The Hilbert space of Chern Simons matter theories

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References

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- Chern Simons theories coupled to dynamical matter fields are of interest for several reasons.
- First, in parity non invariant theories, the one derivative Chern Simons Lagrangian generically dominates the two derivative Yang Mills kinetic term and so governs gauge dynamics at low energies.
- Second, the Chern Simons coupling, $\frac{1}{k}$, does not flow under the renormalization group, so fine tuning matter masses to zero often results in conformal dynamics.
- Third, Chern Simons matter theories host anyonic excitations with 'non half integer' spins whose S matrices display unusual crossing properties
- Fourth, the some of these theories have conjectured AdS/CFT dual descriptions in large N limits.
- Fifth, some of these theories they enjoy invariance under (conjectured) strong weak coupling Bose Fermi duality even without sypersymmetry.

Sixth, and most importantly for this talk, several exact results are available for two interesting limits of these theories.

- (1) When the mass of the matter fields is taken to infinity, our theories reduces to pure Chern Simons theory which has an intricate, beautiful and very thoroughly understood exact solution.
- (2) When all matter fields are in the fundamental, and N and k are taken to infinity with

$$\lambda = \frac{N}{k + \text{sgn}(k)N}$$

held fixed, the theory is once again exactly solvable. Several interesting dynamical quantities have been exactly computed in this limit.

Introduction

- Solvable limits of theories are special. Hilbert Spaces of theories in such limits often admit mathematically elegant descriptions - sometimes given by imposing natural constraints on free theries.
- In this talk I will attempt to present such a description of the Hilbert space of the Chern Simons fundamental matter theories in the solvable large N limits described in the previous transparency. It is possible that the lessons learnt will also have some value away from the large N limit.
- The all orders expression for the thermal free energy of $S^2 \times S^1$ path integral has already been obtained [1]-[11] by analytically summing all planar diagrams. We will use the most complete expressions - presented in [11] S.M, A. Mishra, N. Prabhakar, 2020 as the starting point of the analysis of this talk. Over the next few slides I present this result - which I will later interpret - in some detail.

$S^2 \times S^1$ Partition Function: Structure

- In the limit $N \to \infty$, $k \to \infty$, $\mathcal{V}_2 \to \infty$, with all ratios of these quantities, as well as the temperature T, chemical potential μ, and all masses and couplings held fixed, the S^2 times S^1 partition function $\mathcal{Z}_{S^2 \times S^1}$ is given by an integral over the unitary matrix U, the zero mode of the holonomy around the time circle

$$\mathcal{Z}_{S^2 \times S^1} = \int [dU]_{\rm CS} \; e^{-\mathcal{V}_2 v[\rho]} \;, \qquad (1)$$

- where $[dU]_{CS}$ is the usual Haar measure subject to the constraint

$$\rho(\alpha) \leq \frac{1}{2\pi|\lambda|} \;, \qquad \rho(\alpha) = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \delta(\alpha - \alpha_j) \;. \qquad (2)$$

($e^{i\alpha_j}$ are the eigenvalues of U and $\rho(\alpha)$ is the eigenvalue distribution function).

$S^2 \times S^1$ Partition Function: Theories

- $v[\rho]$ in (1) above depends details of the matter Chern Simons theory under study. In this talk we study the regular fermion and critical boson theory defined by

$$SU(N_F)_{(k_F - \frac{1}{2})} + \int \bar{\psi} D_\mu \gamma^\mu \psi + m_F^{\text{reg}} \bar{\psi}\psi$$

$$U(N_B)_{(k_B, k_B)} + \int \left[D_\mu \bar{\phi} D^\mu \phi + \sigma_B \left(\bar{\phi}\phi + \frac{N_B}{4\pi} m_B^{\text{cri}} \right) \right]$$

(the Chern Simons levels and ranks depicted below are sample examples; we will list all possibilities below; our main results also apply to the more complicated regular boson and critical fermion theories.)

- For each theory, computation reveals that the quantity $v[\rho]$ is given by the extremization, over two auxilliary variables, of a so called off shell free energy. Schematically

$$v[\rho] = \min_{\zeta_i} F(\zeta_i)$$

$S^2 \times S^1$ Partition Function: Fermionic $v[\rho]$,

- In the case of the Regular Fermion theory, $F[\zeta_i]$ is given by

$$F_{\rm RF}(\hat{c}_F, \tilde{\mathcal{C}}) = \frac{N_F T^2}{6\pi}\Bigg[-8\lambda_F^2\tilde{\mathcal{C}}^3 - 3\tilde{\mathcal{C}}\left(\hat{c}_F^2 - \left(2\lambda_F\tilde{\mathcal{C}} + \hat{m}_F\right)^2\right)$$
$$-6\lambda_F\hat{m}_F\tilde{\mathcal{C}}^2 + \hat{c}_F^3$$
$$-3\int_{\hat{c}_F}^{\infty} d\hat{\epsilon}\,\hat{\epsilon}\int_{-\pi}^{\pi} d\alpha\,\rho_F(\alpha)\left(\log\left(1+e^{-\hat{\epsilon}-\hat{\mu}-i\alpha}\right) + \log\left(1+e^{-\hat{\epsilon}+\hat{\mu}+i\alpha}\right)\right)\Bigg]. \quad (3)$$

Here N_F is the gauge rank, λ_F is the t'Hooft coupling, $\tilde{\mathcal{C}}$ and $\hat{c}_F$ are auxilliary variables that have to be extremized over ($\hat{c}_F$ has the interpretation of the thermal mass in units of temperature) and

$$\hat{\mu} = \frac{\mu}{T}, \quad \hat{m}_F = \frac{m_F}{T}$$

- Note that only the third line of (3) depends on $\rho(\theta)$ or μ

$S^2 \times S^1$ Partition Function: Bosonic $v[\rho]$,

- In the case of the Critical Boson theory, $F[\zeta_i]$ is given by

$$
\begin{aligned}
&F_{\rm CB}(c_B, \tilde{\mathcal{S}}) \\
&= \frac{N_B T^2}{6\pi}\Bigg[\frac{3}{2}\hat{c}_B^2 \hat{m}_B^{\rm cri} - 4\lambda_B^2\left(\tilde{\mathcal{S}} - \tfrac{1}{2}\hat{m}_B^{\rm cri}\right)^3 + 6|\lambda_B|\hat{c}_B\left(\tilde{\mathcal{S}} - \tfrac{1}{2}\hat{m}_B^{\rm cri}\right)^2 - \hat{c}_B^3 \\
&+ 3\int_{\hat{c}_B}^{\infty} d\hat{\epsilon}\ \hat{\epsilon}\int_{-\pi}^{\pi} d\alpha\ \rho_B(\alpha)\left(\log(1 - e^{-\hat{\epsilon}+\hat{\mu}+i\alpha}) + \log(1 - e^{-\hat{\epsilon}-\hat{\mu}-i\alpha})\right) \\
&\qquad - \Theta(|\mu| - c_B)\frac{(|\hat{\mu}| - \hat{c}_B)^2(|\hat{\mu}| + 2\hat{c}_B)}{2|\lambda_B|}\Bigg]\ . \qquad (4)
\end{aligned}
$$

Here N_B and λ_B are the gauge rank and t' Hooft coupling, $\tilde{\mathcal{S}}$ and $\hat{c}_B$ are auxilliary variables that have to be extremized over ($\hat{c}_B$ has the interpretation of the thermal mass in units of temperature) and

$$
\hat{\mu} = \frac{\mu}{T}, \qquad \hat{m}_B^{\rm cri} = \frac{m_B^{\rm cri}}{T}
$$

- Note that only last two lines of (3) depends on $\rho(\theta)$ or μ

$S^2 \times S^1$ Partition Function: Interchanging Orders

- In summary for both fermions and bosons

$$\mathcal{Z}_{S^2\times S^1} = \int [dU]_{\rm CS} \min_{\zeta_i} \left[e^{-\mathcal{V}_2 F(\zeta_i,\rho)} \right] , \qquad (5)$$

- At leading order in the large N limit, the integral over U reduces to a saddle point extremization over $\rho(\theta)$. The extremaization over ζ_i and $\rho(\theta)$ can be performed in any order, so (5) can be rewritten as

$$\mathcal{Z}_{S^2\times S^1} = \min_{\zeta_i} \left[\int [dU]_{\rm CS} e^{-\mathcal{V}_2 F(\zeta_i,\rho)} \right] , \qquad (6)$$

- To evaluate $\mathcal{Z}_{S^2\times S^1}$ we must thus
  Step (1): Evaluate $I(\zeta_i) = \int [dU]_{\rm CS} e^{-\mathcal{V}_2 F(\zeta_i,\rho)}$ at fixed ζ_i.
  Step (2): Extremize $I(\zeta_i)$ over ζ_i.
- Step I is universal (indep of details of contact interactions). Step 2 is non universal and accounts for these interactions.

Step I: Fermions

- In Step 1 for fermions we are required to evaluate

$$I_F = \int [dU]_{\rm CS}\, Z^F_{NS}(U)\ ,$$

$$Z^F_{NS}(U) = e^{\frac{N_F T^2 \mathcal{V}_2}{2\pi}\left[\int_{\hat{c}_F}^{\infty} d\hat{\rm e}\ \hat{\rm e} \int_{-\pi}^{\pi} d\alpha\, \rho_F(\alpha)\left(\log\left(1+e^{-\hat{\rm e}-\hat{\mu}-{\rm i}\alpha}\right)+\log\left(1+e^{-\hat{\rm e}+\hat{\mu}+{\rm i}\alpha}\right)\right)\right]}. \tag{7}$$

The exponent of $Z^F_{NS}(U)$ includes all terms in $F^F(\zeta_i)$ that depend on either ρ or μ.

- It is not difficult to verify that

$$Z^F_{NS}(U) = {\rm Tr}_{H_{NS}}\left(Ue^{-\beta(H-\mu Q)}\right)\ , \tag{8}$$

Where the trace is taken over H_{NS} the free Fock Space of free fermions of mass $c_F = \hat{c}_F T$ propagating on a (very large) S^2.

- Consequently

$$I_F = \int [dU]_{\rm CS}\, {\rm Tr}_{H_{NS}}\left(Ue^{-\beta(H-\mu Q)}\right) \tag{9}$$

Step I: Bosons

- In Step 1 for bosons we are required to evaluate

$$
\begin{aligned}
I_B &= \int [dU]_{\rm CS}\; Z^B_{NS}(U) \\
Z^B_{NS}(U) &= e^{-\frac{N_B T^2 \mathcal{V}_2}{2\pi}\int_{\hat{c}_B}^{\infty} d\hat{\epsilon}\;\hat{\epsilon}\int_{-\pi}^{\pi} d\alpha\,\rho_F(\alpha)\left(\log\left(1-e^{-\hat{\epsilon}-\hat{\mu}-i\alpha}\right)+\log\left(1-e^{-\hat{\epsilon}+\hat{\mu}+i\alpha}\right)\right)} \times \\
& \quad e^{-\frac{N_B T^2 \mathcal{V}_2}{2\pi}\Theta(|\mu|-c_B)\;\frac{(|\hat{\mu}|-\hat{c}_B)^2(|\hat{\mu}|+2\hat{c}_B)}{6|\lambda_B|}} \,.
\end{aligned}
\tag{10}
$$

Once again the exponent of $Z_B(U)$ includes all terms in $F^B(\zeta_i)$ that depend on either ρ or μ

- This can be rewritten as

$$
I_B = \int [dU]_{\rm CS}\, {\rm Tr}_{H_{NS}}\left[\left(Ue^{-\beta(H-\mu Q)}\right)\left(e^{-\frac{N_B T^2 \mathcal{V}_2}{2\pi}\Theta(|\mu|-c_B)\;\frac{(|\hat{\mu}|-\hat{c}_B)^2(|\hat{\mu}|+2\hat{c}_B)}{6|\lambda_B|}}\right)\right]
\tag{11}
$$

The second term in the trace is a new element missing in the case of fermions. This term is independent of ρ but depends on μ and is universal (indep of details of contact interactions) and so should be accounted for in step I.

Toy Model for I_F

- Let us momentarily consider a quantity similar to I_F.

$$\tilde{I}_F = \int [dU] \, \mathrm{Tr}_{H_{NS}} \left(U e^{-\beta(H-\mu Q)} \right) \tag{12}$$

  where dU is the usual unmodified Haar measure.
- $\tilde{I}_F$ has a simple and familiar Hilbert Space interpretation. The free fermion Fock Space can be decomposed into a sum over irreducible representations of $U(N_F)$. The integral over U in (13) simply projects this Fock Space onto the $U(N_F)$ singlets.
- In other words

$$\tilde{I}_F = \mathrm{Tr}_{H_{Sing}} \left(e^{-\beta(H-\mu Q)} \right) \tag{13}$$

  where H_{Sing} is the projection H_{NS} to $U(N_F)$ singlets.
- Question: Does I_F have an interpretation similar to $\tilde{I}_F$? And whats the story with I_B?

Hilbert Space for Step I

- The questions posed at the end of the last paragraph have a simple answer. I_F and I_B both have a simple Hilbert Space interpretation. Each of those quantities can be thought of as the large N limit of the free Fermionic / Bosonic Fock space, restricted to the space of WZW (or quantum group) singlets.
- In order to explain why this is true (and also precisely what these words mean), over the next few slides I will make a brief digression to review the types of $SU(N)_k$ and $U(N)_{k,k'}$ Chern Simons theories, their WZW duals and the counting of their conformal blocks.

$SU(N)_k$ and $U(N)_{k,k'}$ Chern Simons theories.

- $SU(N)_k$ Chern Simons theories (and their WZW duals) are relatively famliar. They are characterized by a single level k
- $U(N)_{k,k'}$ Chern Simons theories may be less familiar. These theories are characterized by two levels k and k'. Roughly speaking, k is the level for the $SU(N)$ part of $U(N)$, while Nk' is the level for the $U(1)$ part of $U(N)$. It turns out that consistency forces

$$k' = \kappa + qN$$

where $\kappa = k + \mathrm{sgn}(k)N$ and q is an integer.
- We call the choice $q = 0$ the Type I $U(N)$ theory at level k. We also name the choice $q = -1$ the Type II $U(N)$ theory at level k.

$SU(N)_k$ and $U(N)_{k,k'}$ and duality.

- Pure (matter free) $SU(N)_k$ theories are well known to be level rank dual to Type II $U(|k|)$ Chern simons theories at level $-N$. It is also well known that Type I $U(N)$ pure Chern Simons theories at level k are level rank dual to $U(|k|)$ theories at level $-N$.
- The conjectured Bose Fermi dualities between regular fermion and crtitical Boson Chern Simons matter theories also involve either $SU(N)$ /Type II or Type I / Type I theories on the two sides of the duality.
- As no known level rank duality in relates two $SU(N)$ theories, the study of duality forces us to enlarge our horizons to include $U(N)$ Type II (and then very naturally also Type I) theories rather than focussing only on their $SU(N)$ counterparts.

- Consider pure Chern Simons theory on $S^2 \times S^1$ with m Wilson lines, each at a point on the S^2 but wrapping the S^1 once. The Wilson lines in question transform in the representations $R_1, R_2 \ldots R_m$ where each representation is restricted to be integrable. Over 30 years ago, Witten famously demonstrated that the result of this path integral is an integer which counts the number of WZW conformal blocks on S^2 with primary insertions in the representations $R_1, R_2 \ldots R_m$. This number can be evaluated using the Verlinde formula, but it may also be evaluated directly in 3d.
- Again, almost 30 years ago, Blau and Thompson used a clever gauge fixing to explicitly evaluate the CS path integral on $S^2 \times S^1$. Their explicit $SU(2)_k$ results are easily generlized to $SU(N)_k$. We list our results on the next slide.
- Caution: All formulae overleaf apply only to integrable insertions. Unsatisfying. Would be good to understand better why.

Counting $SU(N)_k$ conformal blocks on S^2

- $$N_{sing} = \frac{1}{N\kappa^{N-1}} \sum_{\{w_i\}} \prod_{i<j} |w_i - w_j|^2 \prod_{p=1}^{m} \chi_{R_p}(w_i) \qquad (14)$$

  Here w_i are the eigenvalues of the $SU(N)$ holonomy,

  $$\prod_{i=1}^{N} w_i = 1, \quad \text{and} \quad |w_i| = 1 \quad \forall i \qquad (15)$$

  Moreover a sum over fluxes in the path integral turns the integrals over holomies into discrete sum taken over all possible distinct choices of N distinct w_i that obey (14) together with the condition

  $$w_i^{\kappa} = w_j^{\kappa} \qquad \forall i, j \qquad (16)$$

- (14) may be verified to infact simply be a rewriting of the Verlinde formula in physically recognizable terms.

Counting $U(N)_{k,k'}$ conformal blocks on S^2

- In this case we find

$$N_{sing} = \frac{1}{(\kappa + qN)\kappa^{N-1}} \sum_{\{w_i\}} \prod_{i<j} |w_i - w_j|^2 \prod_{p=1}^{n} \chi_{R_p}(w_i) \quad (17)$$

where

$$\begin{aligned} |w_i| = 1 \quad \forall i, \quad w_i^\kappa = w_j^\kappa \qquad \forall i,j \\ w_m^\kappa \left(\prod_{i=1}^{N} w_i\right)^q = (-1)^{N+1} \quad \forall m \end{aligned} \quad (18)$$

- Our attempt at generalizing the Blau and Thompson result to $U(N)$ gave us the formula above with $(-1)^{N+1}$ replaced by unity. However recasting the Verlinde formula in this form reveals this factor. Several consistency checks and comparisons with earlier literature give overwhelming evidence this factor is correct. To be done: Correct the naive path integral derivation to reproduce this term.

Counting Type I $U(N)$ conformal blocks on S^2

- The formula of the previous slide simplifies at $q = 0$, i.e. in the Type I theory, to

$$N_{sing} = \frac{1}{\kappa^N} \sum_{\{w_i\}} \prod_{i<j} |w_i - w_j|^2 \prod_{p=1}^{n} \chi_{R_p}(w_i) \qquad (19)$$
$$w_m^\kappa = (-1)^{N+1} \quad \forall m$$

- For simplicity in the rest of this talk we will work with the Type I theory, (our final results apply to every case)
- (19) is a very particular discretization of the Weyl integral formula of classical group theory. In the large N limit the spacing between two eigenvalues, $\frac{1}{2\pi\kappa} \to 0$ the discretization spacing $\to 0$ so (19) reduces to the classical Weyl formula except for one constraint;

$$\rho(\theta) \leq \frac{2\pi N}{\kappa} = \frac{2\pi}{\lambda}.$$

$[dU]_{CS}$ from conformal blocks

- In equations, in the t'Hooft large N limit

$$\frac{1}{\kappa^N}\sum_{\{w_i\}}\prod_{\mathrm{i<j}}|w_i - w_j|^2\prod_{p=1}^{n}\chi_{R_p}(w_i) \to \int [dU]_{CS}\prod_{p=1}^{n}\chi_{R_p}(w_i)$$

- It follows that the quantity I_F above does indeed admit the interpretation we proposed: it is the partition function of the Fock Space restricted to the WZW singlet sector.
- While thats great, it raises the obvious question: what about the Bosons? What is the origin of the extra factor for bosons in step 1? We now now turn to this question

$\theta(\mu - c_B)$... from conformal blocks

- Consider a Type I $U(N)$ Chern Simons theory coupled to fundamental bosons on S^2. Let a index the positive energy solutions of the Klein Gordon equation, with mass c_B, on S^2. Clearly the U twisted partition function over the free bosonic Fock Space is given by a product of partition functions, one for every free particle state
- Explicitly

$$\mathrm{Tr}\left(Ue^{-\beta(H-\mu Q)}\right) = \prod_a \left[\left(\prod_{i_a=1}^{N_B} \frac{1}{1-e^{-\beta(E_a-\mu)}w_{i_a}}\right)\left(\prod_{i_a=1}^{N_B} \frac{1}{1-e^{-\beta(E_a+\mu)}w^*_{i_a}}\right)\right] \tag{20}$$

- Note that

$$\frac{1}{1-e^{-\beta(E_a-\mu)}w_{i_a}} = \sum_{n=0}^{\infty} e^{-n\beta(E_a-\mu)}\chi_n^S(U) \tag{21}$$

Truncation for Bosons

- Now the partition function of the bosonic Fock space restricted to WZW singlets is *not* given simply by

$$\frac{1}{\kappa^N}\sum_{\{w_i\}}\prod_{\mathrm{i<j}}|w_i - w_j|^2 \mathrm{Tr}\left(Ue^{-\beta(H-\mu Q)}\right) \qquad (22)$$

- Terms with $n > k_B$ in (21) are non integrable insertions. Conformal blocks involving such insertions should vanish. (22) does not correctly account for this fact (see comment in red before (14)), which must thus be inserted by hand.
- The correct truncation of the free boson Fock space to WZW singlets is given by

$$I_B^l = \frac{1}{\kappa^{N_B}}\sum_{\{z_i\}}\prod_{\mathrm{pairs}}|w_i - w_j|^2\prod_a\left[\left(\prod_{i_a=1}^{N_B}\frac{1}{1-e^{-\beta(E_a-\mu)}w_{i_a}}\right)\Bigg|_{k_B}\left(\prod_{i_a=1}^{N_B}\frac{1}{1-e^{-\beta(E_a+\mu)}w^*_{i_a}}\right)\Bigg|_{k_B}\right]$$

Implication of the Truncation

- It is not difficult to prove that

$$Q(y) = \prod_{i=1}^{N_B} \frac{1}{1 - w_i y}\Bigg|_{k_B} = \left(1 + (-1)^N y^\kappa\right) \prod_{i=1}^{N_B} \frac{1}{1 - z_i y} \qquad (24)$$

$$= \exp\left(- \mathrm{tr} \ln \left(1 - yU\right) + \ln \left(1 - y^\kappa\right)\right)$$

- In the large N limit it follows that

$$\ln Q(y) = -\mathrm{tr} \ln (1 - yU) + \kappa \Theta(y - 1) \ln w \,. \qquad (25)$$

- In the physical problem of interest $y = e^{-\beta(E-\mu)}$. The second term in (25) is thus nonzero only for states with $E < \mu$. Such states exist only if $c_B < \mu$. Adding up the contribution of all such states (accounting for the density of states) reproduces the extra term in (11) with all factors.

The Bosonic Exclusion Principle

- We have learnt something important here. In large N matter Chern Simons theories, no single particle bosonic state can be occupied more than k_B times. We call this the 'Bosonic Exclusion Principle'. It is the direct level rank dual of a more obvious result for fermionic theories, namely that no single particle fermionic state can be occupied more than N_F.
- Recall that ordinary free boson theories are ill defined at values of the chemical potential greater than the mass, as all states with energies between the mass and the chemical potential are infinitely occupied in such theories. The bosonic exclusion principle cures this singualrity in matter Chern Simons theories, rendering Bosonic theories with chemical potential larger than the mass well defined.

Large Volume Limit

- Let us recap. We have discovered that I_F and I_B, are simply the partition functions over the free Fermion/ Free Boson Fock spaces, subject to the WZW singlet condition. The Bosonic exclusion principle follows from this statemet.
- The WZW singlet condition is an effective interaction between distinct single particle states. As a consequence the final results for I_F and I_B are not free, i.e. the final partition function does not, in general, reduce to a product of partition functions, one for each single particle state.
- Recall, however, that I_F and I_B depend on a parameter, the volume of the S^2, or more precisely $\alpha = \frac{\mathcal{V}_2 T^2}{N}$. It is interesting to study I_F and I_B in the large α limit. In what follows we perform this study at finite N and k, not necessarily in the 't Hooft large N limit, as we expect I_B and I_F to accurately capture the partition function of Chern Simons matter theories in low density limits ($T \ll m$) even at finite N and k.

'Saddle Point' at large Volume

- At finite N and k I_B is given by the formula (23) (I_F is given by a similar formula). The key simplification of the large volume limit is that the summation over choices of eigenvalues in that formula is dominated by a single eigenvalue configuration (this is a sort of saddle point approximation for the summation in that formula).
- The eigenvalue configuration that dominates the sum is

$$w_i = e^{i(\alpha_i)},$$

$$\{\alpha_i\} = \frac{2\pi}{\kappa}\left\{ -\frac{N-1}{2}, -\frac{N-3}{2}, -\frac{N-5}{2}, \ldots \frac{N-5}{2}, \frac{N-3}{2}, \frac{N-1}{2} \right\} \tag{26}$$

This configuration is the correct 'saddle point' for all cases; the $SU(N)$, Type II and Type I theories and for fermions and bosons.

Factorization at large volume

- At general values of the volume, the formula (23) expresses the partition function as a sum over products. As we have explained, at large volume the sum local to a single term, leaving us with a simple product, one term for each single particle state.

$$I_B \propto \prod_a Z_B^a \bar{Z}_B^a , \quad Z_B^a = \left(\prod_{i=1}^{N_B} \frac{1}{1 - e^{-\beta(E_a-\mu)} w_i} \right)\Bigg|_{k_B} , \qquad (27)$$
$$\bar{Z}_B^a = \left(\prod_{i=1}^{N_B} \frac{1}{1 - e^{-\beta(E_a+\mu)} w_i} \right)\Bigg|_{k_B} ,$$

(the eigenvalues that apear in (27) are those listed on the previous slide)

- $$Z_B^a = \sum_{r=0}^{k_B} \chi_n^S(U) \, e^{-r\beta(E_a-\mu)} = \sum_{r=0}^{k_B} d_n^S \, e^{-r\beta(E_a-\mu)} \qquad (28)$$

q numbers and quantum dimensions

- d_n^S is the so called quantum dimenison of the n box symmetric representation. As mentioned above it equals the character of this representation evaluated on our special 'saddle point' unitary matrix. Explicitly

$$\begin{aligned} d_n^S &= \binom{n}{m}_q = \frac{[n]_q!}{[m]_q![n-m]_q!} \\ [m]_q! &= [1]_q[2]_q \cdots [m]_q \\ [r]_q &= \frac{q^{r/2} - q^{-r/2}}{q^{1/2} - q^{-1/2}} \end{aligned} \tag{29}$$

with

$$q = e^{\frac{2\pi i}{\kappa}}$$

- Similar expressions hold for fermions. Using identities involving q factorials, the bosonic and fermionic expressions can be shown to be level rank dual.

Product but not Free

- Ignoring details, for the moment, an immediately striking aspect of the large volume limit is that the partition factorizes (its a product of partition functions, one for each single particle state).
- This feature may appear to suggest that our system is free in the infinite volume limit (whats going on in one single particle state does not affect the partition function of another single particle state).
- While this suggestion sounds initially reasonable, it is not correct. We can see this by noting that the coefficients of $e^{-\beta(E_n-\mu)}$ in the expansion of Z_n above are not integers. It follows that the different Z_n are not partition functions over independently defined Hilbert Spaces.

Explanation of the Product Structure

- We believe that this product structure is a manifestation of an interesting universality in the fusion rule algebra in the large insertion limit.
- Consider any generic collection of integrable representations of the WZW algebra $R_1 \ldots R_n$.
- Let us now sequentially fuse our representations with each other. Once this process is completed let us suppose we are left with n_{R_i} representations of type R_i for each integrable representation R_i.
- In the limit that n is the largest number in the problem, we believe that

$$\frac{n_{R_i}}{n_{R_j}} = \frac{d_{R_i}}{d_{R_j}}$$

Independent of the details of the participating representations R_i.

Explanation of factorization

- This univerality explains the factorization of our partition functions as follows
- The coefficient of $e^{-n\beta(E_n-\mu)}$ in Z_n is actually proportional to the number of sea particles in the representation conjugate to the n box symmetric representation.
- The conjecture of the previous slide explains why this number is independent of the precise state of the 'sea', explaining why the product structure of single particle
- The universality described in our conjecture is tightly connected to the fact that the unitary matrix U localizes on the same universal matrix in the $\mathcal{V}_2 \to \infty$ limit, independent of the temperature, chemical potential and masses together with the fact that $d_R = \chi_R(U)$.

Implications of Factorization

- Recall that

$$Z_B^a = \sum_{m=0}^{k_B} \binom{N_B}{m}_q e^{-r\beta(E_a-\mu)} \qquad (30)$$

- In the limit $\lambda_B \to 0$ $q \to 1$ and

$$\binom{N_B}{m}_q \to \binom{N_B}{m}$$

  Also in this limit k_B so the upper limit on the summation in (30) $\to \infty$. We thus reproduce usual Bose statistics.
- (30) can be thought of as a one parameter deformation of usual Bose thermal 'statistics'; one that changes the details of occupation probabilities at low occupation numbers, and imposes the Bose Exclusion Principle at high occupation numbers.

New Single particle Thermal Statistics

- As a consequence we obtain a one parameter deformation of many of the familiar rules of free statistical physics that we learn about as undergraduates.
- For instance, in the t'Hooft Large N limit we find the following formula for the average occupation number of any given single particle state at temperature T and chemical potential μ

$$\bar{n}_B(\epsilon,\mu) = \frac{1-|\lambda_B|}{2|\lambda_B|} - \frac{1}{\pi|\lambda_B|}\tan^{-1}\left(\frac{e^{\beta(\epsilon-q\mu)}-1}{e^{\beta(\epsilon-q\mu)}+1}\cot\frac{\pi|\lambda_B|}{2}\right),$$

- Generalizing the familiar free boson result

$$\bar{n}_B(\epsilon,\mu) = \frac{1}{e^{\beta(\epsilon-q\mu)}-1}$$

- Similar results apply for fermions and respect duality

Step 2

- We now have a thorough understanding of the physical meaning of Step 1. I now give a brief account of step 2.
- Step 2 captures the effet of contact interactions between our effectively free particles. In the large N limit only forward scattering effect contribute to thermodynamics
- The most imortant qualitative effect of Step 2 is that it determines the thermal mass of scattering particles as a function of T and μ (this comes from extremizing over $\hat{c}_B$. A second effect is that it renormalizes the energy of the multiparticle state away from the sum of energies of single particle in a well understood and algorithmic way.
- Not that interactions captrued by Step 2 do not either cause particles to decay or modify Step 1 occupation number statistics (the last follows because all μ dependence is captured in step 1). Most step 2 is qualitatively similar at all values of λ_B including zero, and applies also to the large N $O(N)$ model.

Conclusions and future directions

- We have demonstrated that the partition function of Large N matter Chern Simons theories is effectively that of a free fock space, constrained to the subspace of WZW singlets, and then renormalized by forward scattering interactions in a mean field like way
- It would be intersting to move this understanding away from the large N limit and also from the large volume limit. It would be fanctastic if we could rewrite the Index of superconformal Chern Simons matter theories in a language similar to this talk (i.e. free theory subject to an effective constraint).
- It would be interesting if the new 'free thermal statistics' of this talk showed up in a real two dimensional material.

Discussion

- The Bose condensate encountered in our analysis is an extremely simple stabilization of the run away instability of free theory. The sharp cut off at k_F plus the Bose exclusion principle gives this phase all the properties of a Fermi Sea.
- It would be interesting to investigate the dynamical implications of the Bose condensation principle. Cut off lasers?
- It would be interesting to better understand how the path integral 'knows' that mixed 'correlators' of non integrable and integrable Wilson lines must vanish, even in the case of pure Chern Simons theory (see remark in red before (14)). It would also be satisfying to reproduce the phase $(-1)^{N+1}$ in (18) directly from the Blau and Thompson path integral.