Scattering Amplitudes in Three Dimensions

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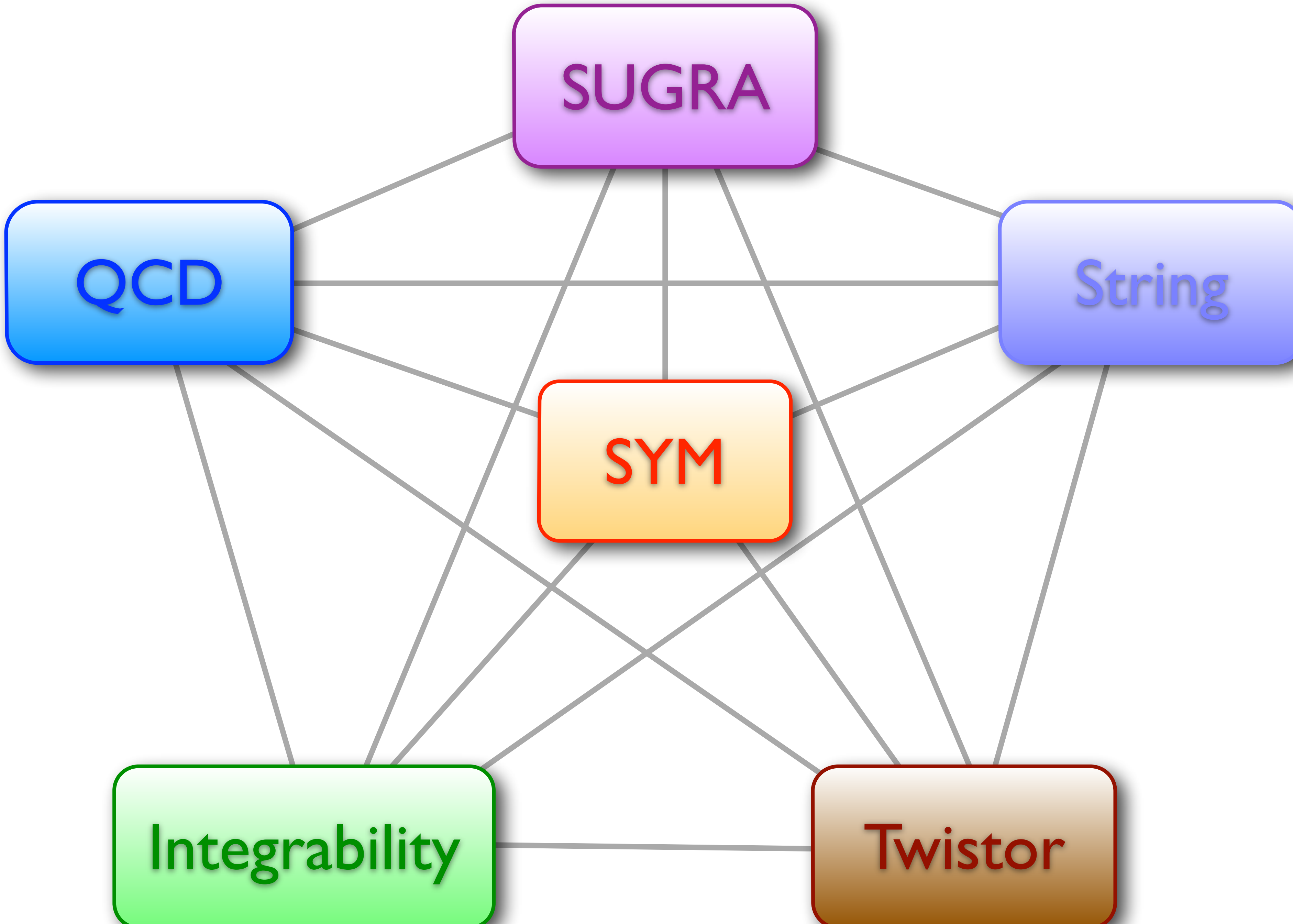
27 June 2014

Strings 2014, Princeton

Scattering Amplitudes

Recent review [Elvang,Huang 13]



Talks by Basso, Cachazo, Dolan, Mafra, Sen, Staudacher, Stieberger, Trnka

Scope of this talk

- ABJM theory / tree amplitudes / planar sector
- Two amplitude-generating integrals

Grassmannian

$$\mathcal{A}_{2k}(\Lambda) = \int \frac{d^{k\times 2k}C}{\mathrm{vol}[\mathrm{GL(k)}]} \frac{\delta(C_{mi}C_{ni}) \prod_{m=1}^{k} \delta^{2|3}(C_{mi}\Lambda_i)}{M_1(C)M_2(C)\cdots M_k(C)}$$

Twistor string

$$\mathcal{A}_{2k}(\Lambda) = \int \frac{d^{2\times 2k}\sigma}{\mathrm{vol}[\mathrm{GL(2)}]} \frac{J\,\Delta \prod_{m=1}^{k} \delta^{2|3}(C_{mi}[\sigma]\Lambda_i)}{(12)(23)\cdots(2k,1)}$$

"in four dimensions" = "for N=4 Super-Yang-Mills"

This talk is based on

SL [1007.4772]

Dongmin Gang, Yu-tin Huang,
Eunkyung Koh, SL, Arthur Lipstein [1012.5032]

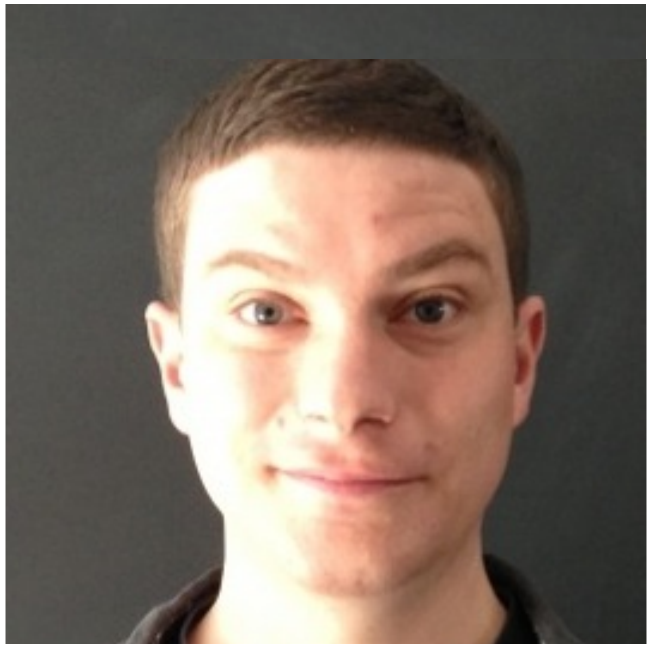
Yu-tin Huang, SL [1207.4851]

Yu-tin Huang, Henrik Johansson, SL [1307.2222]

Joonho Kim, SL [1402.1119]

special thanks to
N. Arkani-Hamed

Collaborators



Henrik Johansson

Yu-tin Huang

Arthur Lipstein

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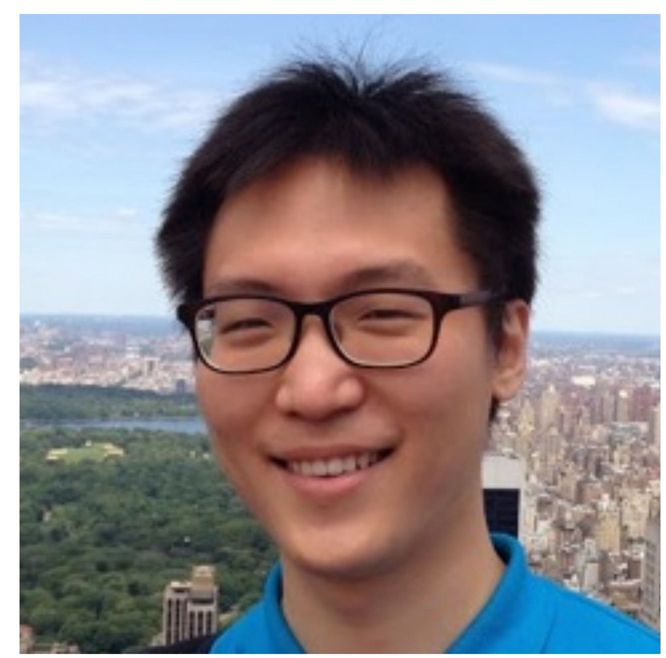
Eunkyung Koh



Joonho Kim

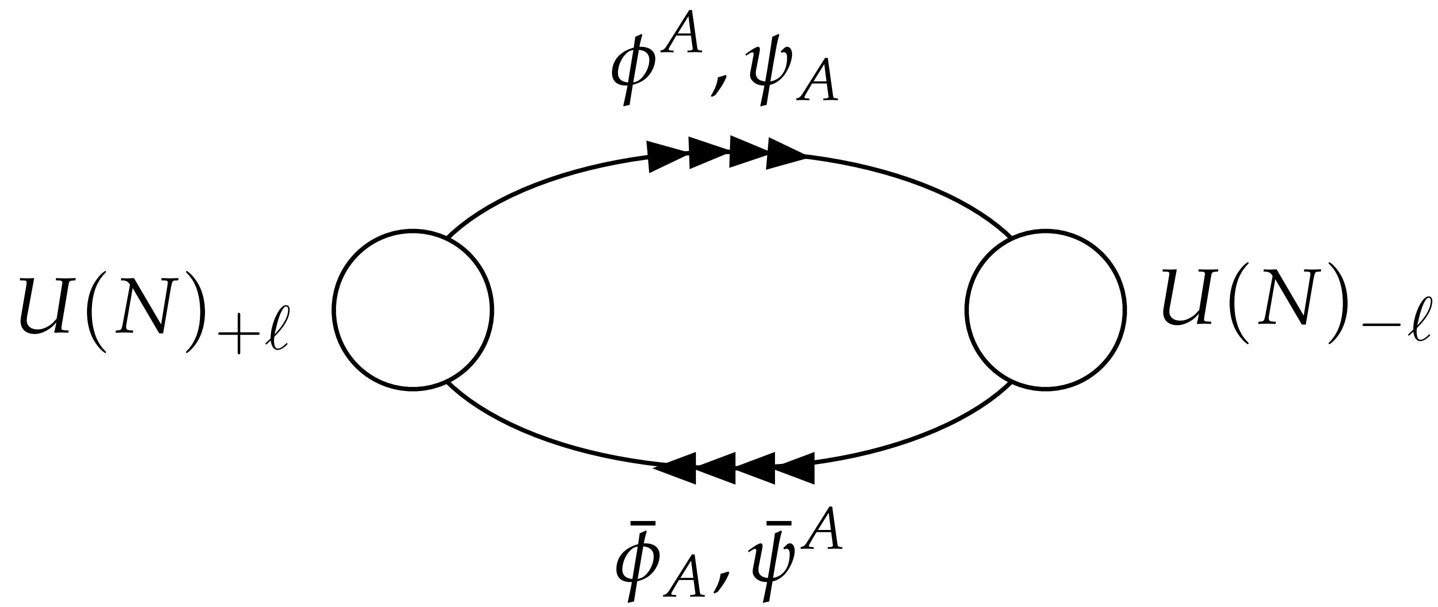
ABJM theory

- ABJM

  $d = 3,\ \mathcal{N} = 6$

  superconformal

  Chern-Simons-matter



$$\mathrm{OSp}(6|4) \supset \mathrm{Sp}(4,\mathbb{R}) \times \mathrm{SO}(6)_R \simeq SO(2,3) \times SU(4)_R$$

- Color-ordered amplitudes of ABJM

  Gauge fields mediate interactions but have no d.o.f.

  Number of external legs must be even ($n = 2k$)

  $\mathcal{A}_{2k}(1, 2, \cdots, 2k-1, 2k)$

Kinematics, on-shell [Bargheer,Loebbert,Meneghelli 10]

- Null momentum in (1+2)d : $p^{\alpha\beta} = p_\mu (C\gamma^\mu)^{\alpha\beta} = \lambda^\alpha \lambda^\beta$

  Lorentz invariants : $\langle ij \rangle = \epsilon_{\alpha\beta} (\lambda_i)^\alpha (\lambda_j)^\beta$

- On-shell super-fields $SO(1,2) \times U(3) \subset SO(2,3) \times SO(6)_R$

$$\Phi = \phi^4 + \eta^I \psi_I + \tfrac{1}{2}\epsilon_{IJK}\eta^I\eta^J\phi^K + \tfrac{1}{6}\epsilon_{IJK}\eta^I\eta^J\eta^K\psi_4\,,$$ (1+3+3+1) = 4 + 4

$$\bar{\Phi} = \bar{\psi}^4 + \eta^I \bar{\phi}_I + \tfrac{1}{2}\epsilon_{IJK}\eta^I\eta^J\bar{\psi}^K + \tfrac{1}{2}\epsilon_{IJK}\eta^I\eta^J\eta^K\bar{\phi}_4\,.$$

- Cyclic symmetry and Lambda-parity

$$\mathcal{A}_{2k}(1,2,\cdots,2k) = (-1)^{k-1}\mathcal{A}_{2k}(3,4,\cdots,2k,1,2)$$

$$\mathcal{A}_{2k}(\cdots,-\Lambda_i,\cdots) = (-1)^i \mathcal{A}_{2k}(\cdots,\Lambda_i,\cdots)$$ $\Lambda = (\lambda^\alpha, \eta^I)$

BCFW and Grassmannian in 4d

Momentum conservation in spinor-helicity

$$\sum_{i=1}^{n} (p_i)_{\bar{\alpha}\alpha} = \sum_{i=1}^{n} \bar{\lambda}_{i\dot{\alpha}} \lambda^i_\alpha = \left(\bar{\lambda}_{1\dot{\alpha}} \cdots \bar{\lambda}_{n\dot{\alpha}} \right) \begin{pmatrix} \lambda^1_\alpha \\ \vdots \\ \lambda^n_\alpha \end{pmatrix}$$

BCFW deformation is a special case of [Britto, Cachazo, Feng, Witten 04-05]

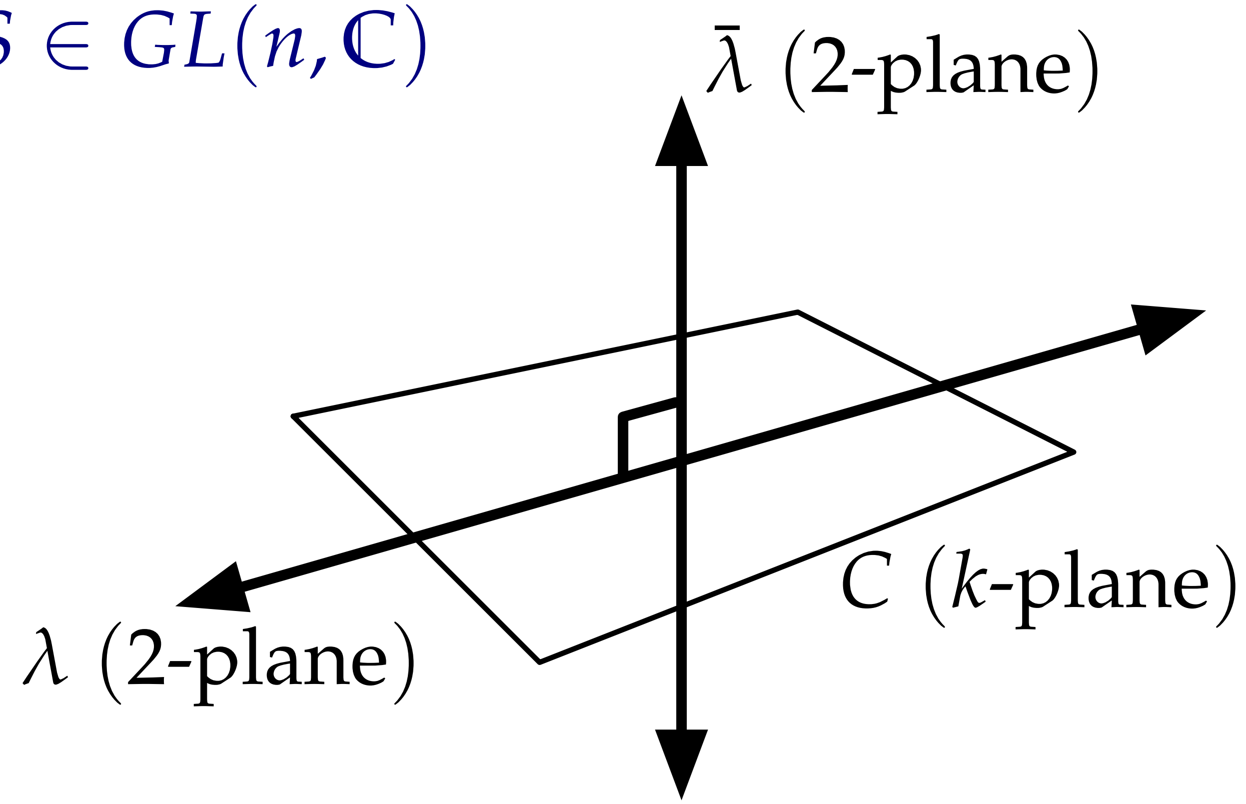
$$\lambda^i \to S^i{}_j \lambda^j \,, \quad \bar{\lambda}_i \to \bar{\lambda}_j (S^{-1})^j{}_i \,, \quad S \in GL(n, \mathbb{C})$$

P-conservation at each vertex:

Grassmannian

$$C_{mi} \in \mathrm{Gr}(k, n)$$

[Arkani-Hamed, Cachazo, Cheung, Kaplan 09]



BCFW and Grassmannian in 3d

Momentum conservation in spinor-helicity

$$\sum_{i=1}^{n} (p_i)_{\alpha\beta} = \sum_{i=1}^{n} \lambda_{i\alpha}\lambda_{i\beta} = (\lambda_{1\alpha} \cdots \lambda_{n\alpha}) \begin{pmatrix} \lambda_{1\beta} \\ \vdots \\ \lambda_{n\beta} \end{pmatrix}$$

BCFW deformation is a special case of

$$\lambda^i \to R^i{}_j \lambda^j \,, \quad R \in O(n, \mathbb{C})$$

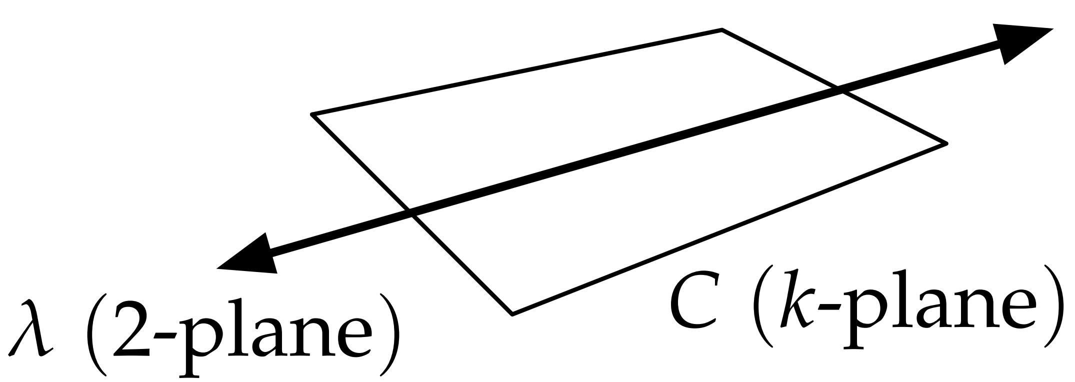
[Gang,Huang,Koh,SL,Lipstein 10]

P-conservation at each vertex:

orthogonal Grassmannian

(a.k.a. isotropic Gr) [SL 10]

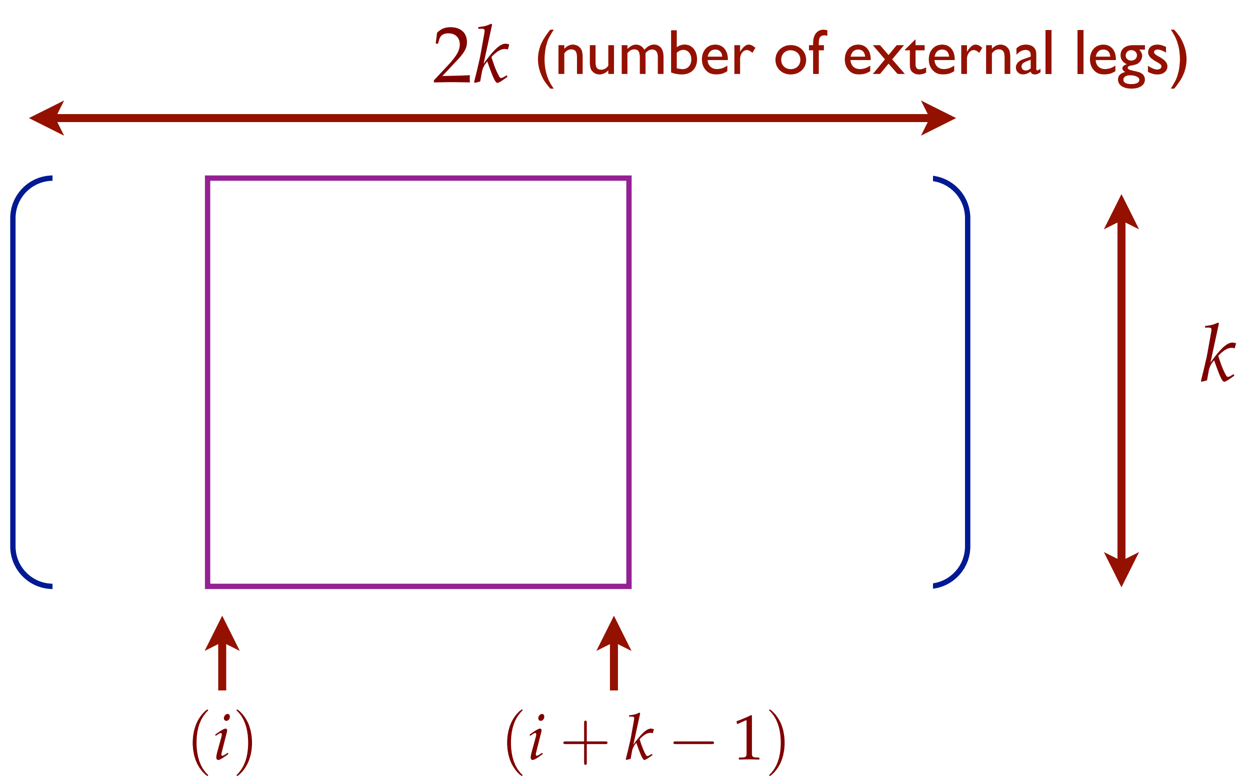
$$\sum_i C_{mi} C_{ni} = 0 \,, \quad C_{mi} \in OG(k, 2k)$$



Grassmannian Integral in 3d [SL 10]

$$\mathcal{A}_{2k}(\Lambda) = \int \frac{d^{k\times 2k}C}{\mathrm{vol[GL(k)]}} \frac{\delta(C_{mi}C_{ni}) \prod_{m=1}^{k} \delta^{2|3}(C_{mi}\Lambda_i)}{M_1(C)M_2(C)\cdots M_k(C)}$$

$C =$



$GL(k) \times O(2k)$

N=6 SUSY crucial

$$M_i(C) = \epsilon^{m_1\cdots m_k} C_{m_1(i)} C_{m_2(i+1)} \cdots C_{m_k(i+k-1)}$$

Grassmannian Integral in 3d

$$\mathcal{A}_{2k}(\Lambda) = \int \frac{d^{k\times 2k}C}{\mathrm{vol}[\mathrm{GL(k)}]} \frac{\delta(C_{mi}C_{ni}) \prod_{m=1}^{k} \delta^{2|3}(C_{mi}\Lambda_i)}{M_1(C)M_2(C)\cdots M_k(C)}$$

- Superconformal symmetry

$$\Lambda\frac{\partial}{\partial\Lambda} \rightarrow \text{ manifest} \qquad \frac{\partial^2}{\partial\Lambda\partial\Lambda} \rightarrow \text{ killed by } \delta(C\cdot C^T)$$

$$C^T\widehat{C} + \widehat{C}^T C = I_{2k\times 2k} \qquad \Lambda\Lambda \rightarrow \Lambda^T\cdot(C^T\widehat{C} + \widehat{C}^T C)\cdot\Lambda = 0$$

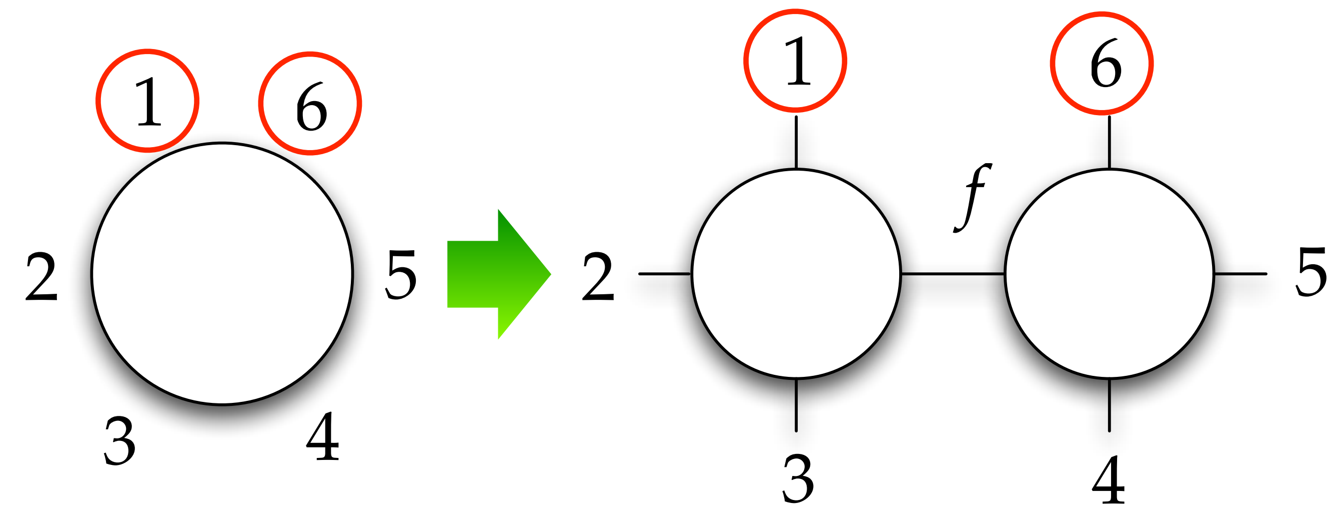
- Cyclic symmetry $\mathcal{A}_{2k}(1,2,\cdots,2k) = (-1)^{k-1}\mathcal{A}_{2k}(3,4,\cdots,2k,1,2)$

$$C\cdot C^T = 0 \implies M_i M_{i+1} = (-1)^{k-1} M_{i+k} M_{i+1+k}$$

6-point amplitude [Bargheer,Loebbert,Meneghelli 10][Gang,Huang,Koh,SL,Lipstein 10]

- Factorization gauge

$$C = \begin{pmatrix} 1 & 0 & 0 & c_{14} & c_{15} & c_{16} \\ 0 & 1 & 0 & c_{24} & c_{25} & c_{26} \\ 0 & 0 & 1 & c_{34} & c_{35} & c_{36} \end{pmatrix}$$



- 6-fermion amplitude

$$A_{6\psi} = +i\frac{\left(\langle 13\rangle\langle 46\rangle + i\langle 2|p_{123}|5\rangle\right)^3}{(p_{123})^2\left(\langle 23\rangle\langle 56\rangle + i\langle 1|p_{123}|4\rangle\right)\left(\langle 12\rangle\langle 45\rangle + i\langle 3|p_{123}|6\rangle\right)} - i\frac{\left(\langle 13\rangle\langle 46\rangle - i\langle 2|p_{123}|5\rangle\right)^3}{(p_{123})^2\left(\langle 23\rangle\langle 56\rangle - i\langle 1|p_{123}|4\rangle\right)\left(\langle 12\rangle\langle 45\rangle - i\langle 3|p_{123}|6\rangle\right)}.$$

shows the factorization channel clearly: Check via recursion relation!

$$A_6(1,2,3,4,5,6) \to A_4(1,2,3,f)\frac{1}{(p_{123})^2}A_4(f,4,5,6) + (\text{regular}).$$

From Grassmannian to twistor string - in 4d

- Grassmannian : contour integral

deformation of integrand

partial localization

Veronese map: $\mathrm{Gr}(2,n) \to \mathrm{Gr}(k,n)$

$$\sigma \quad \to \quad C[\sigma] = \begin{pmatrix} a_1^{k-1} & a_2^{k-1} & \cdots & a_n^{k-1} \\ a_1^{k-2}b_1 & a_2^{k-2}b_2 & \cdots & a_n^{k-2}b_n \\ \vdots & \vdots & & \vdots \\ b_1^{k-1} & b_2^{k-1} & \cdots & b_n^{k-1} \end{pmatrix}$$

[Arkani-Hamed,Bourjailly,Cachazo,Trnka 09]
[Bourjailly,Trnka,Volovich,Wen 10]

[Witten 03][Roiban,Spradlin,Volovich 04]

- Twistor string : integration over degree $(k-1)$ curves in $\mathbb{CP}^{3|4}$.

$$\sigma = \begin{pmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \end{pmatrix} / GL(2,\mathbb{C}) \in \mathrm{Gr}(2,n) \simeq \begin{matrix} n \text{ points on } \mathbb{CP}^1 \\ / \, SL(2,\mathbb{C}) \end{matrix}$$

Twistor string integral for ABJM amplitudes [Huang, SL 12]

$$\mathcal{A}_{2k}(\Lambda) = \int \frac{d^{2\times 2k}\sigma}{\text{vol}[\text{GL}(2)]} \frac{J\,\Delta\,\prod_{m=1}^{k}\delta^{2|3}(C_{mi}[\sigma]\Lambda_i)}{(12)(23)\cdots(2k,1)}$$

$$\sigma = \begin{pmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \end{pmatrix}, \quad (ij) = a_i b_j - a_j b_i\,, \quad C_{mi}[\sigma] = a_i^{k-m} b_i^{m-1}$$

$$\Delta = \prod_{j=1}^{2k-1} \delta\left(\sum_{i=1}^{2k} a_i^{2k-1-j} b_i^j\right), \quad J = \frac{\det_{1\le i,j\le 2k-1}\left(a_i^{2k-1-j} b_i^j\right)}{\prod_{1\le i<j\le k}(2i-1,2j-1)}$$

$\dim[\text{Gr}(2,2k)] - \dim(\Delta) = 2k-3$

cyclic symmetry

8-point amplitude

- Deform the denominator while preserving the residue

$$\frac{1}{M_1 M_2 M_3 M_4} \to \frac{(4631)(4712)}{M_1 M_3 [M_2 M_4 (4631)(4712) - M_1 M_3 (4167)(4175)]}$$

- Deformed denominator and part of the null constraint $\delta(C_{mi} C_{ni})$ localizes the integral to the image of the Veronese map.

- General prescription for the contour?

- Twistor string realization?

Twistor string realization

- Rational map for 4d SYM and SUGRA
  dressing by wave function
  dimensional reduction

[Cachazo,He,Yuan 13]

- Open string theory on $\mathbb{CP}^{2,2|4,1}$
  truncated to ABJM states

[Engelund,Roiban 14]

KK and BCJ in 4d

- Kleiss-Kuijf identities [Kleiss-Kuijf 89]

$$\mathcal{A}_n(p_i, h_i, a_i) = \sum_{\sigma \in S_n/\mathbb{Z}_n} \mathrm{Tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}}) \mathcal{A}_n(\sigma(1^{h_1}), \cdots, \sigma(n^{h_n}))$$

$$f^{abc} = \mathrm{Tr}(T^a T^b T^c) - \mathrm{Tr}(T^b T^a T^c), \quad f^{abe} f_e{}^{cd} + f^{bce} f_e{}^{ad} + f^{cae} f_e{}^{bd} = 0$$

#(independent amplitudes) : $(n-1)! \to (n-2)!$

- Bern-Carrasco-Johansson identities [Bern,Carrasco,Johansson 08]

$$\mathcal{A}_4 = \frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}$$

$$c_s + c_t + c_u = 0 \quad \text{color (Jacobi)}$$

$$n_s + n_t + n_u = 0 \quad \text{kinematics}$$

$$\left(\mathcal{A}_n(\text{gauge}) = \sum_I \frac{c_I n_I}{D_I} \implies \mathcal{M}_n(\text{gravity}) = \sum_I \frac{n_I \tilde{n}_I}{D_I} \right) \quad \text{BCJ doubling}$$

#(independent amplitudes) : $(n-2)! \to (n-3)!$

KK in 3d

- KK identity follows from bi-fundamental algebra [Bagger,Lambert 08]

$$f^{\bar{a}b\bar{c}d} = \mathrm{Tr}(\bar{M}^{\bar{a}} M^b \bar{M}^{\bar{c}} M^d - \bar{M}^{\bar{c}} M^b \bar{M}^{\bar{a}} M^d)$$

"structure constant"

$$f^{\bar{a}b\bar{c}g} f_g{}^{d\bar{e}f} - f^{\bar{a}d\bar{c}g} f_g{}^{b\bar{e}f} = f^{\bar{e}f\bar{a}g} f_g{}^{b\bar{c}d} - f^{\bar{e}f\bar{c}g} f_g{}^{b\bar{a}d}$$

"fundamental identity"

- Independent color basis counted; no closed-form formula known

external legs	4	6	8	10	12	$n = 2k$
cyclic/reflection	1	6	72	1440	43200	$k!(k-1)!/2$
KK identity	1	5	57	1144	*	*

KK from twistor string [Huang,SL 12][Huang,Johansson,SL 13]

$$\mathcal{A}_{2k}(\Lambda) = \int \frac{d^{2\times 2k}\sigma}{\text{vol}[\text{GL}(2)]} \frac{J\,\Delta\,\prod_{m=1}^{k}\delta^{2|3}(C_{mi}[\sigma]\Lambda_i)}{(12)(23)\cdots(2k,1)}$$

- Under (non-cyclic) permutations, the only non-trivial part is

$$D_{2k}(1,2,\ldots,2k) = \frac{1}{(12)(34)\cdots(2k,1)}$$

- All KK identities can be constructed
  by recursive use of elementary identities:

$$\frac{1}{(ij)} + \frac{1}{(ji)} = 0$$

$$\frac{1}{(ij)(jk)(kl)} + \text{(cyclic)} = \frac{(ij)+(jk)+(kl)+(li)}{(ij)(jk)(kl)(li)} = 0$$

BCJ in 3d

- Yang-Mills : BCJ relation in any dimension (on-shell)
- BLG : BCJ relation in 3d only

[Bargheer,He,McLoughlin 12]

[Huang,Johansson 12]

- 4- and 6-point
  BLG doubling for YM and CSm agree!
- 8-point of higher
  BCJ for BLG, but NOT ABJM

SUSY truncation

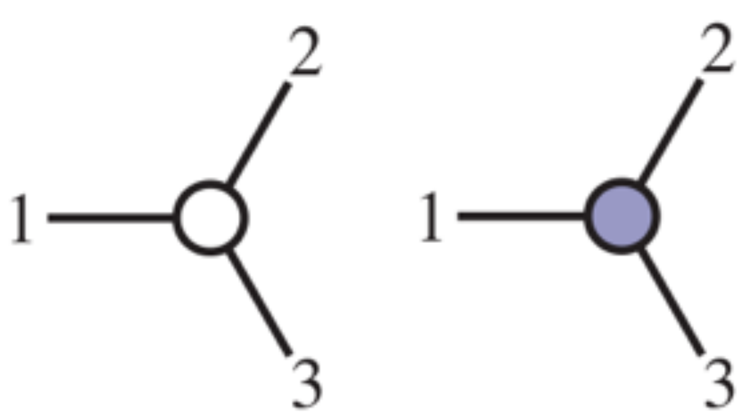
- Bonus relations and more

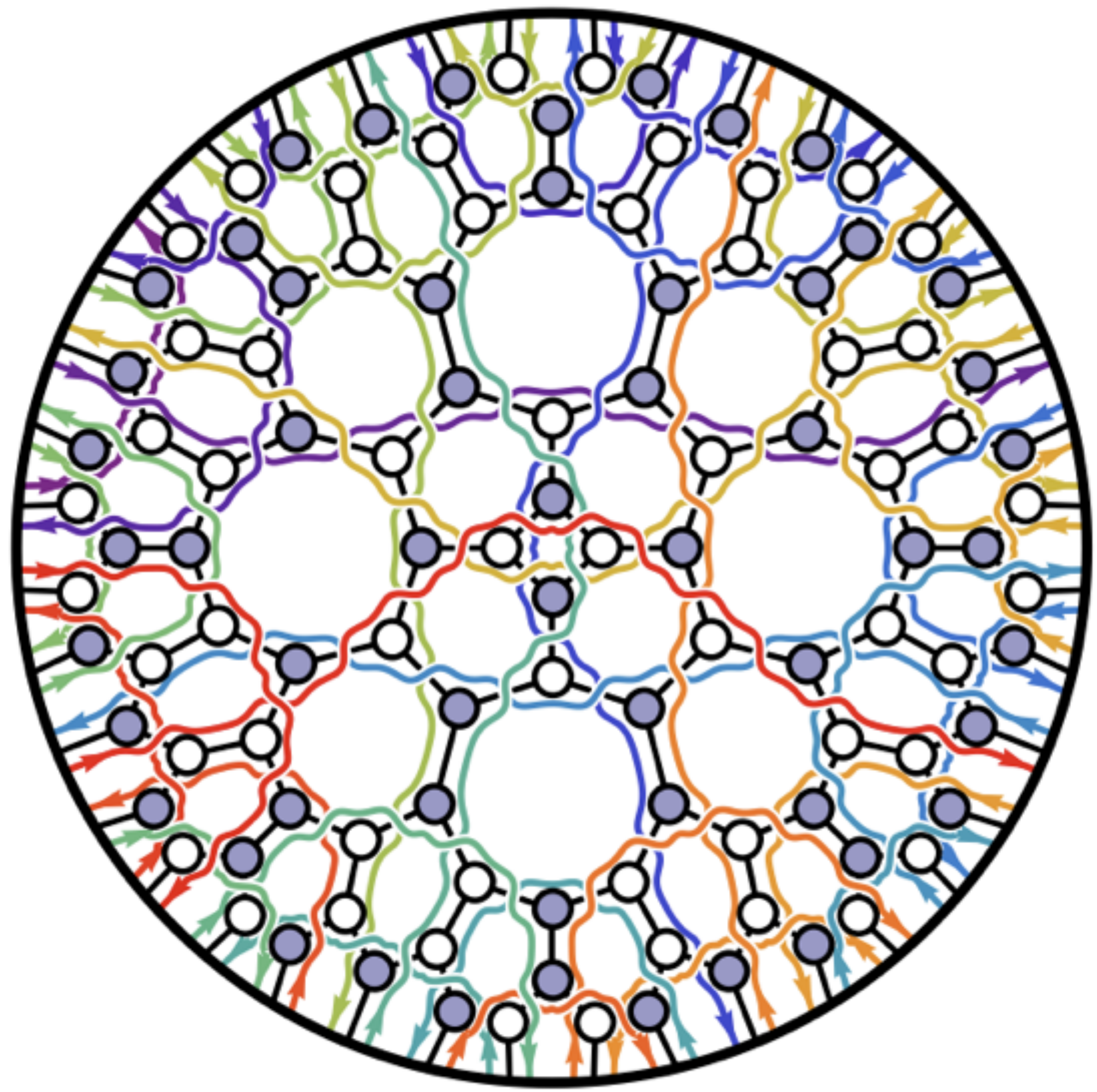
On-shell diagrams and positive Grassmannian

[Arkani-Hamed,Bourjaily, Cachazo,Goncharov,Postnikov,Trnka 12]

All tree amplitudes & loop integrands from 3-point amplitudes via “BCFW bridging”

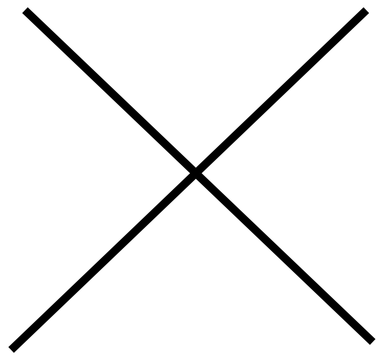
Permutation, positive Grassmannian





On-shell diagrams in 3d [ABCGPT 12][Huang, Wen1309][Kim, SL 1402]

- Building blocks



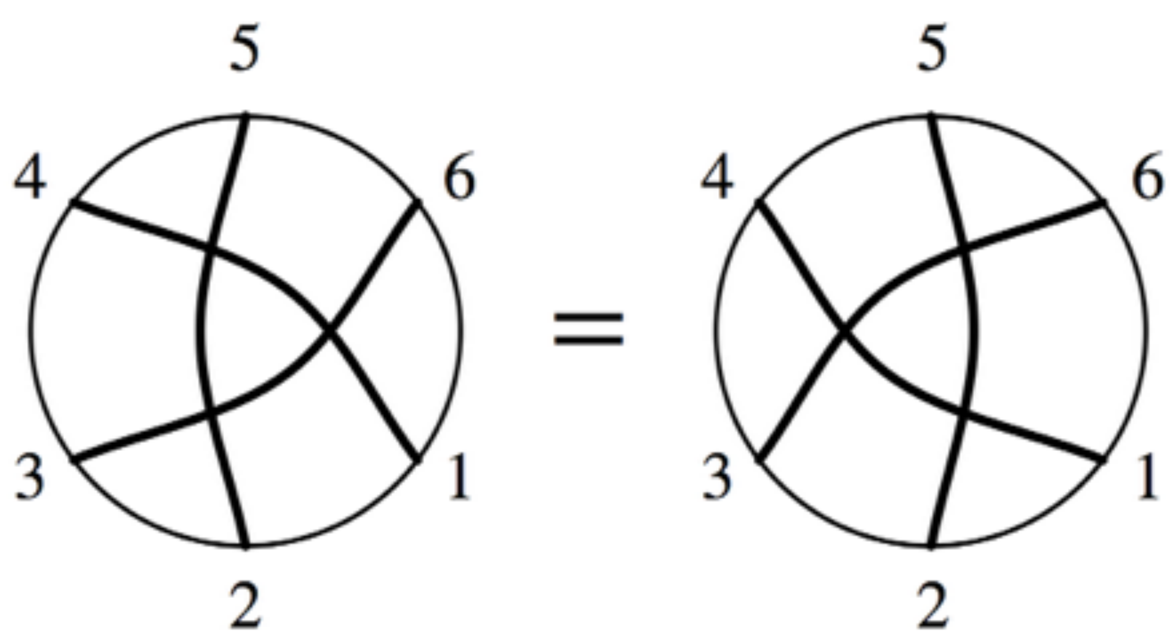
$$\frac{\delta^3(P)\delta^6(Q)}{\langle 14\rangle\langle 34\rangle}$$

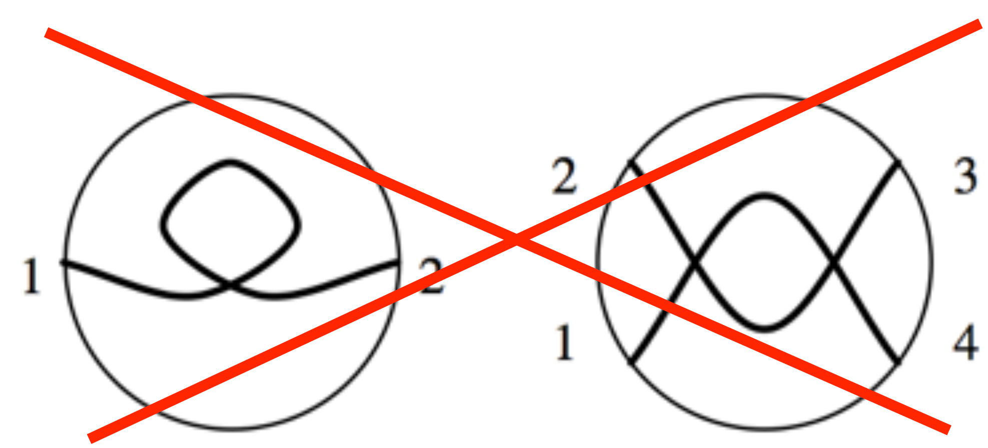
4-point on-shell amplitude

$$\int d^2\lambda d^3\eta$$

unique integral preserving superconformal symmetry

- Pairing of external particles up to Yang-Baxter equivalence





Positive orthogonal Grassmannian [Huang, Wen1309][Kim, SL 1402]

- Complex OG(k) = moduli space of null k-planes in $\mathbb{C}^{2k}$.

$$= O(2k)/U(k)$$

Pure spinor

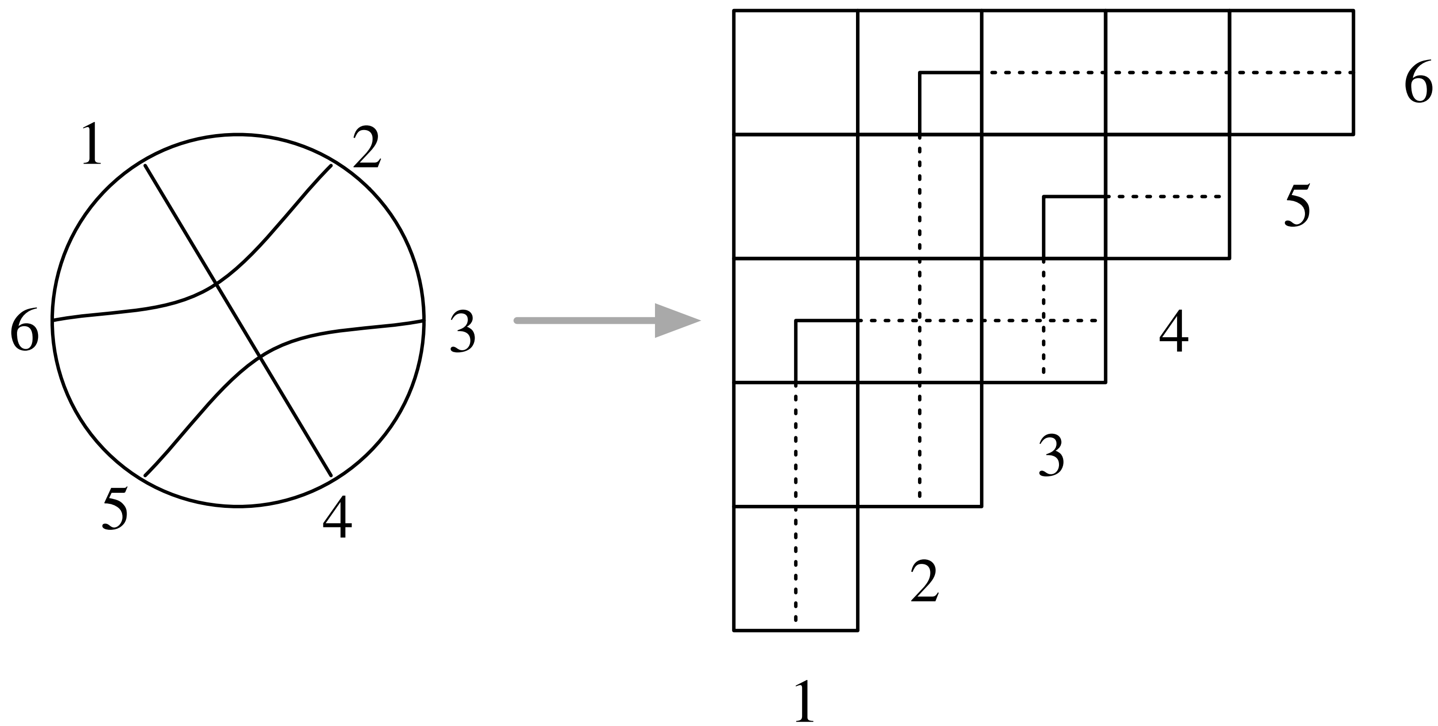
- Real OG(k)

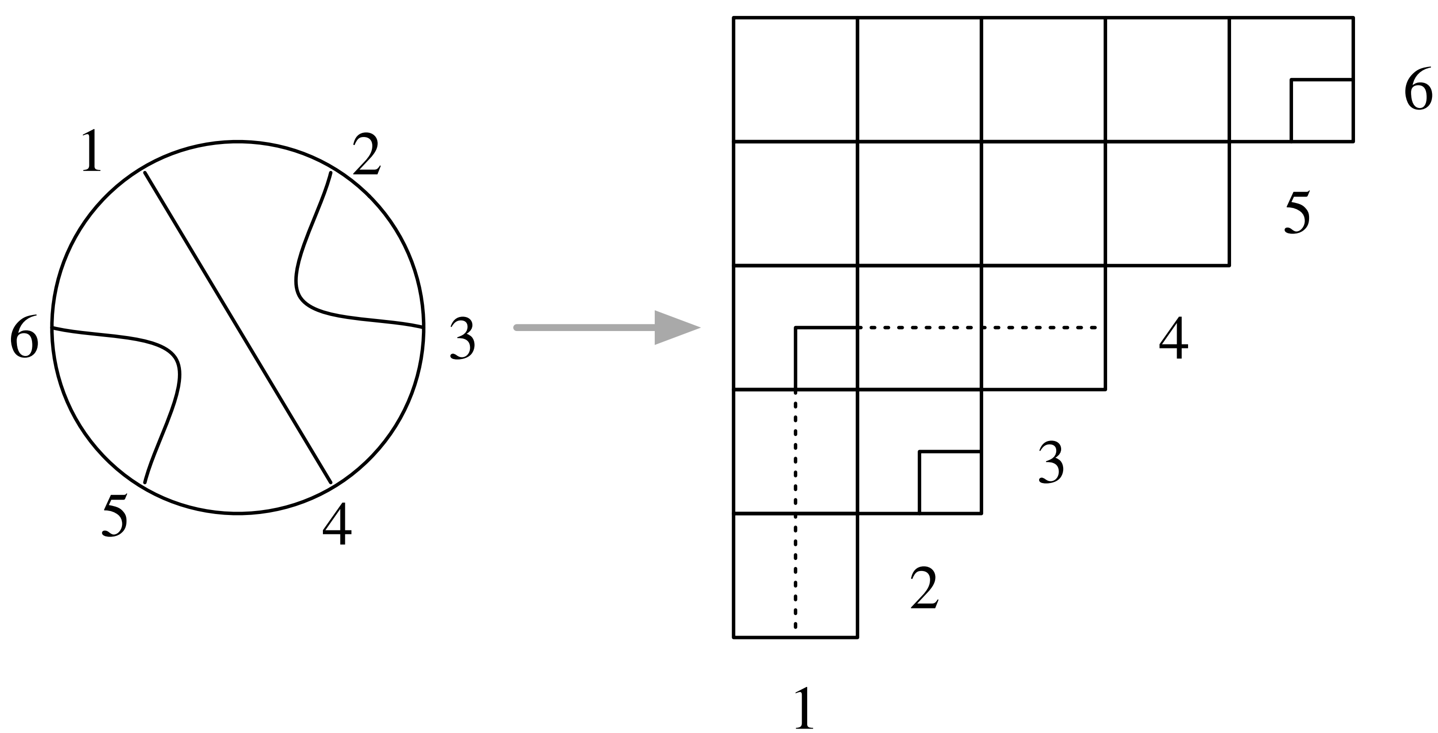
$$\eta = \text{diag}(-,+,\cdots,-,+)\,, \quad C\cdot\eta\cdot C^T = 0\,, \quad C_{mi} \in \mathbb{R}$$

- Positive OG(k)

All ordered minors are positive

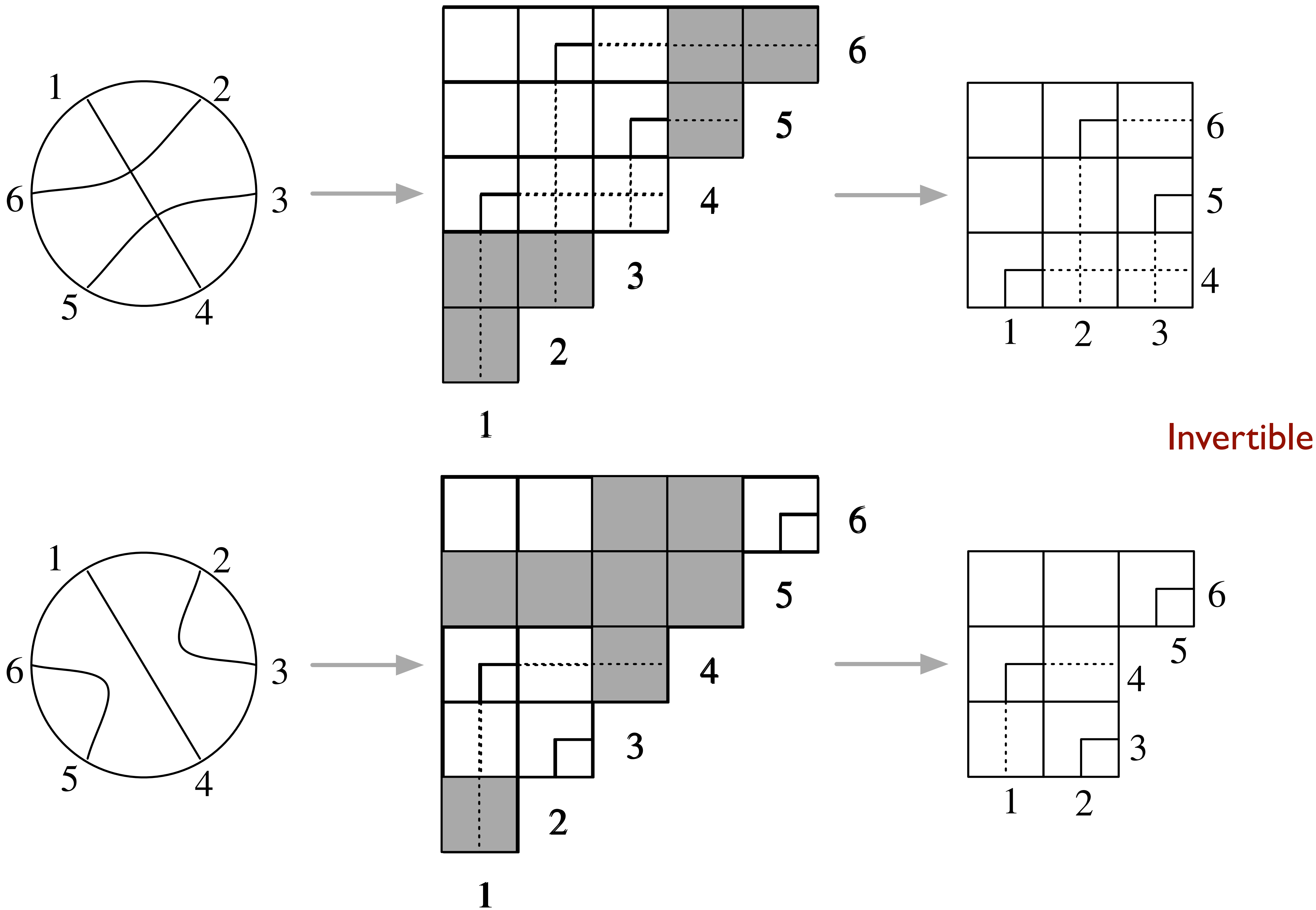
OG tableaux [Joonho Kim, SL 1402]





Fix Yang-Baxter ambiguity
at the expense of
manifest cyclic symmetry

OG tableaux [Joonho Kim, SL 1402]

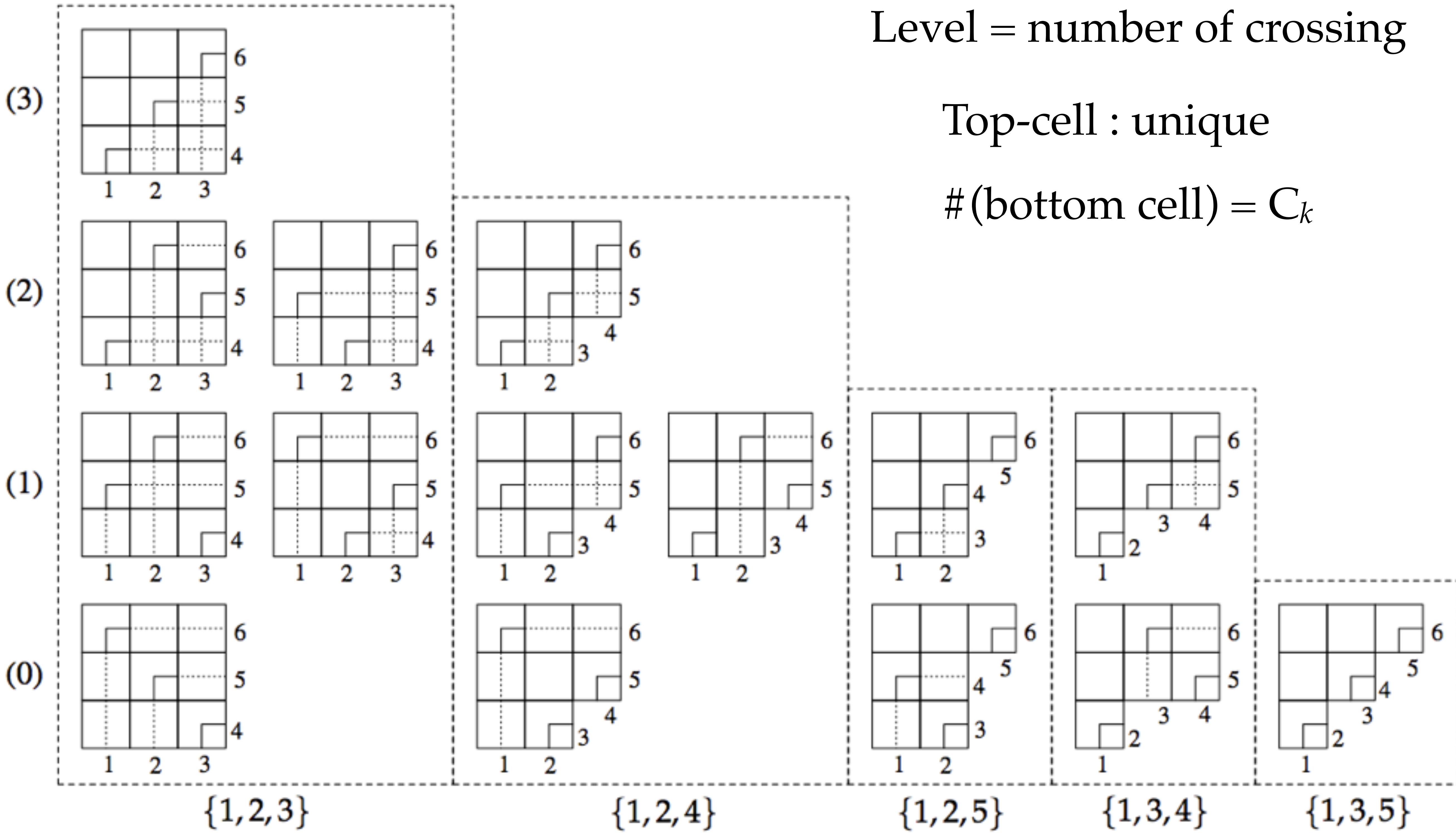


OG tableaux

Level = number of crossing

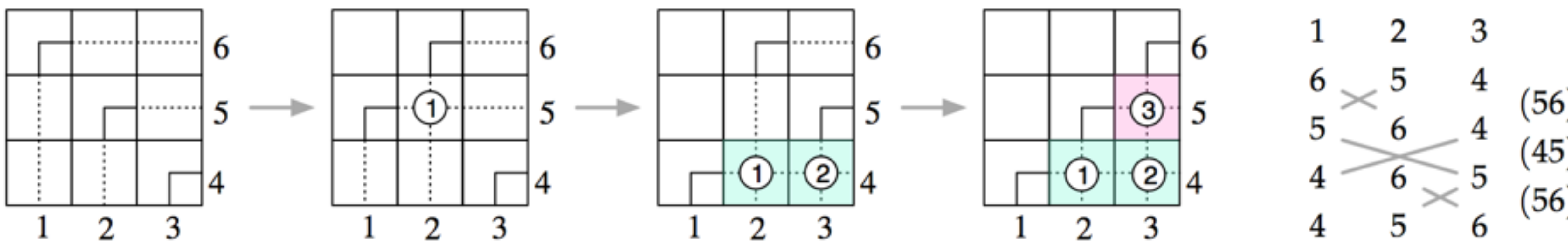
Top-cell : unique

#(bottom cell) = C_k



Canonically positive BCFW coordinates

- Explicit coordinate charts for POG



$$(C_4, C_5, C_6) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1 & s_1 \\ 0 & s_1 & c_1 \end{pmatrix} \begin{pmatrix} c_2 & s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_3 & s_3 \\ 0 & s_3 & c_3 \end{pmatrix}$$

$$= \begin{pmatrix} s_1 s_2 & s_1 c_2 c_3 + c_1 s_3 & c_1 c_3 + s_1 c_2 s_3 \\ -c_1 s_2 & -c_1 c_2 c_3 - s_1 s_3 & -s_1 c_3 - c_1 c_2 s_3 \\ c_2 & s_2 c_3 & s_2 s_3 \end{pmatrix} .$$

$$s_i = \sinh t_i\,, \quad c_i = \cosh t_i\,, \quad t_i \in \mathbb{R}_+$$

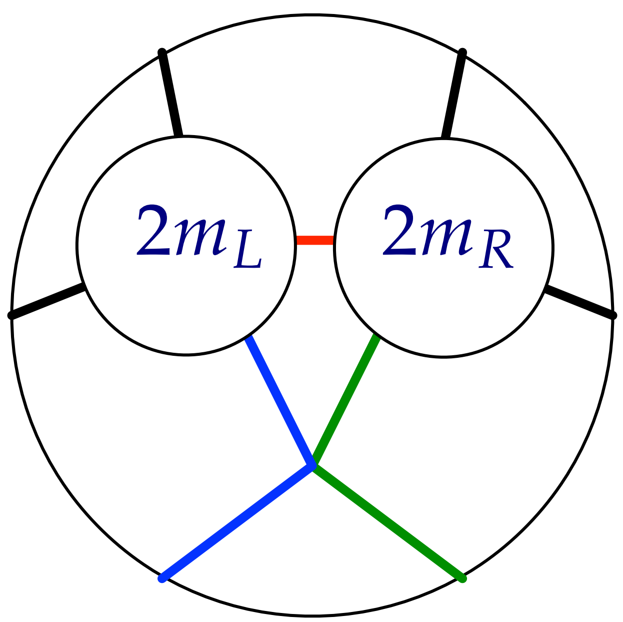
BCFW coordinates and recursion

- Integration measure

$$\int \prod_a \frac{dt_a}{\sinh t_a} = \int \prod_a \frac{dz_a}{z_a} = \int \prod_a d\log z_a \qquad \left(z_a = \tanh \frac{t_a}{2} \right)$$

- Integration contour problem solved

$$\mathcal{A}_{2k} = \sum_{m_L + m_R = k+1}$$



- Alternative coordinates and loop BCFW

[Huang,Wen,Xie 1402]

Combinatorics and topology of POG

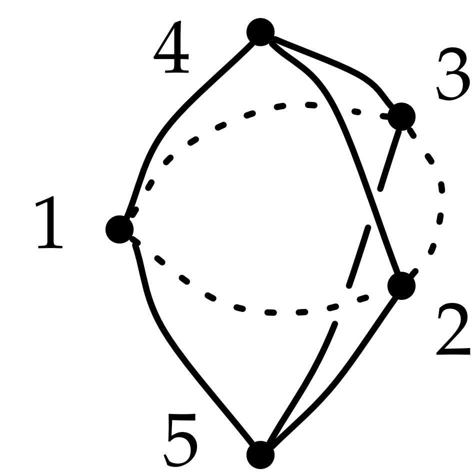
- Generating function

$$T_k(q) = \sum_{l=0}^{k(k-1)/2} T_{k,l} q^l = \frac{1}{(1-q)^k} \sum_{j=-k}^{k} (-1)^j \binom{2k}{k+j} q^{j(j-1)/2}$$

[Riordan 75]

$$T_2(q) = 2 + q$$

$$T_3(q) = 5 + 6q + 3q^2 + q^3$$



- Euler characteristic

$$\chi_k = T_k(-1) = \sum_l (-1)^l T_{k,l} = 1$$

Combinatorics - Eulerian poset : proven! [Lam 1406]

Topology - Ball : conjectured

Open problems - trees

- Dual superconformal (Yangian) symmetry [Drummond,Ferro 10][SL 10] [Bargheer,Loebbert,Meneghelli 10] [Huang,Lipstein 10]

  Momentum twistor ? [Hodges 09]

- Amplitude / Wilson-loop duality [Bianchi,Griguolo,Leoni,Penati,Seminara 14] [Bianchi,Giribet,Leoni,Penati 13]

  Fermionic T-duality in AdS4 ? [Adam,Dekel,Oz 10][Bakhmatov 11] [Bakhmatov,OColgain,Yavartanoo 11]

- Amplituhedron ? [Arkani-Hamed,Trnka 13]

- Scattering equation ? [Cachzo,He,Yuan 13]

- Symplectic (a.k.a. Lagrangian) Grassmannian in 5d / 6d ?

Open problems - loops

Thanks to feedback from Yu-tin Huang

- IR divergence

  4-point 3-loop and 6-point 2-loop [Caron-Huot Huang 12]

  [Chen,Huang 11][Bianchi,Leoni,Mauri,Penati,Santambrogio 11][Bianchi,Leoni 14]

- Unitarity

  1-loop = 0, 2-loop $\neq 0$ [Agarwal,Beisert,McLoughlin 08][Chen,Huang 11] [Bianchi,Leoni,Mauri,Penati,Santambrogio 11]

  Modified unitarity equation? [Jain,Mandlik,Minwalla,Takimi,Wadia,Yokoyama 14]

- Analyticity [Bargheer,Beisert,Loebbert,McLoughlin 12]

  6-point, 1-loop contains $\mathrm{sgn}(\theta_2 - \theta_1)$

  Non-perturbative resolution? [Caron-Huot Huang 12]