Chiral Algebras and the Superconformal Bootstrap in Four and Six Dimensions

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Based on work with
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SuperConformal Field Theories in $d > 2$

Fast-growing body of results:

- Many new models, most with no known Lagrangian description.
- A hodgepodge of techniques:
  localization, integrability, effective actions on moduli space.
  Powerful but with limited scope.

  Conformal symmetry not fully used.

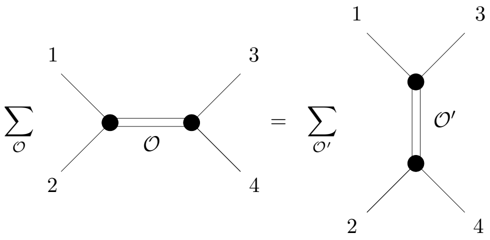
We advocate a more systematic and universal approach.

Conformal Bootstrap

Abstract algebra of local operators

$$\mathcal{O}_1(x)\mathcal{O}_2(0) = \sum_k c_{12k}(x)\mathcal{O}_k(0)$$

subject to unitarity and crossing constraints



Since 2008, successful *numerical* approach in any d.
See Simmons-Duffin's talk.

Two sorts of questions

What is the space of consistent SCFTs in $d \leqslant 6$?

For maximal susy, well-known list of theories.
Is the list complete?

What is the list with less susy?

Can we bootstrap concrete models?

The bootstrap should be particularly powerful for models uniquely cornered by few discrete data.
Only method presently available for finite N, non-Lagrangian theories, such as the $6d$ (2,0) SCFT.

More technically, not clear how much susy can really help.

A natural question:

> *Do the bootstrap equations in $d > 2$ admit a solvable truncation for superconformal theories?*

The answer is Yes for large classes of theories:

(A) Any $d = 4$, $\mathcal{N} \geqslant 2$ or $d = 6$, $\mathcal{N} = (2, 0)$ SCFT admits a subsector $\cong$ $2d$ chiral algebra.

(B) Any $d = 3$, $\mathcal{N} \geqslant 4$ SCFT admits a subsector $\cong$ $1d$ TQFT.

Beem Lemos Liendo Peelaers LR van Rees, Beem LR van Rees

In this talk, we'll focus on the rich structures of (A).

Bootstrapping in two steps

For this class of SCFTs, crossing equations split into

(1) Equations that depend only on the intermediate BPS operators.
Captured by the $2d$ chiral algebra.

(2) Equations that also include intermediate non-BPS operators.

(1) are tractable and determine an infinite amount of CFT data, given flavor symmetries and central charges.

This is essential input to the full-fledged bootstrap (2), which can be studied numerically.

Meromorphy in $\mathcal{N} = 2$ or $(2,0)$ SCFTs

Fix a plane $\mathbb{R}^2 \subset \mathbb{R}^d$, parametrized by $(z, \bar{z})$.

Claim : $\exists$ subsector $\mathcal{A}_\chi = \{\mathcal{O}_i(z_i, \bar{z}_i)\}$ with meromorphic

$$\langle \mathcal{O}_1(z_1, \bar{z}_1)\, \mathcal{O}_2(z_2, \bar{z}_2) \ldots \mathcal{O}_n(z_n, \bar{z}_n) \rangle = f(z_i)\,.$$

Rationale: $\mathcal{A}_\chi \equiv$ cohomology of a nilpotent $\mathbb{Q}$,

$$\mathbb{Q} = \mathcal{Q} + \mathcal{S}\,,$$

$\mathcal{Q}$ Poincaré, $\mathcal{S}$ conformal supercharges.

$\bar{z}$ dependence is $\mathbb{Q}$-exact: cohomology classes $[\mathcal{O}(z, \bar{z})]_{\mathbb{Q}} \rightsquigarrow \mathcal{O}(z)$.

Analogous to the $d = 4$, $\mathcal{N} = 1$ chiral ring:
cohomology classes $[\mathcal{O}(x)]_{\tilde{\mathcal{Q}}_{\dot{\alpha}}}$ are x-independent.

Cohomology

At the origin of $\mathbb{R}^2$, $\mathbb{Q}$-cohomology $\mathcal{A}_\chi$ easy to describe.

$\mathcal{O}(0,0) \in \mathcal{A}_\chi \leftrightarrow \mathcal{O}$ obeys the chirality condition

$$\frac{\Delta - \ell}{2} = R$$

Δ conformal dimension, ℓ angular momentum on $\mathbb{R}^2$,
R Cartan generator of $SU(2)_R \subset$ full $\mathcal{R}$ symmetry

- $\mathcal{R} = SU(2)_R \times U(1)_r$ for $d = 4$, $\mathcal{N} = 2$
- $\mathcal{R} = SO(5)$ for $(2,0)$: $SU(2)_R \cong SO(3)_R \subset SO(5)$.

$$[\mathbb{Q}\,, \mathfrak{sl}(2)] = 0 \quad \text{but} \quad [\mathbb{Q}\,, \overline{\mathfrak{sl}(2)}] \neq 0$$

To define $\mathbb{Q}$-closed operators $\mathcal{O}(z, \bar{z})$ away from origin, we twist the right-moving generators by $SU(2)_R$,

$$\widehat{L}_{-1} = \bar{L}_{-1} + \mathcal{R}^- \,, \quad \widehat{L}_0 = \bar{L}_0 - \mathcal{R} \,, \quad \widehat{L}_1 = \bar{L}_1 - \mathcal{R}^+ \,.$$

$$\widehat{\mathfrak{sl}(2)} = \{\mathbb{Q}, \dots\}$$

$\mathbb{Q}$-closed operators are "twisted-translated"

$$\mathcal{O}(z, \bar{z}) = e^{zL_{-1} + \bar{z}\widehat{L}_{-1}}\, \mathcal{O}(0)\, e^{-zL_{-1} - \bar{z}\widehat{L}_{-1}} \,.$$

$SU(2)_R$ orientation correlated with position on $\mathbb{R}^2$.

Chirality condition $\frac{\Delta - \ell}{2} - R = 0 \Leftrightarrow \widehat{L}_0 = 0$

By the usual formal argument, the $\bar{z}$ dependence is exact,

$$[\mathcal{O}(z,\bar{z})]_{\mathbb{Q}} \quad \rightsquigarrow \quad \mathcal{O}(z)\ .$$

Cohomology classes define left-moving $2d$ operators $\mathcal{O}_i(z)$, with conformal weight

$$h = R + \ell.$$

They are closed under OPE,

$$\mathcal{O}_1(z)\mathcal{O}_2(0) = \sum_k \frac{c_{12k}}{z^{h_1+h_2-h_k}}\,\mathcal{O}_k(0)\,.$$

$\mathcal{A}_\chi$ has the structure of a $2d$ chiral algebra

Example: free $(2,0)$ tensor multiplet

$$\Phi_I \,, \quad \lambda_{aA} \,, \quad \omega^+_{ab}$$

$I = SO(5)_R$ vector index.

Scalar in $SO(3)_R \subset SO(5)_R$ h.w. is only field obeying $\Delta - \ell = 2R$

$$\Phi_{h.w.} = \frac{\Phi_1 + i\Phi_2}{\sqrt{2}} \,, \quad \Delta = 2R = 2 \,, \quad \ell = 0 \,.$$

Cohomology class of twisted-translated field

$$\Phi(z) := \left[\Phi_{h.w.}(z,\bar{z}) + \bar{z}\Phi_3(z,\bar{z}) + \bar{z}^2\Phi^*_{h.w.}(z,\bar{z})\right]_{\mathbb{Q}}$$

$$\Phi(z)\,\Phi(0) \sim \bar{z}^2\Phi^*_{h.w.}(z,\bar{z})\,\Phi_{h.w.}(0) \sim \frac{\bar{z}^2}{z^2\,\bar{z}^2} = \frac{1}{z^2} \,.$$

$\Phi(z)$ is an $\mathfrak{u}(1)$ affine current, $\Phi(z) \rightsquigarrow J_{\mathfrak{u}(1)}(z)$.

χ_6 : 6d (2,0) SCFT $\longrightarrow$ 2d Chiral Algebra.

- Global $\mathfrak{sl}(2) \to$ Virasoro, indeed $T(z) := [\mathcal{O}_{\mathbf{14}}(z, \bar{z})]_{\mathbb{Q}}$, with $\mathcal{O}_{\mathbf{14}}$ the stress-tensor multiplet superprimary.

$$c_{2d} = c_{6d}$$

  in normalizations where c_{6d} (free tensor) $\equiv 1$

- All $\frac{1}{2}$-BPS operators $(\Delta = 2R)$ are in $\mathbb{Q}$ cohomology.

  Generators of the $\frac{1}{2}$-BPS ring $\to$ generators of the chiral algebra.

- Some semi-short multiplets also play a role.

Chiral algebra for $(2,0)$ theory of type A_{N-1}

One $\frac{1}{2}$-BPS generator each of dimension $\Delta = 4, 6, \ldots 2N$

$$\Downarrow$$

One chiral algebra generator each of dimension $h = 2, 3, \ldots N$.

Most economical scenario: these are all the generators.
Check: the superconformal index computed by Kim[3] is reproduced.

Plausibly a unique solution to crossing for this set of generators.

- The chiral algebra of the A_{N-1} theory is $\mathcal{W}_N$, with

$$c_{2d} = 4N^3 - 3N - 1\,.$$

General claim

- For the $(2,0)$ SCFT labelled by the simply-laced Lie algebra $\mathfrak{g}$, the chiral algebra is $\mathcal{W}_{\mathfrak{g}}$, with

$$c_{2d}(\mathfrak{g}) = 4 d_{\mathfrak{g}} h^{\vee}_{\mathfrak{g}} + r_{\mathfrak{g}} \ .$$

Connection with the AGT correspondence.
$c_{2d}(\mathfrak{g})$ matches Toda central charge for $b = 1$.

Half-BPS 3pt functions of $(2,0)$ SCFT

OPE of $\mathcal{W}_{\mathfrak{g}}$ generators $\Rightarrow$ half-BPS 3pt functions of SCFT.

Let us check the result at large N.

$W_{N\to\infty}$ with $c_{2d} \sim 4N^3$ $\rightarrow$ a *classical* W-algebra.
(Gaberdiel Hartman, Campoleoni Fredenhagen Pfenninger)

We find

$$C(k_1,k_2,k_3) = \frac{2^{2\alpha-2}}{(\pi N)^{\frac{3}{2}}}\Gamma\left(\frac{\alpha}{2}\right)\left(\frac{\Gamma\left(\frac{k_{123}+1}{2}\right)\Gamma\left(\frac{k_{231}+1}{2}\right)\Gamma\left(\frac{k_{312}+1}{2}\right)}{\sqrt{\Gamma(2k_1-1)\Gamma(2k_2-1)\Gamma(2k_3-1)}}\right)$$

$k_{ijk} \equiv k_i + k_j - k_k$, $\alpha \equiv k_1 + k_2 + k_3$,
in precise agreement with calculation in $11d$ sugra on $AdS_7 \times S^4$!
(Corrado Florea McNees, Bastianelli Zucchini)

$1/N$ corrections in W_N OPE $\Rightarrow$ quantum M-theory corrections.

χ_4 : 4d $\mathcal{N} = 2$ SCFT $\longrightarrow$ 2d Chiral Algebra.

- Global $\mathfrak{sl}(2) \to$ Virasoro
  $T(z) := [\mathcal{J}_R(z, \bar{z})]_{\mathbb{Q}}$, the $SU(2)_R$ conserved current.

$$c_{2d} = -12\, c_{4d}$$

  $c_{4d} \equiv$ Weyl^2 conformal anomaly coefficient.

- Global flavor $\to$ Affine symmetry
  $J(z) := [M(z, \bar{z})]_{\mathbb{Q}}$, the moment map operator.

$$k_{2d} = -\frac{k_{4d}}{2}$$

- $4d$ Higgs branch generators $\to$ chiral algebra generators.
  Higgs branch relations $\equiv$ chiral algebra null states!

Bootstrap of the full 4pt function

$$\mathcal{A}^{\mathcal{I}_1\mathcal{I}_2\mathcal{I}_3\mathcal{I}_4}(z,\bar{z}) = \langle\, \mathcal{O}^{\mathcal{I}_1}(0)\; \mathcal{O}^{\mathcal{I}_2}(z,\bar{z})\; \mathcal{O}^{\mathcal{I}_3}(1)\; \mathcal{O}^{\mathcal{I}_4}(\infty)\,\rangle$$

$\mathcal{I}=$ index of some $SU(2)_R$ irrep.

Associated chiral algebra correlator

$$f(z)=\langle\mathcal{O}(0)\mathcal{O}(z)\mathcal{O}(1)\mathcal{O}(\infty)\rangle\,,\; \mathcal{O}(z) = [u_{\mathcal{I}}(\bar{z})\mathcal{O}^{\mathcal{I}}(z,\bar{z})]_{\mathbb{Q}}\;.$$

Double-OPE expansion

$$\mathcal{A}(z,\bar{z}) = \sum p_i^{\text{short}}\; \mathcal{G}_i^{\text{short}}(z,\bar{z}) + \sum p_k^{\text{long}}\; \mathcal{G}_k^{\text{long}}(z,\bar{z})$$

$\mathcal{G}_i =$ superconformal blocks $=\sum_{\text{finite}}$ conformal blocks $G_{\Delta,\ell}$.

The short part can be entirely reconstructed from $f(z)$.

Symmetries & central charges $\mathbf{c}$

$\downarrow$

Chiral algebra correlator $f(z; \mathbf{c})$

$\downarrow$

Short spectrum and OPE coefficients $p_i^{\text{short}}(\mathbf{c})$
(unique assuming no higher-spin symmetry)

$\downarrow$

$\mathcal{A}^{\text{short}}(z, \bar{z}; \mathbf{c})$

$\downarrow$

Finally, numerical bootstrap of $\mathcal{A}^{\text{long}}(z, \bar{z})$

Unitarity $\Rightarrow p_i^{\rm short}(\mathbf{c}) \geqslant 0 \Rightarrow$ novel bounds on central charges.

For example, in any interacting $d = 4$, $\mathcal{N} = 2$ SCFT with flavor group G_F,

$$\frac{\dim G_F}{c_{4d}} \geqslant \frac{24 h^\vee}{k_{4d}} - 12 \ .$$

c_{4d} = Weyl^2 conformal anomaly,
k_{4d} = flavor central charge.

Bootstrap Sum Rule

$$\sum_{\text{long } superprimaries} p_{\Delta,\ell}\mathcal{F}_{\Delta,\ell}(z,\bar{z}) + \mathcal{F}^{\text{short}}(z,\bar{z};c) = 0$$

$\mathcal{F}_{\Delta,\ell} \equiv \mathcal{G}_{\Delta,\ell} - \mathcal{G}^{\times}_{\Delta,\ell}$ is the *super*conformal block minus its crossing.

Contrast with sum rule from Rattazzi Rychkov Tonni Vichi

$$\sum_{\text{primaries}} p_{\Delta,\ell}\, F_{\Delta,\ell}(z,\bar{z}) + F^{\text{identity}}(z,\bar{z}) = 0$$

$F_{\Delta,\ell} \equiv G_{\Delta,\ell} - G^{\times}_{\Delta,\ell}$ is the conformal block minus its crossing.

Three paradigmatic cases

- $d = 6$, $(2, 0)$: stress-tensor-multiplet 4pt function.
  Beem Lemos LR van Rees, to appear
- $d = 4$, $\mathcal{N} = 4$: stress-tensor multiplet 4pt function.
  Beem LR van Rees
  Alday Bissi
- $d = 4$, $\mathcal{N} = 2$: moment-map 4pt function.
  Beem Lemos Liendo LR van Rees, to appear

Bootstrap of stress-tensor multiplet 4pt in $(2,0)$

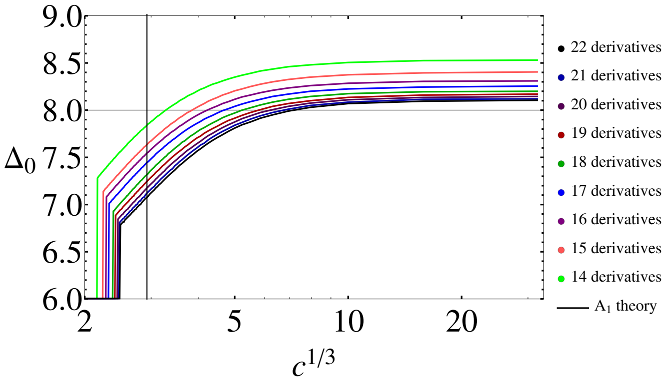


Figure : Upper bound for the dimension Δ_0 of the leading-twist unprotected operator of spin $\ell = 0$, as a function of the anomaly c. Within numerical errors, the bound at large c agrees with the dimension (=8) of the "double-trace" operator : $\mathcal{O}_{\mathbf{14}}\mathcal{O}_{\mathbf{14}}$:

Bootstrap of stress-tensor multiplet 4pt in $(2,0)$

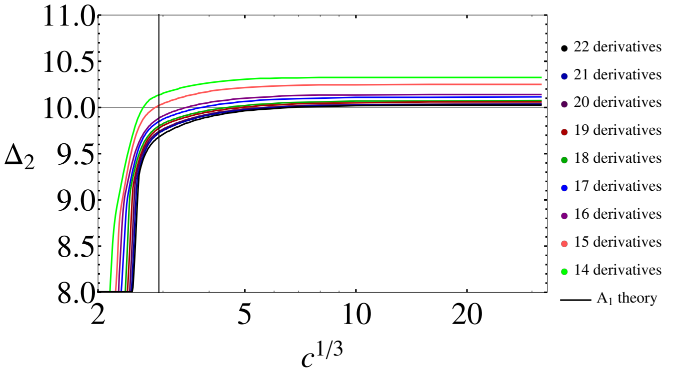


Figure : Upper bound for the dimension Δ_2 of the leading-twist unprotected operator of spin $\ell = 2$, as a function of the anomaly c. Within numerical errors, the bound at large c agrees with the dimension (=10) of the "double-trace" operator : $\mathcal{O}_{\mathbf{14}} \partial^2 \mathcal{O}_{\mathbf{14}}$:

Bootstrap of stress-tensor multiplet 4pt in $(2,0)$

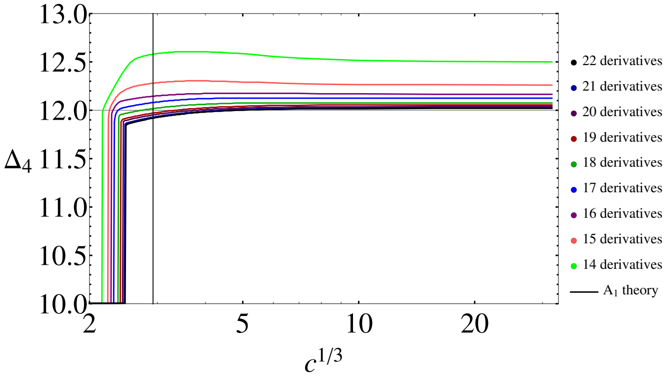


Figure : Upper bound for the dimension Δ_4 of the leading-twist unprotected operator of spin $\ell = 4$, as a function of the anomaly c. Within numerical errors, the bound at large c agrees with the dimension (=12) of the "double-trace" operator : $\mathcal{O}_{\mathbf{14}}\partial^4\mathcal{O}_{\mathbf{14}}$:

Bootstrap of stress-tensor multiplet 4pt in $\mathcal{N} = 4$

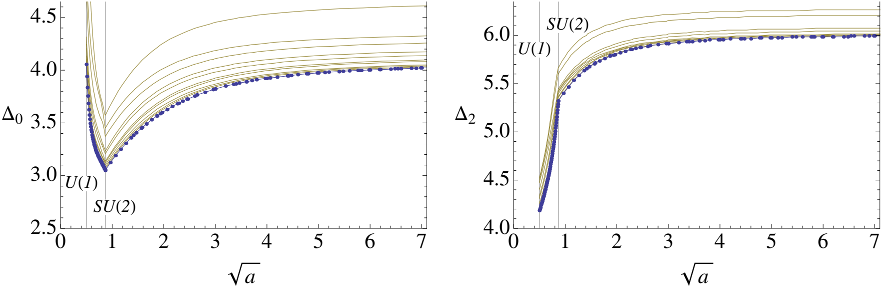


Figure : Bounds for the scaling dimension of the leading-twist unprotected operator of spin $\ell = 0, 2$, as a function of the anomaly a. For $a \to \infty$, saturated by $AdS_5 \times S^5$ sugra, including $1/a$ corrections. In planar $\mathcal{N} = 4$ SYM for large 't Hooft coupling, leading-twist unprotected operators are double-traces of the form $\mathcal{O}_s = \mathcal{O}_{\mathbf{20'}} \partial^s \mathcal{O}_{\mathbf{20'}}$.

Bootstrap of moment map 4pt in $d = 4$, $\mathcal{N} = 2$

Input:

flavor group G_F, flavor central charge k, conformal anomaly c.

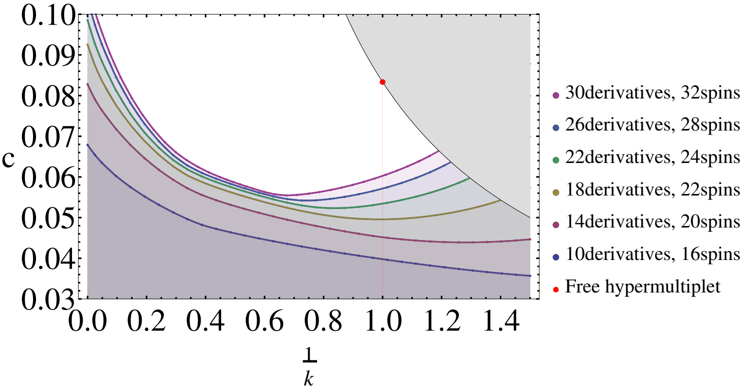


Figure : Exclusion plot in the plane $(\frac{1}{k}, c)$ for a general $\mathcal{N} = 2$ SCFT with $G_F = SU(2)$ flavor symmetry.

Outlook: miniboostrap

- Chiral algebras of the $\mathcal{N} = 2$ SCFTs of class $\mathcal{S}$.
  Generalized TQFT structure.
  Interesting purely mathematical conjectures.
  Beem Peelaers LR van Rees, to appear
- For a given SCFT $\mathcal{T}$, develop systematic tools to characterize $\chi[\mathcal{T}]$ in terms of generators.
- Classification of SCFTs related to classification of "special" chiral algebras.
- Add non-local operators.
  Particularly interesting in $d = 6$: a derivation of AGT?

Outlook: numerical boostrap

- (2, 0) bootstrap: in progress. Stay tuned.
- Exploration of landscape of $\mathcal{N} = 2$ SCFTs, especially non-Lagrangian ones.
- More $d = 4$, $\mathcal{N} = 4$.

Intriguing interplay of mathematical physics and numerical experimentation.