion].
- This talk is mostly about internal global symmetries in bosonic systems. For an example of a gravitational anomaly, see [Tachikawa's talk]. For anomalies in fermionic theories, see [Gomis' and Putrov's talks].
- This talk will be structured from a field theory/high energy physics perspective. See [Wen's discussion] for a CMT perspective on topological phases.

Global vs. gauge symmetry

- Global symmetry acts nontrivially on operators.
- It is an intrinsic property of the quantum system.
- It's therefore important to characterize global symmetries abstractly and invariantly, without referring to any Lagrangian description.

- Gauge "symmetry" leaves all operators invariant. It's a redundancy.
- It's ambiguous --- there can be a gauge symmetry in one duality frame but not in another. (E.g. 2+1d $U(1)$ gauge theory is dual to a free compact scalar field theory.)

The adjective "global" doesn't mean that it necessarily acts globally on the whole space.

Ordinary global symmetry

An internal ordinary global symmetry $g \in G$ in d spacetime dimensions can be characterized by its symmetry operator $U_g\left(M^{(d-1)}\right)$.

Some general properties:

- It is supported on a codimension-**1** manifold $M^{(d-1)}$ in spacetime. For example, it can be supported over the whole space at a fixed time.
- It is topological under deformation of $M^{(d-1)}$. In particular, it is conserved under time evolution.
- The fusion between these operators obeys the group multiplication law
$$U_{g_1}\left(M^{(d-1)}\right)U_{g_2}\left(M^{(d-1)}\right) = U_{g_1 g_2}\left(M^{(d-1)}\right)$$

Ordinary global symmetry

Properties of symmetry op.	Ordinary symmetry $U_g\left(M^{(d-1)}\right)$	**Example**: $U(1)$ $\exp(i\theta \oint_{M^{(d-1)}} j^{(d-1)})$
Codimension in spacetime	1	$j^{(d-1)}$ is a $d-1$-form
Topological	yes	$j^{(d-1)}$ is closed, $dj^{(d-1)}=0$
Fusion rule	group $U_{g_1}U_{g_2}=U_{g_1 g_2}$	$U(1)$ $U_{\theta_1}U_{\theta_2}=U_{\theta_1+\theta_2}$

Next, we generalize the ordinary global symmetry by modifying the above conditions.

Global symmetries and generalizations

Properties of symmetry op.	Ordinary symmetry	Higher-form symmetry	Subsystem symmetry	Non-invertible operator
Codimension in spacetime	1	$q+1$	$q+1$	$q+1$
Topological	yes	yes	not completely but conserved in time	yes
Fusion rule	group $g_1 \times g_2 = g_3$	group $g_1 \times g_2 = g_3$	group $g_1 \times g_2 = g_3$	fusion ring $a \times b = \sum_c N_{ab}^c c$

Global symmetries and generalizations

Properties of symmetry op.	Ordinary symmetry	Higher-form symmetry	Subsystem symmetry	Non-invertible operator
Codimension in spacetime	1	$q+1$	$q+1$	$q+1$
Topological	yes	yes	not completely but conserved in time	yes
Fusion rule	group $g_1 \times g_2 = g_3$	group $g_1 \times g_2 = g_3$	group $g_1 \times g_2 = g_3$	fusion ring $a \times b = \sum_c N_{ab}^c c$

Global symmetries and generalizations

Properties of symmetry op.	Ordinary symmetry	Higher-form symmetry	Subsystem symmetry	Non-invertible operator
Codimension in spacetime	1	$q+1$	$q+1$	$q+1$
Topological	yes	yes	not completely but conserved in time	yes
Fusion rule	group $g_1 \times g_2 = g_3$	group $g_1 \times g_2 = g_3$	group $g_1 \times g_2 = g_3$	fusion ring $a \times b = \sum_c N_{ab}^c c$

Global symmetries and generalizations

Properties of symmetry op.	Ordinary symmetry	Higher-form symmetry	Subsystem symmetry	Non-invertible operator
Codimension in spacetime	1	$q+1$	$q+1$	$q+1$
Topological	yes	yes	not completely but conserved in time	yes
Fusion rule	group $g_1 \times g_2 = g_3$	group $g_1 \times g_2 = g_3$	group $g_1 \times g_2 = g_3$	fusion ring $a \times b = \sum_c N_{ab}^c c$

Global symmetries and generalizations

Properties of symmetry op.	Ordinary symmetry	Higher-form symmetry	Subsystem symmetry	Non-invertible operator
Codimension in spacetime	1	$q+1$	$q+1$	$q+1$
Topological	yes	yes	not completely but conserved in time	yes
Fusion rule	group $g_1 \times g_2 = g_3$	group $g_1 \times g_2 = g_3$	group $g_1 \times g_2 = g_3$	fusion ring $a \times b = \sum_c N_{ab}^c c$

Possibly more generalizations by combining different columns

Higher-Form Symmetry

Global symmetries and generalizations

Properties of symmetry op.	Ordinary symmetry	Higher-form symmetry	Subsystem symmetry	Non-invertible operator
Codimension in spacetime	1	$q+1$	$q+1$	$q+1$
Topological	yes	yes	not completely but conserved in time	yes
Fusion rule	group $g_1 \times g_2 = g_3$	group $g_1 \times g_2 = g_3$	group $g_1 \times g_2 = g_3$	fusion ring $a \times b = \sum_c N_{ab}^c c$

Higher-form global symmetry

[Gaiotto-Kapustin-Seiberg-Willett 2014,…]

Properties of symmetry op.	q-form symmetry $U_g\left(M^{(d-q-1)}\right)$	**Example**: $U(1)$ $\exp\left(i\theta \oint_{M^{(d-q-1)}} j^{(d-q-1)}\right)$
Codimension in spacetime	$q+1$	$j^{(d-q-1)}$ is a $d-q-1$-form
Topological	yes	$j^{(d-q-1)}$ is closed, $dj^{(d-q-1)}=0$
Fusion rule	group $U_{g_1}U_{g_2}=U_{g_1g_2}$	$U(1)$ $U_{\theta_1}U_{\theta_2}=U_{\theta_1+\theta_2}$

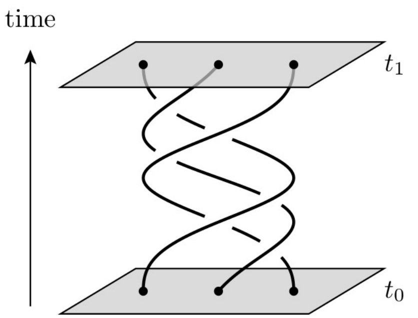
The charged objects are q-dimensional.

Higher-form symmetries and anomalies

[Gaiotto-Kapustin-Seiberg-Willett 2014,…]

- Higher-form global symmetries can also have ‘t Hooft anomalies or mixed anomalies with ordinary global symmetries.
- Such anomalies have to be matched along the RG flow. Nontrivial anomalies imply that the low energy phase can **NOT** be trivially gapped with a non-degenerate ground state.
- Example: 3+1d $SU(2)$ pure gauge theory at $\theta = \pi$ has a mixed anomaly between CP and the $\mathbb{Z}_2$ one-form center symmetry. The low energy phase cannot be trivially gapped. (Contrast with the expectation at $\theta = 0$.) [Gaiotto-Kapustin-Komargodski-Seiberg 2017]

Higher-form symmetries in TQFT



- Topological Quantum Field Theory (TQFT) (in 2+1d or above) is generally characterized by the extended topological operators, a subset of which generates the higher-form global symmetries.
- The Wilson lines of a 2+1d abelian Chern-Simons theory generate the 1-form global symmetry. They arise from the anyons in the microscopic lattice model.
- The braiding between anyons are interpreted as the 't Hooft anomaly of the 1-form global symmetry [Gaiotto-Kapustin-Seiberg-Willett 2014, Gomis-Komargodski-Seiberg 2016, Hsin-Lam-Seiberg 2018][see also Kapustin-Thorngren 2013].
- Each gapped boundary of the abelian CS theory is characterized by a Lagrangian, non-anomalous 1-form symmetry subgroup [Kapustin-Saulina 2011,…][See Komargodski's talk].

1-form symmetries in 2+1d TQFT

- Example: 2+1d $\mathbb{Z}_2$ gauge theory described by two 1-form gauge fields $a, \hat{a}$ [Maldacena-Moore-Seiberg 2001, Banks-Seiberg 2010, Kapustin-Seiberg 2014]:

$$\mathcal{L} = \frac{2}{2\pi} a d\hat{a}$$

This is the low energy continuum field theory of the toric code [Kitaev 1997].

- The Wilson lines $U = \exp\left(i \oint_L a\right), \quad \widehat{U} = \exp\left(i \oint_L \hat{a}\right)$ generate a $\mathbb{Z}_2 \times \mathbb{Z}_2$ 1-form global symmetry. They depend only on the topology of the line L.
- The nontrivial braiding between $U, \widehat{U}$ implies the mixed 't Hooft anomaly of this $\mathbb{Z}_2 \times \mathbb{Z}_2$ 1-form global symmetry.

1-form symmetries in 2+1d TQFT

- Let the space be a two-torus with A-cycle L_A and B-cycle L_B. The anomaly implies (similarly A, B exchanged)

$$U(L_A)\widehat{U}(L_B) = -\widehat{U}(L_B)U(L_A)$$

which leads to a **4**-fold ground state degeneracy on the torus.

- This ground state degeneracy is robust because there is no local operator perturbation that can lift it.
- It's a topological phase characterized by its 1-form global symmetry and anomaly.
- See [Nussinov-Ortiz 2007, …, Wen 2018, Wen's discussion] for parallel discussions from the lattice perspective.

Other topics

- Higher-group symmetry: mixture of higher-form symmetries of different degrees [Kapustin-Thorngren 2013, Cordova-Dumitrescu-Intriligator 2018-2020, Benini-Cordova-Hsin 2018,…].

- Higher-form symmetries in supersymmetric field theories from string/M/F-theory [Schafer-Nameki's talk].

- Spontaneous breaking of higher-form symmetries [McGreevy's talk].

Subsystem Symmetry

Global symmetries and generalizations

Properties of symmetry op.	Ordinary symmetry	Higher-form symmetry	Subsystem symmetry	Non-invertible operator
Codimension in spacetime	1	$q+1$	$q+1$	$q+1$
Topological	yes	yes	not completely but conserved in time	yes
Fusion rule	group $g_1 \times g_2 = g_3$	group $g_1 \times g_2 = g_3$	group $g_1 \times g_2 = g_3$	fusion ring $a \times b = \sum_c N_{ab}^c c$

Subsystem global symmetry

[..., Lawler-Fradkin 2004,...]

- The symmetry operator of a subsystem global symmetry can be supported on certain higher-codimensional manifolds L in space (E.g. straight lines on a plane).
- Unlike the higher-form symmetry, the subsystem symmetry operator depends **NOT** only on the topology of the manifold L.
- It is conserved in time. The system is not Lorentz invariant.
- It acts on the Hilbert space --- it's a global symmetry rather than a gauge symmetry. It's also not "in-between global and gauge."

Higher-form symmetry	Subsystem symmetry
$U_g(L) = U_g(L')$ if $L \sim L'$	$U_g(L) \neq U_g(L')$ for some $L \sim L'$ $\partial_t U_g(L) = 0$

See [Seiberg 2019, Qi-Radzihovsky-Hermele 2020] for related discussions in a different context.

UV/IR mixing

- There are many interesting lattice models exhibiting subsystem global symmetries. The number of subsystem symmetry operators generally depends on the number of lattice points. This leads to dramatic consequences on the low energy description.
- Observables vary at the lattice scale a, and hence they are discontinuous in the continuum limit --- UV/IR mixing [Seiberg-SHS 2020, Seiberg's discussion].
- Reminiscent of the UV/IR mixing in the little string theory [Seiberg 1997] and field theory on a non-commutative space [Minwalla-Van Raamsdonk-Seiberg 1999].

$U(1)$ subsystem symmetry

- Consider a 2+1-dimensional field theory based on a free compact scalar $\phi \sim \phi + 2\pi$ [Paramekanti-Balents-Fisher 2002,…]:

$$S = \int dtdxdy \left(\frac{\mu_0}{2}(\partial_t\phi)^2 - \frac{1}{2\mu}\left(\partial_x\partial_y\phi\right)^2\right)$$

- It has a $U(1)$ subsystem global symmetry:

$$\phi(x,y,t) \to \phi(x,y,t) + \alpha_x(x) + \alpha_y(y)$$

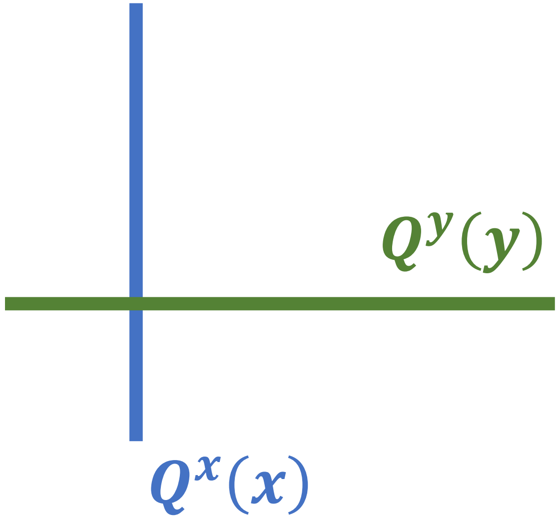
This a global symmetry, rather than a gauge symmetry.

- Noether currents: $j_t = \mu_0\partial_t\phi,\ j_{xy} = -\frac{1}{\mu}\partial_x\partial_y\phi$

$$\partial_t j_t = \partial_x\partial_y j_{xy}$$

$U(1)$ subsystem symmetry

$$\partial_t j_t = \partial_x \partial_y j_{xy}$$



- Conserved charge

$$Q^x(x) = \oint dy\, j_t$$

$$\partial_t Q^x(x) = \oint dy \partial_t j_t = \oint dy \partial_y (\partial_x j_{xy}) = 0$$

- Independent conserved charge $Q^x(x)$ at every point in x (similarly in y).
- Infinitely many such conserved charges in the continuum.
- UV/IR mixing in various observables such as correlation functions [Seiberg-SHS 2020, Gorantla-Lam-Seiberg-SHS to appear].

Fractons and subsystem symmetry

- Fracton [Chamon 2005, Haah 2011,…] is a large class of lattice spin models with many peculiar features:

1. Large ground state degeneracy that typically grows exponentially in the linear size of the system.
2. The ground state degeneracy is robust: small deformations by local operators cannot lift the degeneracy in perturbation theory.
3. Excitations have restricted mobility.

- The key common feature of these models is the exact or emergent subsystem global symmetry.
- Novel topological phases that do not admit a conventional continuum field theory description.

Fractons and subsystem symmetry

- One simple 3+1d gapped fracton model: X-cube model [Vijay-Haah-Fu 2016].
- Ground state degeneracy on a torus with periodic boundary conditions:

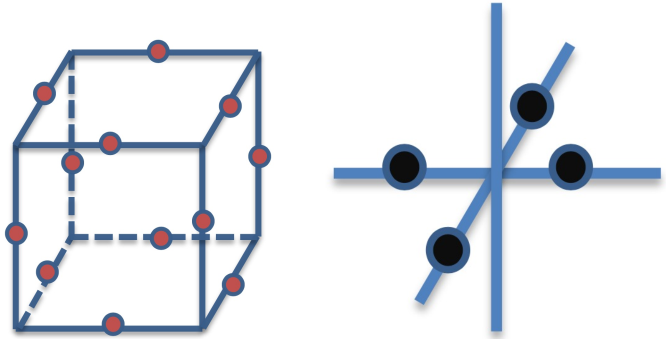
$$2^{2L^x+2L^y+2L^z-3}$$

where L^i is the number of lattice sites in the i-direction. It becomes infinite in the continuum limit, reflecting UV/IR mixing.

- The continuum field theory for the X-cube model takes the form of a BF-type action involving two kinds of nonrelativistic tensor gauge fields $A, \hat{A}$ (indices suppressed) [Slagle-Kim 2017, Seiberg-SHS 2020]

$$\mathcal{L} = \frac{2}{2\pi}(A_0\hat{B} + A\hat{E})$$

where $\hat{B}, \hat{E}$ are the gauge invariant field strengths for $\hat{A}$.



Fractons and subsystem symmetry

- Subsystem symmetry operators of the X-cube model (logical operators):

$$U \sim \exp\left(i \oint_{strip} A\right) \, , \hat{U} \sim \exp\left(i \oint_{line} \hat{A}\right)$$

independent operator for each strip and line along the x, y, z directions (with certain relations among them).

- The subsystem symmetry operators form $2L^x + 2L^y + 2L^z - 3$ pairs of the clock and shift algebra $U\hat{U} = -\hat{U}U \, , \; U^2 = \hat{U}^2 = 1$. This leads to $2^{2L^x+2L^y+2L^z-3}$ ground states.
- This nontrivial algebra can be viewed as an anomaly of the subsystem symmetry [Seiberg-SHS 2020, Burnell-Devakul-Gorantla-Lam-SHS to appear].
- In other more exotic gapped fracton models such as the Haah code [Haah 2011], the subsystem symmetry operators are supported on fractal geometric objects (rather than strips and lines) on the lattice.

Generalized global symmetries in topological phases of matter

Lattice model	(2+1)d toric code	(3+1)d X-cube model
Torus ground states	2^2	$2^{2L^x+2L^y+2L^z-3}$
Excitations	Anyons	Fractons, …
Low energy QFT	$\mathbb{Z}_2$ gauge theory $\frac{2}{2\pi}\, ad\hat{a}$	$\mathbb{Z}_2$ tensor gauge theory $\frac{2}{2\pi}(A_0\hat{B}+A\hat{E})$
Global symmetry	1-form symmetry	Subsystem symmetry

There are hybrid models that mix the higher-form symmetry and the subsystem symmetry [Tantivasadakarn-Ji-Vijay 2021, Hsin-Slagle 2021].

Non-invertible Topological Operators

Global symmetries and generalizations

Properties of symmetry op.	Ordinary symmetry	Higher-form symmetry	Subsystem symmetry	Non-invertible operator
Codimension in spacetime	1	$q+1$	$q+1$	$q+1$
Topological	yes	yes	not completely but conserved in time	yes
Fusion rule	group $g_1 \times g_2 = g_3$	group $g_1 \times g_2 = g_3$	group $g_1 \times g_2 = g_3$	fusion ring $a \times b = \sum_c N_{ab}^c c$

Gauging in 1+1d

- If we gauge a non-anomalous $\mathbb{Z}_N^{(0)}$ 0-form global symmetry of a 1+1d bosonic theory T, the original $\mathbb{Z}_N^{(0)}$ symmetry is gauged and disappears, but we gain a new symmetry $\mathbb{Z}_N^{(0)}$ in the orbifold theory T' [Vafa 1986].
- We can then gauge $\mathbb{Z}_N^{(0)}$ in T' to retrieve T:

$$\text{1+1d:}\quad \mathbb{Z}_N^{(0)} \underset{gauging}{\longleftrightarrow} \mathbb{Z}_N^{(0)}$$

- T and T' are two different theories. They might happen to be isomorphic to each other (e.g. Ising CFT), but $\mathbb{Z}_N^{(0)}$ and $\mathbb{Z}_N^{(0)}$ do not coexist at the same time in a well-defined 1+1d theory.

Generalized symmetries from gauging

Higher-form symmetries and non-invertible topological operators can arise naturally from gauging ordinary global symmetries:

(1+1)d: $\mathbb{Z}_N^{(0)} \underset{gauging}{\longleftrightarrow} \mathbb{Z}_N^{(0)}$

[Vafa 1986]

Higher-form symmetries

Non-invertible top. operators

$(D+1)$d: $\mathbb{Z}_N^{(q)} \underset{gauging}{\longleftrightarrow} \mathbb{Z}_N^{(D-q-1)}$

[Gaiotto-Kapustin-Seiberg-Willett 2014, Tachikawa 2017]

(1+1)d: $G^{(0)} \underset{\text{``}gauging\text{''}}{\longleftrightarrow} Rep(G)$

[Brunner-Carqueville-Plencner 2014,…, Bhardwaj-Tachikawa 2017, Chang-Lin-SHS-Wang-Yin 2018]

The story is more intricate with more general tangential structures.

Non-invertible topological lines in 1+1d

- More generally, topological lines are extended operators that do not necessarily obey a group-like fusion rule:

$$L_a \times L_b = \sum_c N_{ab}^c L_c$$

← More than one term on RHS

- A non-invertible line L does **not** have an inverse such that $L \times L^{-1} = 1$.
- In recent years, non-invertible topological lines have been discussed from a modern perspective as generalizations of ordinary global symmetries [Bhardwaj-Tachikawa 2017, Chang-Lin-SHS-Wang-Yin 2018,…].

Many different names: topological symmetry, non-symmetry line, fusion category symmetry, algebraic higher symmetry, non-invertible symmetry…

Non-invertible topological lines in 1+1d

- Non-invertible topological lines are everywhere in 1+1d:

1. They have a long history in RCFT -- Verlinde lines [Verlinde 1988, Moore-Seiberg 1988-1989, Petkova-Zuber 2000, (Frohlich)-Fuchs-Runkel-Schweigert 2002-2006,…].
2. Wilson lines $Rep(G)$ from gauging a non-abelian global symmetry $G^{(0)}$.
3. From anomalous global symmetries in a fermionic theory after GSO [Thorngren 2018, Ji-SHS-Wen 2019].

- Lattice realization in condensed matter system: golden chain [Feiguin-Trebst-Ludwig-Troyer-Kitaev-Wang-Freedman 2006]. There is an operator that commutes with the lattice Hamiltonian but obeys the Fibonacci fusion rule:

$$W \times W = 1 + W$$

Non-invertible topological lines in 1+1d

- The existence of certain non-invertible lines imply that the low-energy phase can **NOT** be trivially gapped. Similar consequences as 't Hooft anomalies for ordinary global symmetries.
- Example: Tricritical Ising CFT perturbed by the subleading magnetic field σ'. It explicitly breaks $\mathbb{Z}_2$, but preserves a non-invertible line:

$$W \times W = 1 + W$$

The low energy phase is gapped with two-fold degenerate vacua [Zamolodchikov 1990]. The degeneracy is not explained by any symmetry. Rather, it's a consequence of the non-invertible line W [Chang-Lin-SHS-Wang-Yin 2018].

- More constraints on RG flows in 1+1d [Thorngren-Wang 2021].

Non-invertible topological operators

- Constraints on 1+1d adjoint QCD from non-invertible lines [Komargodski-Ohmori-Roumpedakis-Seifnashri 2020]. See also [Gomis' talk].
- Bulk 2+1d interpretation of the non-invertible lines [Thorngren-Wang 2019, Gaiotto-Kulp 2020].
- Categorical generalization of the Monster Moonshine [Lin-SHS 2019].
- Completeness of spectrum and the absence of certain topological operators [Rudelius-SHS 2020, Heidenreich-McNamara-Montero-Reece-Rudelius-Valenzuela 2021] [See Valenzuela's review talk on Swampland].
- Algebraic higher symmetry [Wen 2018, Ji-Wen 2019, Kong-Lan-Wen-Zhang-Zheng 2020 x2].
- Topological operators in the algebraic approach to QFT [Casini's talk].

Conclusion

- We have discussed three generalizations of ordinary global symmetries:

1. Higher-form symmetries
2. Subsystem symmetries
3. Non-invertible topological operators

- 1 and 3 can arise naturally from gauging ordinary global symmetries.
- 2 arises naturally from seemingly innocent lattice models such as fractons. It leads to UV/IR mixing.

Conclusion

- This more general perspective of global symmetry unifies many known phenomena into a coherent framework.
  - Generalized global symmetries and their anomalies provide an invariant characterization of many topological phases of matter such as fractons.
- More importantly, they lead to new results that are otherwise obscured.
  - Generalizations of the ‘t Hooft anomaly matching condition lead to nontrivial constraints on renormalization group flows.
- Many more to be explored! Collaboration between high energy physicists, condensed matter physicists, and mathematicians.

Global symmetries and generalizations

Properties of symmetry op.	Ordinary symmetry	Higher-form symmetry	Subsystem symmetry	Non-invertible operator
Codimension in spacetime	1	$q+1$	$q+1$	$q+1$
Topological	yes	yes	not completely but conserved in time	yes
Fusion rule	group $g_1 \times g_2 = g_3$	group $g_1 \times g_2 = g_3$	group $g_1 \times g_2 = g_3$	fusion ring $a \times b = \sum_c N_{ab}^c c$

Thank you for listening!