wall crossing redux

PILJIN YI

KORFA INSTITUTE for ADVANCED STUDY

STRINGS 2013

Denef 2002
Denef+Moore 2007
de Boer+El-Showk+Messamah+Van den Bleeken 2008
Sungay Lee+P.Y. 2011
Heeyeon Kim+Jaemo Park+Zhaolong Wang+P.Y. 2011
Sen 2011
Bena+Berkooz+de Boer+El-Showk+Van den Bleeken 2012
Seungjoo Lee+Zhaolong Wang+P.Y. 2012
Manschot+Pioline+Sen 2010/2011/2012/2013

2008 Kontsevich+Soibelman
2008/2009/2010/2011/2012 Gaiotto+Moore+Neitzke
2007/2009 Derksen+Weyman+Zelevinsky
2011 Keller
2011 Alim+Cecotti+Cordova+Espahbodi+Rastogi+Vafa
2012 Xie
2012 Andriyash+Denef+Jafferis+Moore
2012 Chuang+Diaconescu+Manschot+Moore+Soibelman

constructive

wall crossing redux

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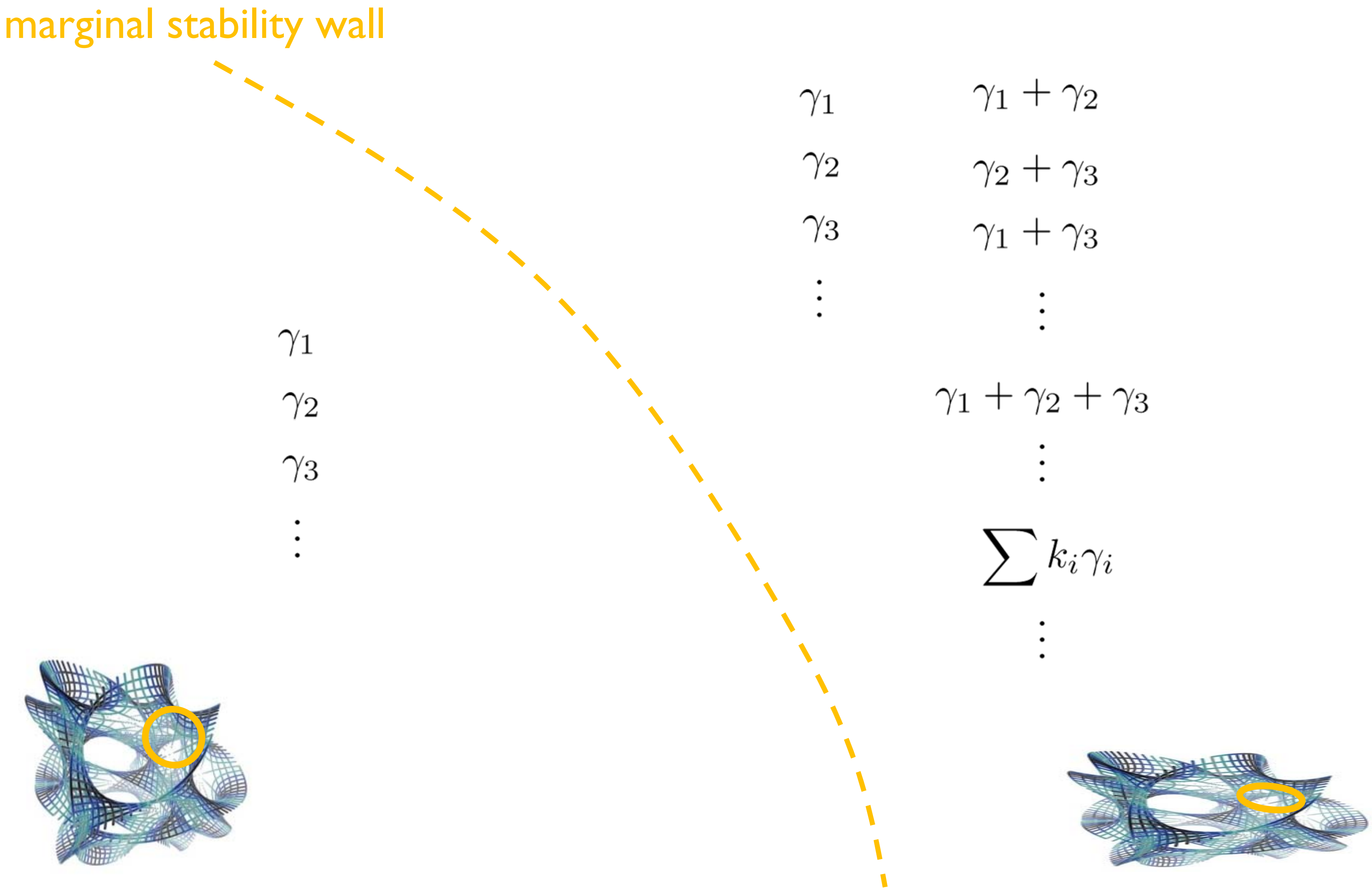
Manschot+Pioline+Sen 2010/2011/2012/2013

Zhaolong Wang Heeyeon Kim Seung-Joo Lee Sungjay Lee

wall-crossing of BPS states with 4 (or less) supersymmetries



first, some pre-history

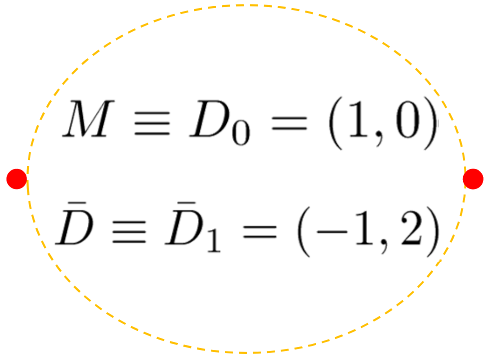
Gauntlett+Harvey 1995
Bilal+Ferrari 1996-1997

prototype : D=4 N=2 SU(2) → U(1) Seiberg-Witten

$N_f = 0$

$W = (0, 2)$

$\bar{D}_n = (-1, 2n)$

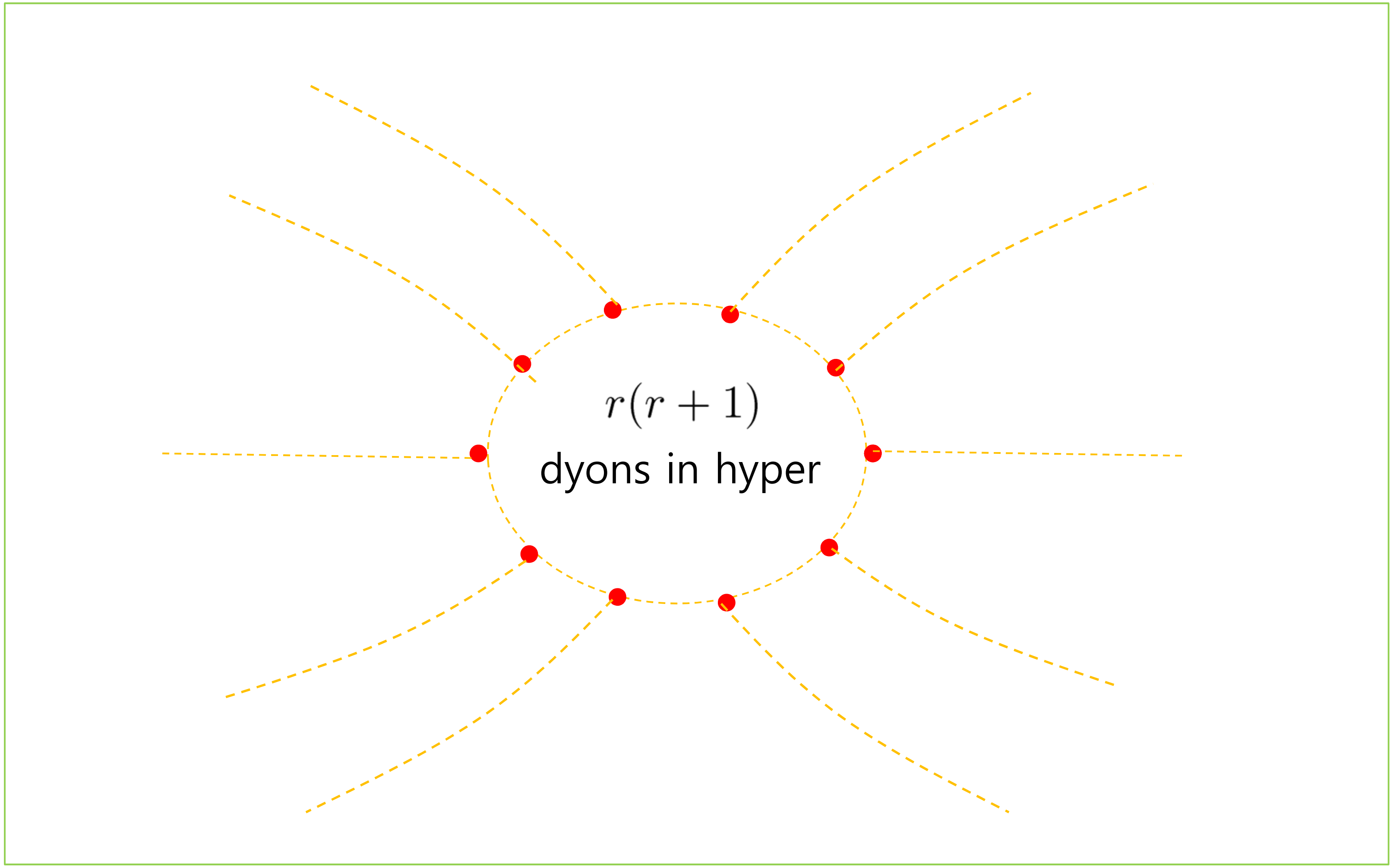


$M \equiv D_0 = (1, 0)$

$\bar{D} \equiv \bar{D}_1 = (-1, 2)$

Lerche 2000

D=4 N=2 SU(r+1) → U(1)^r Seiberg-Witten



1998 Lee + P.Y.

N=4 SU(n) ¼ BPS states via semiclassical multi-center dyon solitons

← 1997 Bergman

¼ BPS states as open string-web, and decay thereof

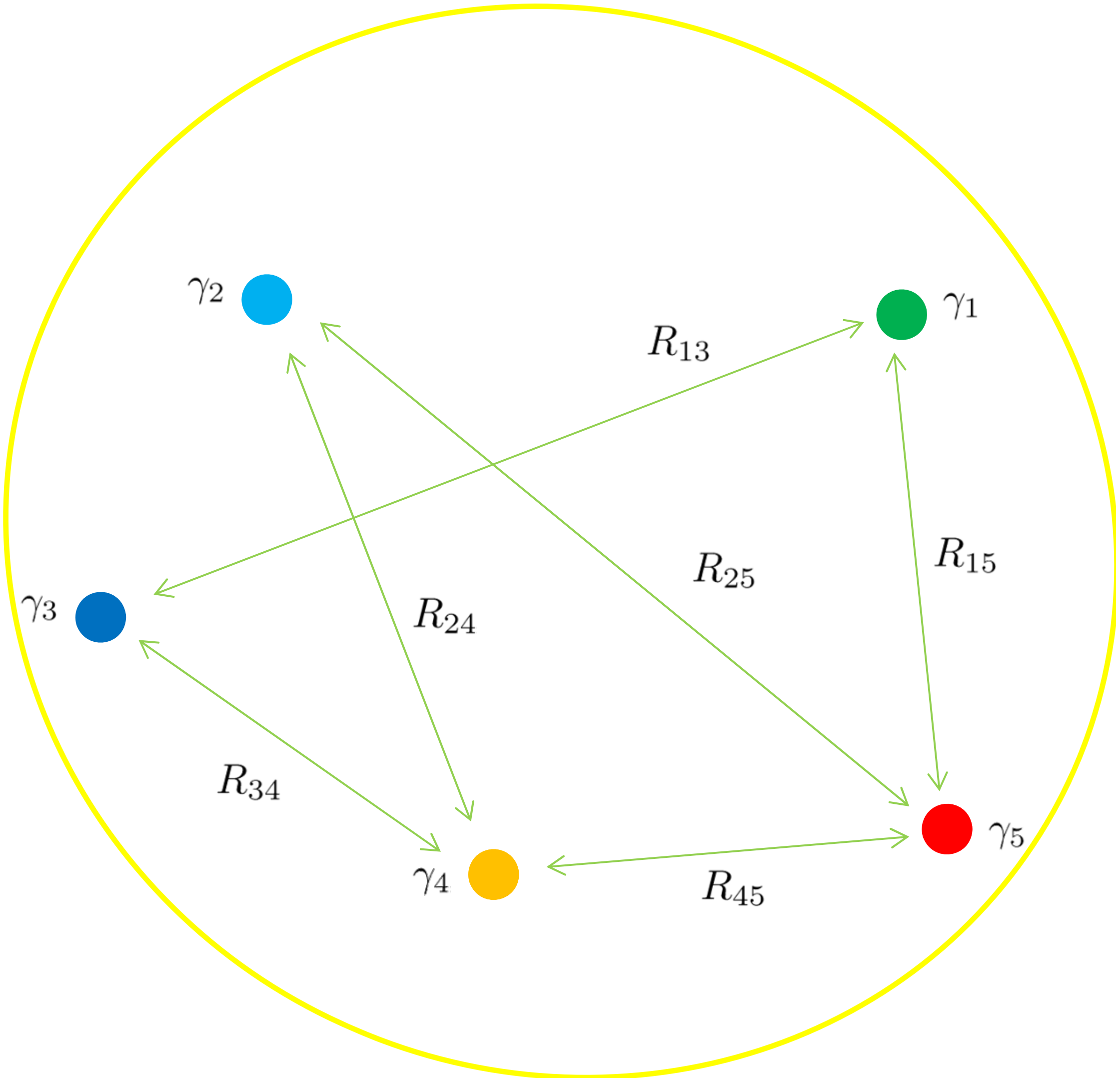
← 1997 Dasgupta + Muhki / Sen

string junctions

generic 4 SUSY BPS "particles" are loose bound states of charge centers

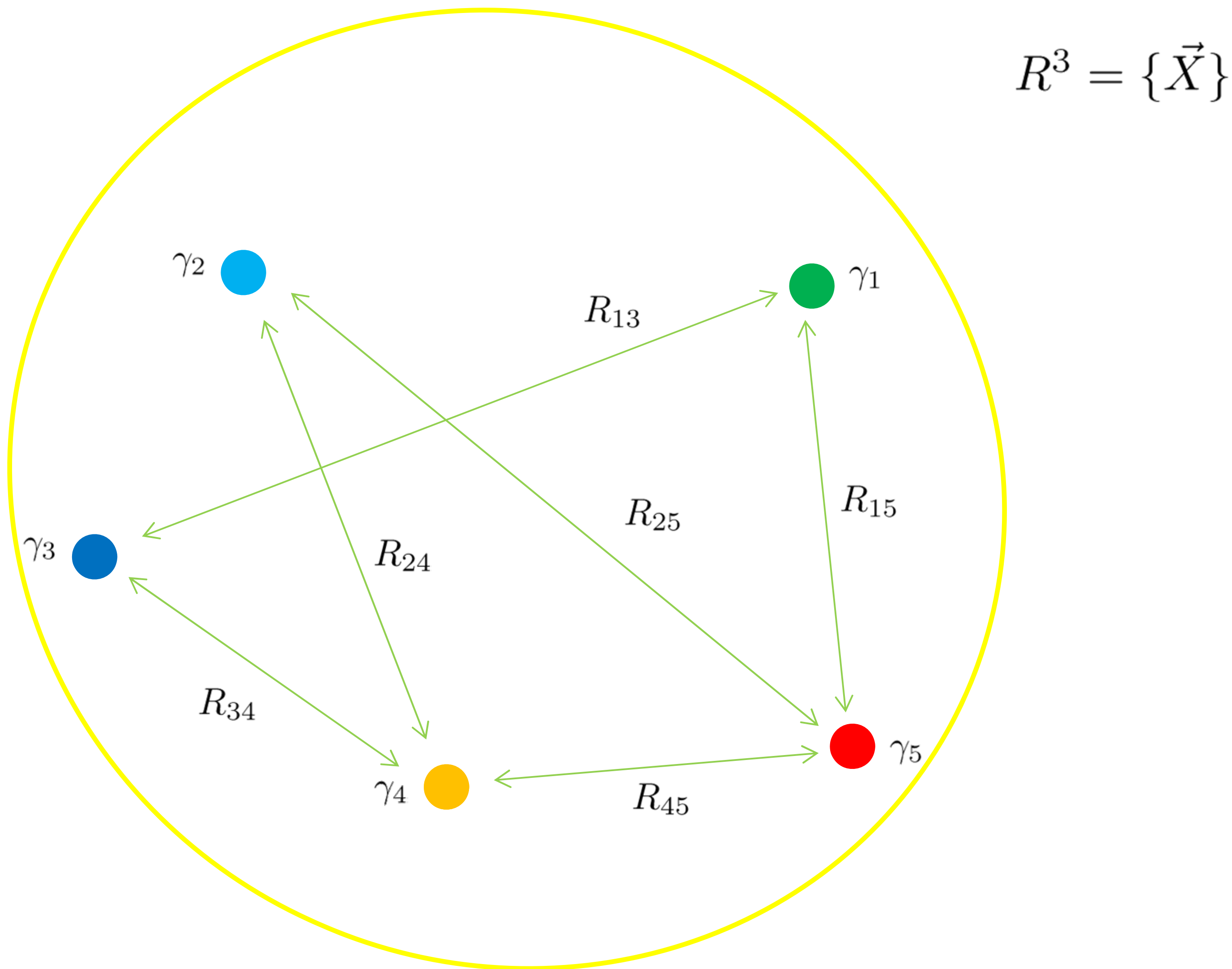
wall-crossing ← one or more distances diverge

$R^3 = \{\vec{X}\}$



generic 4 SUSY BPS “particles” are loose bound states of charge centers

wall-crossing ~ supersymmetric Schroedinger problem



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N=2 SU(n) BPS state counting via semi-classical multi-center monopole dynamics

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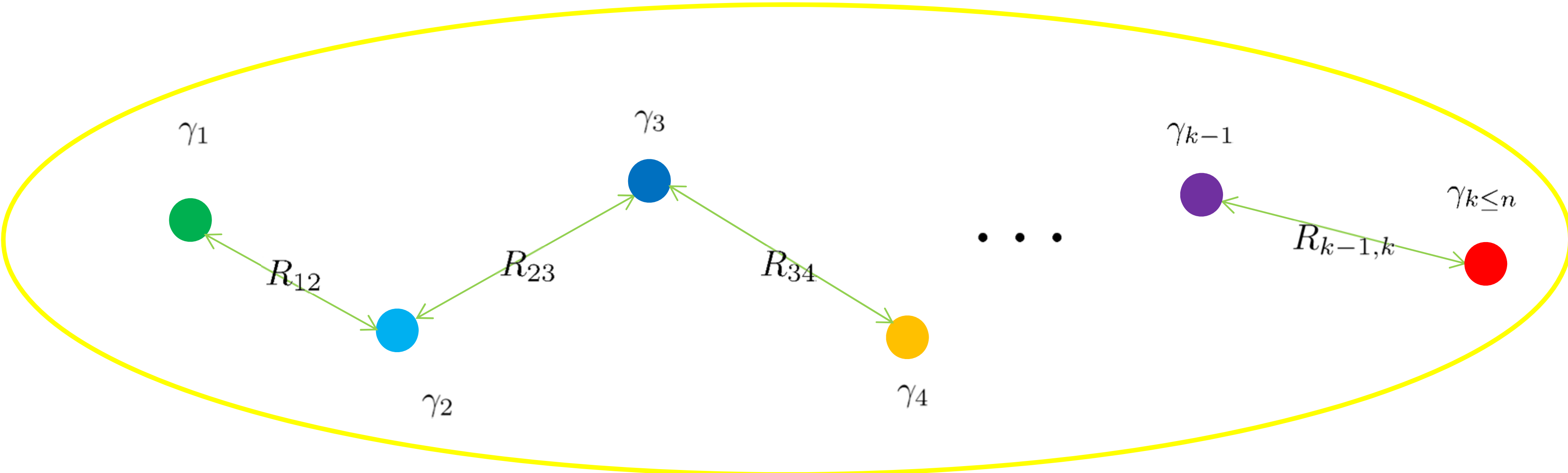


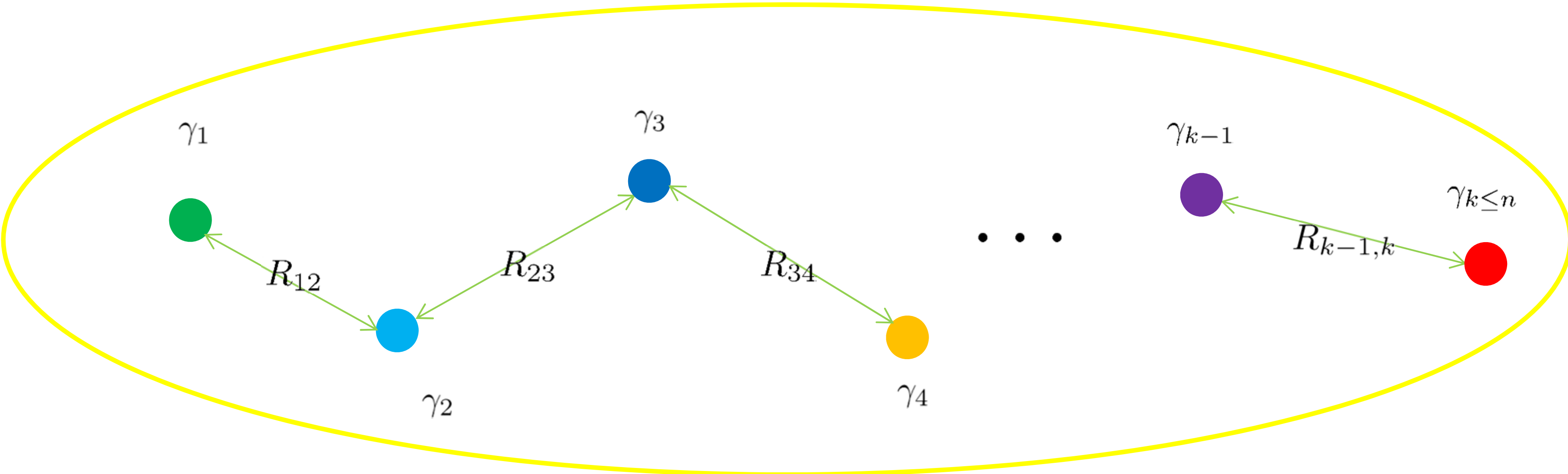
Gauntlett + Kim + Park + P.Y.
Gauntlett + Kim + Lee + P.Y.
Stern + P.Y.
1999-2000

the BPS supermultiplet of the bound state

$$[1/2\,\text{hyper}] \otimes_i [(|\langle\gamma_i, \gamma_{i+1}\rangle| - 1)/2]$$

→ in effect, N=2 SUSY multi-particle "primitive wall-crossing formula"

$$\Omega = \prod_i (-1)^{\langle\gamma_i,\gamma_{i+1}\rangle - 1} |\langle\gamma_i, \gamma_{i+1}\rangle|$$



Gauntlett + Kim + Park + P.Y.
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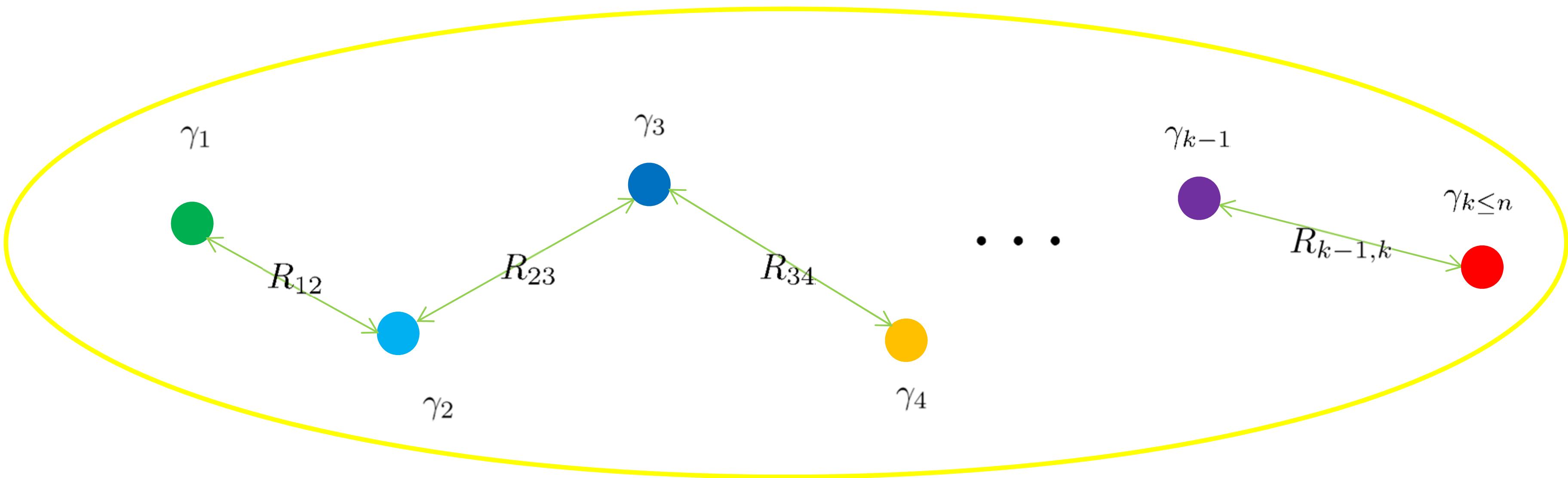
Stern + P.Y
1999-2000

for Seiberg-Witten theories of rank > 1,
arbtrarily high-spin BPS dyons in the weakly coupled regime &
the accompanying infinite number of marginal stability walls

qualitative difference of BPS spectra between rank 1 vs. rank > 1



Chuang + Diaconescu + Manschot
+ Moore + Soibelman 2012

Gauntlett + Kim + Park + P.Y.
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1999-2000

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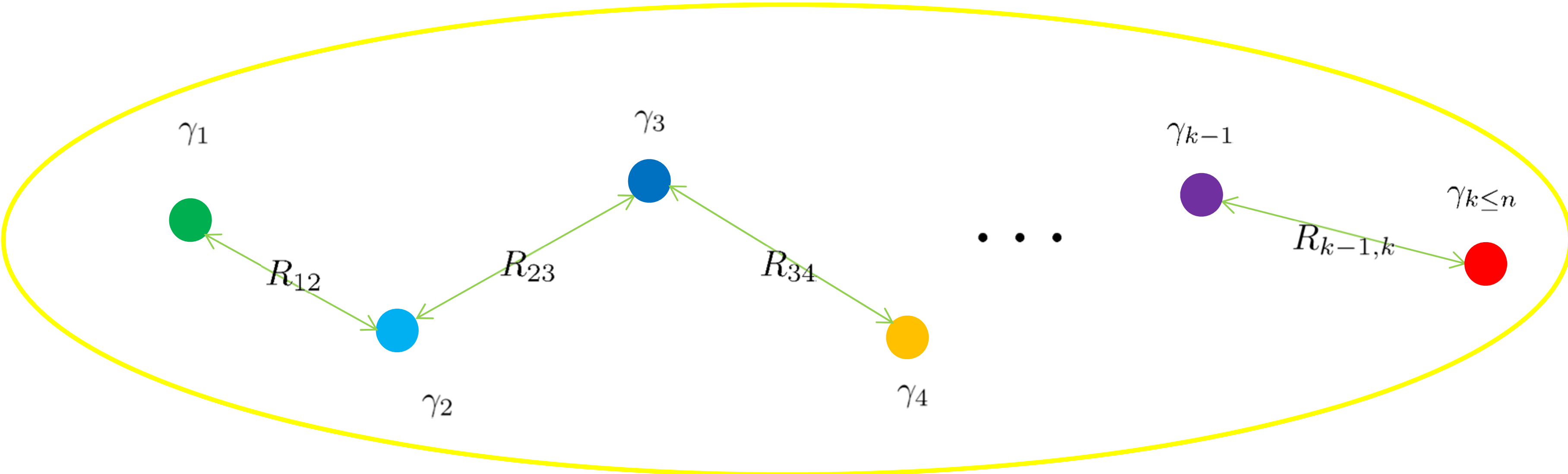

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N=2 SU(n) BPS state counting via semi-classical multi-center monopole dynamics

also,

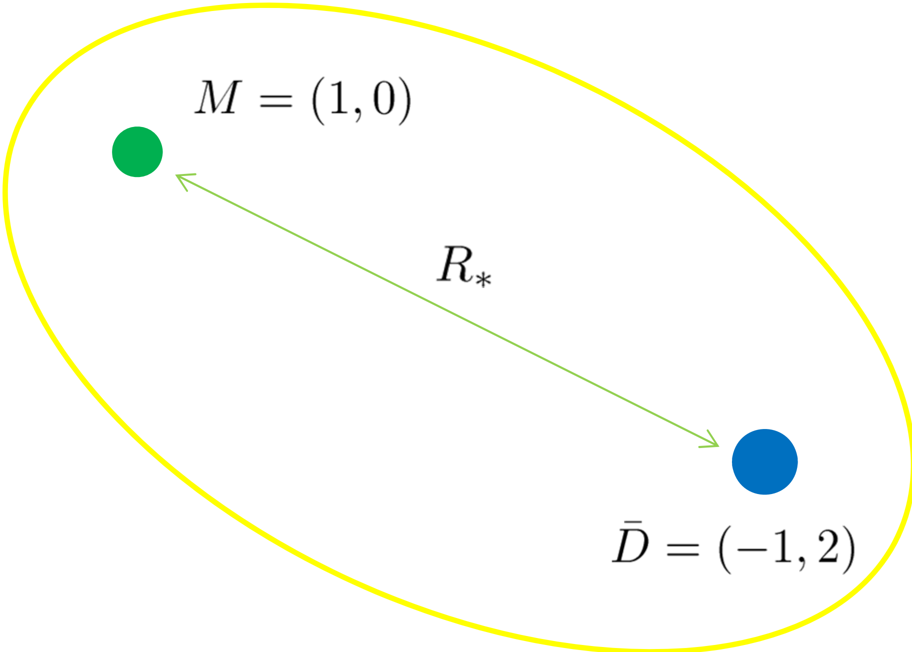
2001 Argyres + Narayan / Ritz + Shifman + Vainshtein + Voloshin
UV-incomplete string-web picture for N=2 BPS dyons in Seiberg-Witten description
→ again, multi-particle picture of BPS states in strongly coupled regime

SU(2) Seiberg-Witten

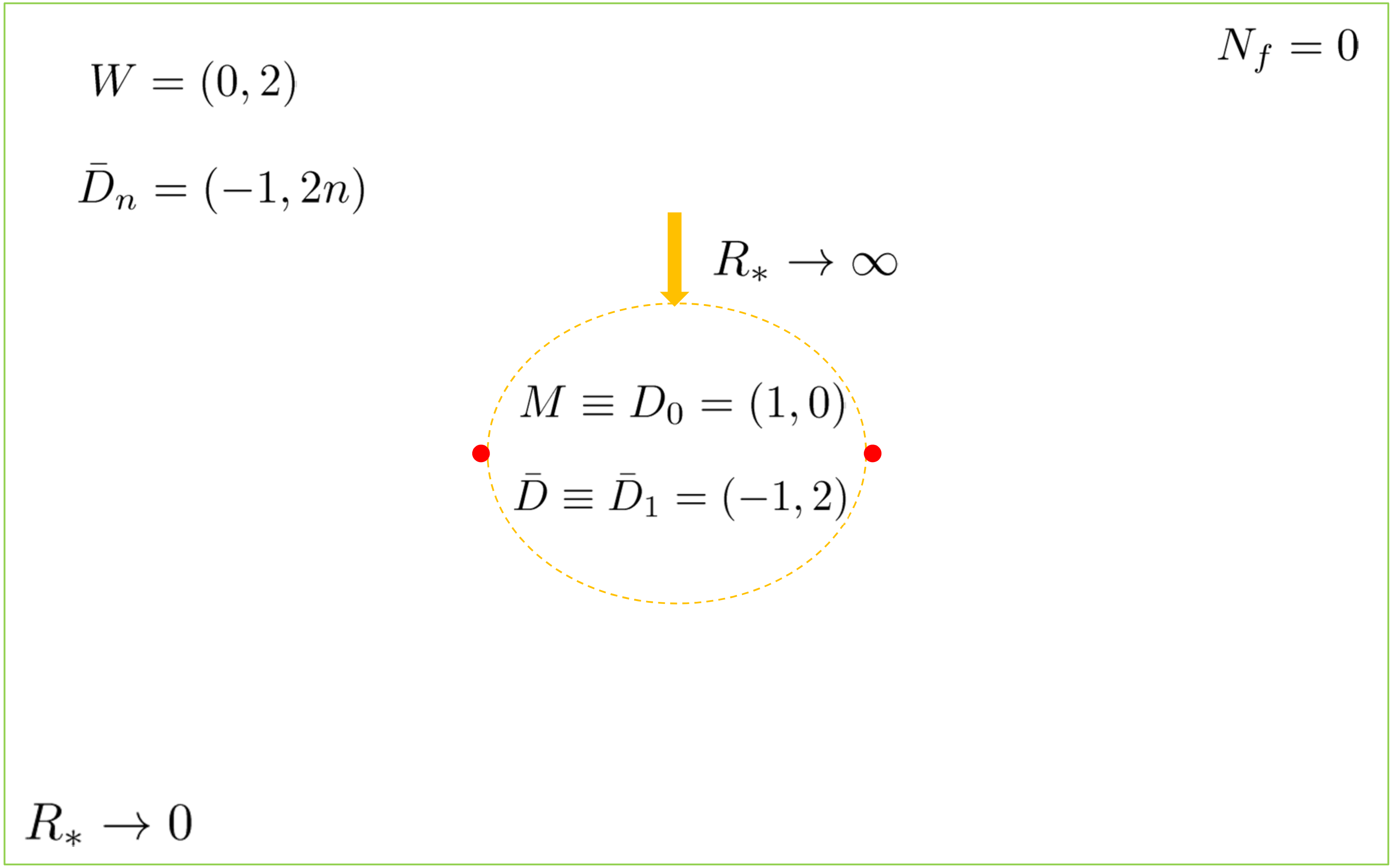
$$W^+ = M + \bar{D}$$

vector multiplet

$\Omega = -2$



SU(2) Seiberg-Witten



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N=2 SU(n) BPS state counting via semi-classical multi-center monopole dynamics

2000 Denef

N=2 supergravity via classical multi-center black holes attractor solutions

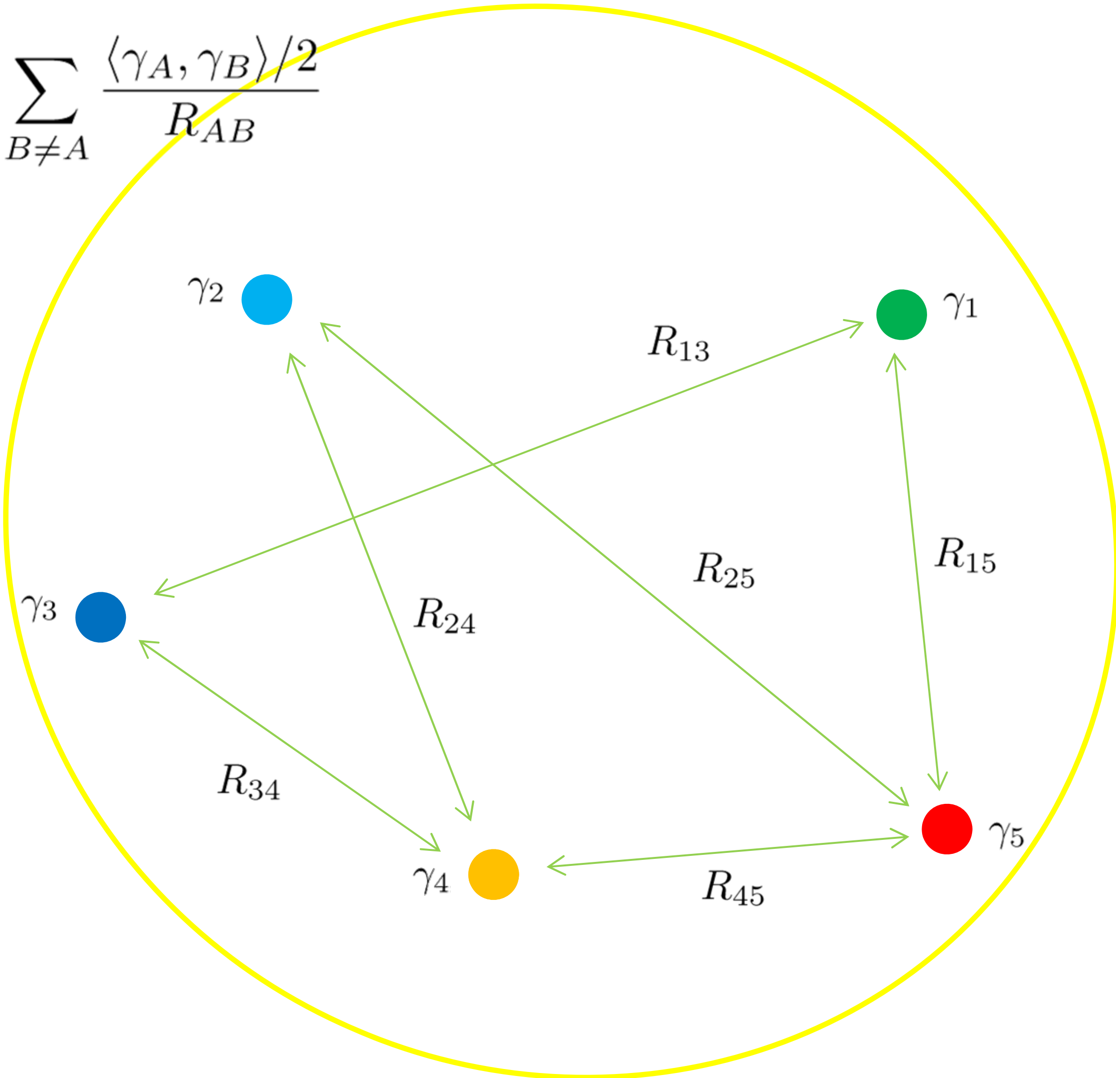
$$\mathrm{Im}[\zeta^{-1} Z_{\gamma_A}] = \sum_{B\neq A} \frac{\langle \gamma_A, \gamma_B \rangle /2}{R_{AB}} \qquad \zeta \equiv \frac{\sum_A Z_{\gamma_A}}{|\sum_A Z_{\gamma_A}|}$$

2001 Argyres + Narayan / Ritz + Shifman + Vainshtein + Voloshin

UV-incomplete string-web picture for N=2 BPS dyons generic 4 SUSY BPS black hole solutions are loose bound states of many charged singe-center BPS black holes

$$\mathrm{Im}[\zeta^{-1} Z_{\gamma_A}] = \sum_{B\neq A} \frac{\langle \gamma_A, \gamma_B \rangle /2}{R_{AB}}$$

$$R^3 = \{\vec{X}\}$$



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UV-incomplete string-web picture for N=2 BPS dyons

2002 Denef / 2007 Denef + Moore

quiver dynamics of BPS states / (semi-)primitive wall-crossing formula ....

KS, GMN, and …

real-space wall-crossing

quivers & quiver invariants

quiver mutation

Kontsevich-Soibelman

& Gaiotto-Moore-Neitzke

Schwinger product

$$[V_\alpha, V_\beta] = (-1)^{\langle\alpha,\beta\rangle}\langle\alpha,\beta\rangle V_{\alpha+\beta}$$

$$K_\gamma \equiv \exp\left(\sum_n \frac{V_{n\gamma}}{n^2}\right)$$

a marginal stability wall

Kontsevich + Soibelman 2009

+ side

$$\prod_\gamma K_\gamma^{\Omega^+(\gamma)} = \prod_\gamma{}' K_\gamma^{\Omega^-(\gamma)}$$

− side

with the 2nd helicity trace for N=2 BPS states

$$\Omega = -\frac{1}{2}\mathrm{tr}\,(-1)^{2J_3}(2J_3)^2$$

$$\to (-1)^{2l} \times (2l+1) \;= \mathrm{tr}'(-1)^{2J_3}$$

on [a spin ½ + two spin 0]
x [angular momentum l multiplet]

$$[V_\alpha, V_\beta] = (-1)^{\langle \alpha, \beta \rangle} \langle \alpha, \beta \rangle V_{\alpha+\beta}$$

$$K_\gamma \equiv \exp\left(\sum_n \frac{V_{n\gamma}}{n^2} \right)$$

$$\bar{\Omega}(\Gamma) = \sum_{p|\Gamma} \Omega(\Gamma/p)/p^2$$

a marginal stability wall

+ side

$$\prod_\gamma \left(e^{V_\gamma}\right)^{\bar{\Omega}^+(\gamma)} = \prod_\gamma{}' \left(e^{V_\gamma}\right)^{\bar{\Omega}^-(\gamma)}$$

\- side

is KS true physically ?

how to see from BPS state building/counting ?

why rational invariants ?

$$\bar{\Omega}(\Gamma) = \sum_{p|\Gamma} \Omega(\Gamma/p)/p^2$$

input data ?

$$\Omega^+(\gamma) = \Omega^-(\gamma) \neq 0$$

is KS true physically ?

how to see from BPS state building/counting ?

why rational invariants ?

$$\bar{\Omega}(\Gamma) = \sum_{p|\Gamma} \Omega(\Gamma/p)/p^2$$

or, more generally

$$\Omega^{+}(\gamma) \neq 0 \neq \Omega^{-}(\gamma)$$

is KS true for SW ?

how to see from BPS state building/counting ?

why rational invariants ?

$$\bar{\Omega}(\Gamma) = \sum_{p|\Gamma} \Omega(\Gamma/p)/p^2$$

input data ?

$$\Omega^+(\gamma) \neq 0 \neq \Omega^-(\gamma)$$

SW on a circle, Stokes phenomena,
(2,0), Hitchin system, Darboux/Fock-Goncharov,
framed BPS state, spectrum generator, BPS quiver,
spectral network, …

Gaiotto + Moore + Neiztke 2008, 2009, 2010, 2011, 2012
(also, Chen + Dorey + Petunin 2010 for GMN 2008)
Alim+Cecotti+Cordova+Espahbodi+Rastogi+Vafa 2011
Xie 2012
Chuang+Diaconescu+Manschot+Moore+Soibelman 2012

……

(i) proves KS for general Seiberg-Witten theories

continuity of Darboux coordinates

$Z_\alpha(u^+;\theta) = Z_\alpha(u^-;\theta)$ across marginal stability walls

$$\prod_\gamma K_\gamma^{\Omega^+(\gamma)} = \prod_\gamma{}' K_\gamma^{\Omega^-(\gamma)}$$

KS operator string = Stokes' phenomena

Gaiotto + Moore + Neiztke 2008, 2009
also, see Chen + Dorey + Petunin 2010

(i) proves KS for general Seiberg-Witten theories

discontinuous spectra = continuity of vacuum physics

$$Z_\alpha(u^+;\theta) \qquad\qquad Z_\alpha(u^-;\theta)$$

$$Z_\alpha(u^+;\theta+\pi) = Z_\alpha(u^+;\theta)\prod_\gamma K_\gamma^{\Omega^+(\gamma)}$$

$$Z_\alpha(u^-;\theta+\pi) = Z_\alpha(u^-;\theta){\prod_\gamma}' K_\gamma^{\Omega^-(\gamma)}$$

Gaiotto + Moore + Neiztke 2008, 2009
also, see Chen + Dorey + Petunin 2010

(ii) spectrum generator

Gaiotto + Moore + Neiztke 2009

independent computation of the two sets of such holomorphic quantity from which the KS operator string can be in principle extracted

$$Z_\alpha(u;0) \qquad\qquad Z_\alpha(u;\pi)$$

(ii) spectrum generator

Gaiotto + Moore + Neiztke 2009

independent computation of the two sets of such holomorphic quantity from which the KS operator string can be in principle extracted

$$Z_\alpha(u;0) \qquad\qquad Z_\alpha(u;\pi)$$

$$= Z_\alpha(u;0)\boxed{\prod_\gamma K_\gamma^{\Omega(\gamma)}}$$

(iii) hypermultiplets as geodesic segments on the Gaiotto curve

explicit construction of SU(2) theory spectra

Gaiotto+Moore+Neiztke 2009

how to read off BPS quivers from triangulation

Gaiotto+Moore+Neitzke 2009
Alim+Cecotti+Cordova+Espahbodi+Rastogi+Vafa 2011

also more recently, SU(n>2) partial spectra & higher spin states

Gaiotto+Moore+Neitzke 2012
Xie 2012
Chuang+Diconescu+Manschot+Moore+Soibelman 2012
Galakhov+Longhi+Mainiero+Moore+Neitzke 2013

is KS true for all physical wall-crossing ?

how to see from BPS state building/counting ?

why rational invariants ?

$$\bar{\Omega}(\Gamma) = \sum_{p|\Gamma} \Omega(\Gamma/p)/p^2$$

input data ?

$$\Omega^{+}(\gamma) \neq 0 \neq \Omega^{-}(\gamma)$$

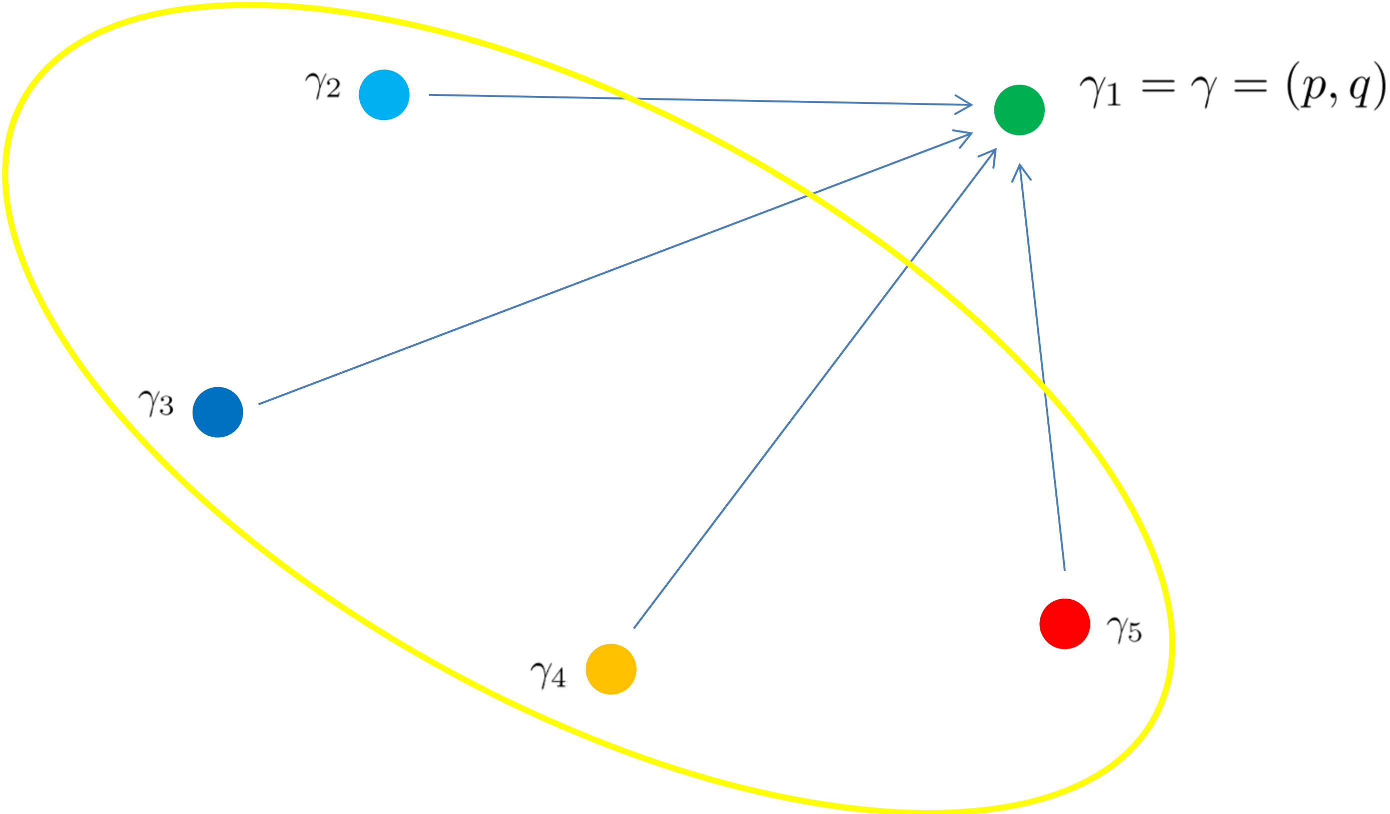
real space wall-crossing

Denef 2002
Denef + Moore 2007
de Boer + El-Showk + Messamah +Van den Bleeken 2008
Manschot+Pioline+ Sen 2010/2011
Lee+P.Y. / Kim+Park+Wang+P.Y. 2011
Sen 2011

BPS dyon Schroedinger problem for Seiberg-Witten theories

→ each dyon feels the rest via long-range tails

$$\mathcal{Z}_{\gamma=(p,q)} \equiv [p^i \phi_D^i + q^i \phi^i]\Big|_{\gamma_{A'=2,3,4,\dots}} \qquad R^3 = \{\vec{X}\}$$



for which UV-incomplete long distance solutions will do

Seiberg+Witten 1994
Chalmers + Rocek + von Unge 1996
Ritz + Shifman + Vainshtein + Voloshin 2001
Argyres + Narayan 2001
...

$$F_a^i = i\zeta^{-1}\partial_a \phi^i$$

$$(F_D)_a^i = i\zeta^{-1}\partial_a \phi_D^i$$

$$F_a^i \equiv B_a^i + iE_a^i \qquad \mathrm{Re}\int_{S^2} F^i = 4\pi m^i$$

$$(F_D)_a^i \equiv \tau^{ij} F_j^a \qquad \mathrm{Re}\int_{S^2} F_D^i = -4\pi n^i$$

⬇

$$\mathrm{Im}[\zeta^{-1}\mathcal{Z}_\gamma] = \mathcal{K}_\gamma \equiv \mathrm{Im}[\zeta^{-1}Z_\gamma] - \sum_{A'} \frac{\langle \gamma, \gamma_{A'}\rangle/2}{|\vec{x} - \vec{x}_{A'}|}$$

each SW dyon feels the rest via long-range tails

Lee+P.Y. 2011

cf) Ritz+Vainshtein 2008

$$\mathcal{L}_\gamma = -|\mathcal{Z}_\gamma|\sqrt{1-\dot{\vec{x}}^2} + \mathrm{Re}[\zeta^{-1}\mathcal{Z}_\gamma] - \dot{\vec{x}}\cdot\vec{W}$$

$$\simeq \frac{1}{2}|\mathcal{Z}_\gamma|\dot{\vec{x}}^2 - (|\mathcal{Z}_\gamma| - \mathrm{Re}[\zeta^{-1}\mathcal{Z}_\gamma]) - \dot{\vec{x}}\cdot\vec{W}$$

$$\simeq \frac{1}{2}|\mathcal{Z}_\gamma|\dot{\vec{x}}^2 - \frac{(\mathrm{Im}[\zeta^{-1}\mathcal{Z}_\gamma])^2}{2|\mathcal{Z}_\gamma|} - \dot{\vec{x}}\cdot\vec{W}$$

$$\vec{\partial}\times\vec{W} \equiv \vec{\partial}\,\mathrm{Im}\,[\zeta^{-1}\mathcal{Z}_\gamma]$$

$$\zeta^{-1}\mathcal{Z}_\gamma = |\mathcal{Z}_\gamma|e^{i\alpha}, \quad |\alpha| \ll 1$$

treat all charge-centers on equal footing & supersymmetrize

step (1) real space N=4 susy quantum mechanics for Seiberg-Witten dyons near marginal stability wall

Lee+P.Y. 2011
Kim+Park+Wang+P.Y, 2011

$$\int dt\, \mathcal{L}_{kinetic} = \int dt \int d\theta^2 d\bar{\theta}^2 F(\hat{\Phi}^{Aa})$$

$$\int dt\, \mathcal{L}_{potential} = \int dt \int d\theta \;\left(i\mathcal{K}(\Phi)_A \Lambda^A - iW(\Phi)_{Aa} D\Phi^{Aa}\right)$$

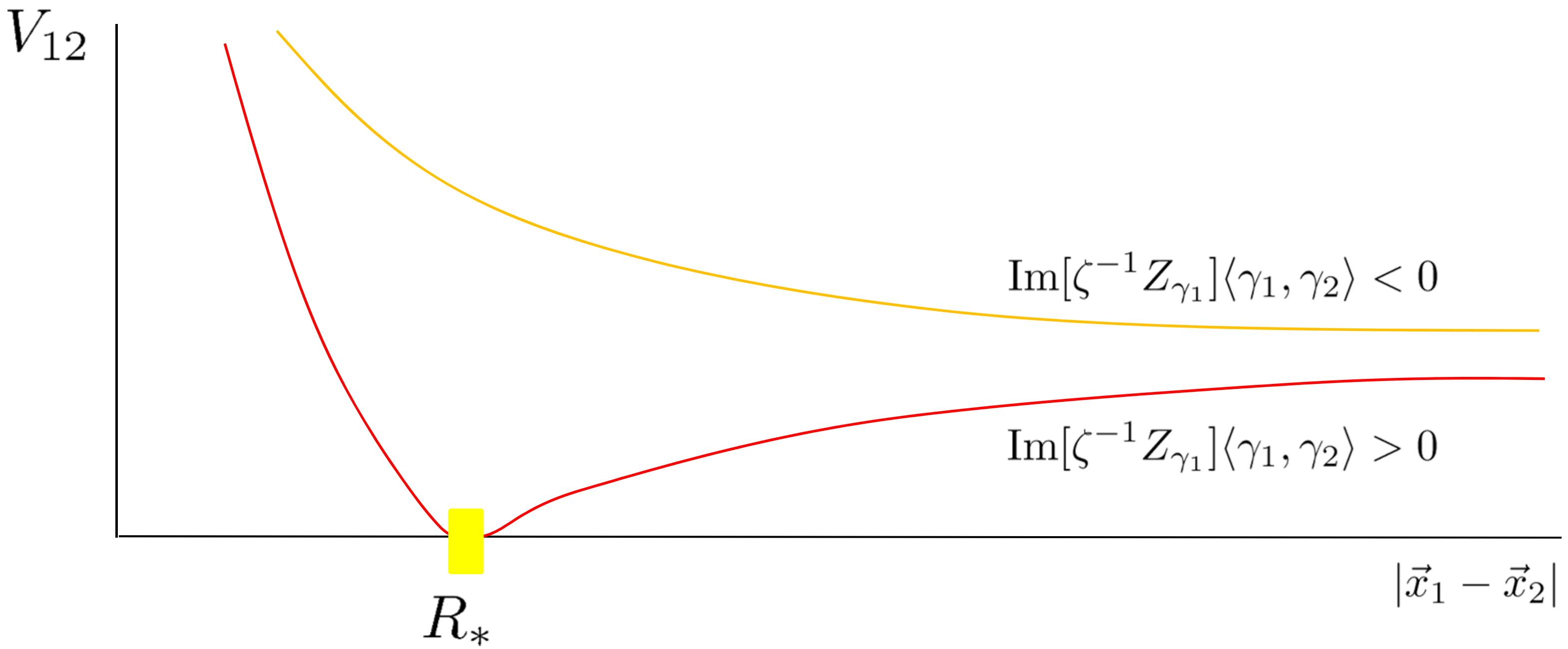
$$\mathcal{K}_A = \mathrm{Im}[\zeta^{-1} Z_{\gamma_A}] - \sum_{B\neq A} \frac{\langle \gamma_A, \gamma_B \rangle /2}{|\vec{x}_A - \vec{x}_B|}$$

$$\vec{W}_A = \sum_{B\neq A} \frac{\langle \gamma_A, \gamma_B \rangle}{2}\; \vec{W}^{4\pi}_{Dirac}(\vec{x}_A - \vec{x}_B)$$

$$F(\vec{x}_A) \simeq \sum_A |Z_A||\vec{x}_A|^2 = \sum_A m_A |\vec{x}_A|^2 \qquad \text{asymptotically}$$

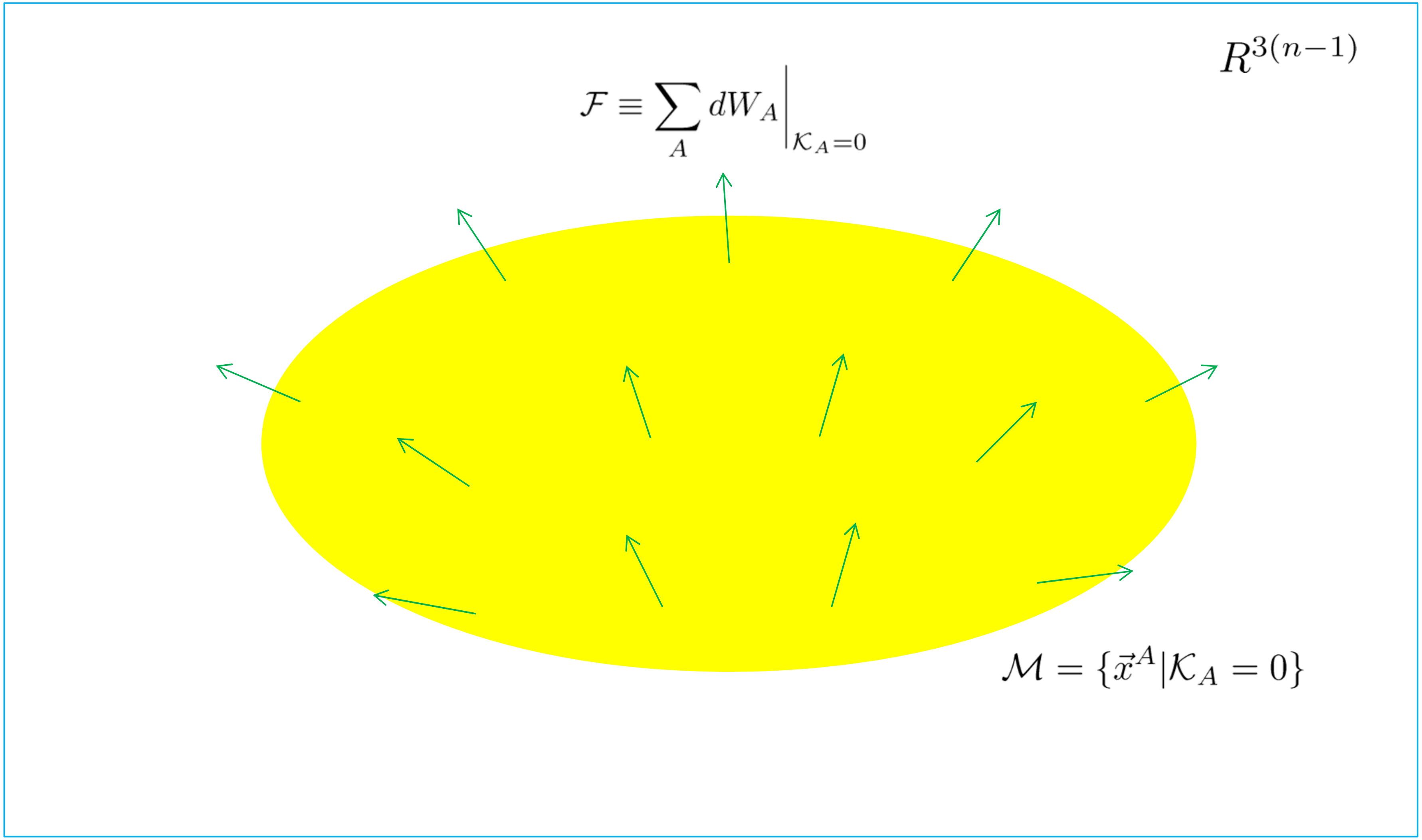
~ ab initio, for SW theories, reproduction of Denef's Coulomb phase dynamics

$$V(\{\vec{x}_{12}\}) \sim \left(\mathrm{Im}[\zeta^{-1} Z_{\gamma_1}] - \frac{\langle \gamma_1, \gamma_2 \rangle / 2}{|\vec{x}_1 - \vec{x}_2|} \right)^2$$



deform & localize N=4 3(n-1) dimensional dynamics

→ N=1 2(n-1) dim nonlinear sigma model with U(1) bundle



reduces to a N=1 Dirac index

of a nonlinear sigma model on the manifold $\mathcal{K}_A = 0$

Kim+Park+Wang+P.Y. 2011

3(n-1) bosons + 4(n-1) fermions → 2(n-1) bosons + 2(n-1) fermions

$$\mathcal{L}_{deformed}^{for\ index\ only}\Big|_{L\to\infty} \quad \to \quad \mathcal{L}_{index}$$

$$\mathcal{L}_{index} \;\simeq \frac{1}{2} g_{\mu\nu}\dot{z}^\mu \dot{z}^\nu - \dot{x}^\mu \cdot \mathcal{A}_\mu + \frac{i}{2} g_{\mu\nu}\psi^\mu \left(\dot{\psi}^\nu + \dot{z}^\alpha \Gamma^\nu_{\alpha\beta}\psi^\beta\right) + i\mathcal{F}_{\mu\nu}\psi^\mu\psi^\mu$$

$$\mathcal{F} = d\mathcal{A} \equiv \sum_A dW_A\Big|_{\mathcal{K}_A=0}$$

step (2) an index theorem before quantum statistics

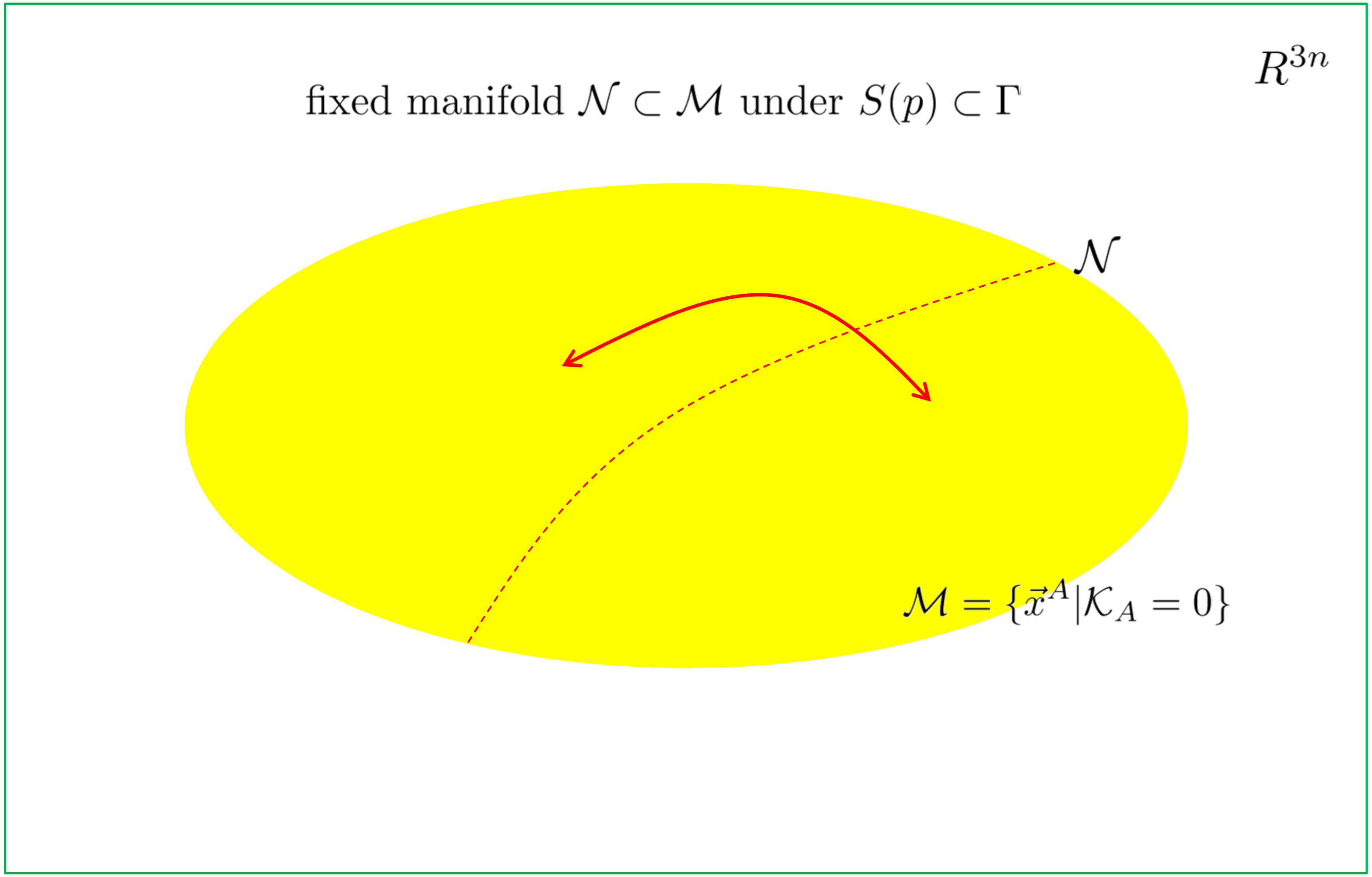
Manschot+Pioline+ Sen 2010/2011
Kim+Park+Wang+P.Y. 2011

$$\Omega_{\text{before statistics}} = (-1)^{\sum_{A>B}\langle\gamma_A,\gamma_B\rangle+n-1} \boxed{I_n(\{\gamma_A\})} \prod_A \Omega(\gamma_A)$$

$$I_n(\{\gamma_A\}) = \text{tr}\left[(-1)^F e^{-\beta H}\right] = \int_{\mathcal{M}=\{\vec{x}_A \,|\, \mathcal{K}_A=0\}/R^3} ch(\mathcal{F}) \wedge \mathbf{A}(\mathcal{M})$$

$$\mathcal{F} \equiv \sum_{A>B} \frac{\langle\gamma_A,\gamma_B\rangle}{2}\, \mathcal{F}^{4\pi}_{Dirac}(\vec{x}_A - \vec{x}_B)\Big|_{\mathcal{K}_A=0}$$

Bose/Fermi statistics → an iterative sum over fixed submanifolds under permutation of identical particles



imposing Bose/Fermi statistics → orbifolding of the index

$$\text{fixed manifold } \mathcal{N} \subset \mathcal{M} \text{ under } S(p) \subset \Gamma$$

$$\Gamma' = \Gamma / S(p) \qquad \mathcal{P}' = \sum_{\sigma \in \Gamma'} (\pm 1)^{|\sigma|} \sigma$$

$$\mathrm{tr}\,(-1)^F e^{-\beta H} \mathcal{P}$$

$$= \mathrm{tr}_{\mathcal{M}/\Gamma - \mathcal{N}}\,(-1)^F e^{-\beta H} \mathcal{P} + \boxed{\Delta_{\mathcal{N}}}\, \mathrm{tr}_{\mathcal{N}/\Gamma' - \mathcal{N}'}\,(-1)^F e^{-\beta H} \mathcal{P}' + \cdots$$

for p identical particles & with internal degeneracy

→ rational invariants !!!

P.Y. 1997

Green + Gutperle 1997

Kim + Park + Wang + P.Y. 2011

fixed manifold $\mathcal{N} \subset \mathcal{M}$ under $S(p) \subset \Gamma$

$$\Delta_{\mathcal{N}}(p\gamma) = \mathrm{tr}_{\mathcal{N}^\perp} \left[(-1)^{F^\perp} e^{-\beta H^\perp} \mathcal{P}^{(\pm)}_{S(p)} \right] \quad = \quad \frac{\Omega(\gamma)}{p^2}$$

↓

$$\bar{\Omega}(\Gamma) = \sum_{p|\Gamma} \Omega(\Gamma/p)/p^2$$

cf) Manschot + Pioline + Sen 2010/2011

step (3) universal wall-crossing formula from ab initio, real space dynamics of charge centers

Manschot+Pioline+Sen 2011
Kim+Park+Wang+P.Y. 2011

$$\Omega^-\left(\sum \gamma_A\right) - \Omega^+\left(\sum \gamma_A\right) = (-1)^{\sum_{A>B}\langle\gamma_A,\gamma_B\rangle+n-1}\frac{\prod_A \bar{\Omega}^-(\gamma_A)}{|\Gamma|}\int_{\mathcal{M}} ch(\mathcal{F})\wedge\mathcal{A}(M)$$

$$\vdots$$

$$+(-1)^{\sum_{A'>B'}\langle\gamma'_{A'},\gamma'_{B'}\rangle+n'-1}\frac{\prod_{A'} \bar{\Omega}(\gamma'_{A'})}{|\Gamma'|}\int_{\mathcal{M}'} ch(\mathcal{F}')\wedge\mathcal{A}(M')$$

$$\vdots$$

$$+(-1)^{\sum_{A''>B''}\langle\gamma''_{A''},\gamma''_{B''}\rangle+n''-1}\frac{\prod_{A''} \bar{\Omega}(\gamma''_{A''})}{|\Gamma''|}\int_{\mathcal{M}''} ch(\mathcal{F}'')\wedge\mathcal{A}(M'')$$

$$\vdots$$

$$\bar{\Omega}(\gamma) = \sum_{p|\gamma}\Omega(\gamma/p)/p^2$$

$$\sum_{A=1}^{n}\gamma_A = \sum_{A'=1}^{n'}\gamma'_{A'} = \sum_{A''=1}^{n''}\gamma''_{A''} = \cdots$$

= sum over all partition of charges, with rational invariants, with particles in each partition treated as if distinguishable

Manschot+Pioline+Sen 2011
Kim+Park+Wang+P.Y. 2011

$$\Omega^-\left(\sum\gamma_A\right) - \Omega^+\left(\sum\gamma_A\right) = (-1)^{\sum_{A>B}\langle\gamma_A,\gamma_B\rangle+n-1}\frac{\prod_A\bar{\Omega}^-(\gamma_A)}{|\Gamma|}\int_{\mathcal{M}} ch(\mathcal{F})\wedge\mathcal{A}(M)$$

$$\vdots$$

$$+(-1)^{\sum_{A'>B'}\langle\gamma'_{A'},\gamma'_{B'}\rangle+n'-1}\frac{\prod_{A'}\bar{\Omega}(\gamma'_{A'})}{|\Gamma'|}\int_{\mathcal{M}'} ch(\mathcal{F}')\wedge\mathcal{A}(M')$$

$$\vdots$$

$$+(-1)^{\sum_{A''>B''}\langle\gamma''_{A''},\gamma''_{B''}\rangle+n''-1}\frac{\prod_{A''}\bar{\Omega}(\gamma''_{A''})}{|\Gamma''|}\int_{\mathcal{M}''} ch(\mathcal{F}'')\wedge\mathcal{A}(M'')$$

$$\vdots$$

$$\bar{\Omega}(\gamma) = \sum_{p|\gamma}\Omega(\gamma/p)/p^2$$

$$\sum_{A=1}^{n}\gamma_A = \sum_{A'=1}^{n'}\gamma'_{A'} = \sum_{A''=1}^{n''}\gamma''_{A''} = \cdots$$

step (4) evaluate & compare

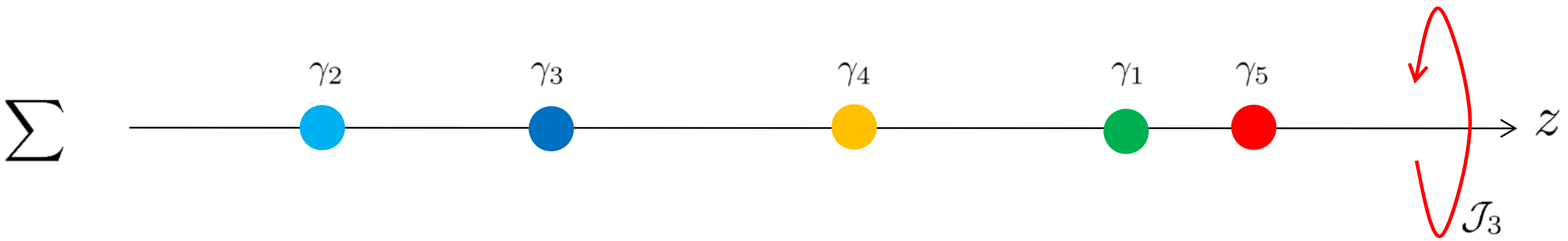
Manschot+Pioline+Sen 2010/2011

?

Jan's talk, tomorrow, for equivariant evaluation

$$\Omega = (-1)^{\sum_{A>B}\langle\gamma_A,\gamma_B\rangle+n-1}\mathrm{tr}((-1)^F y^{2\mathcal{J}_3})$$

and also for how to include quiver invariants to the counting



$$\mathrm{Im}[\zeta^{-1}Z_{\gamma_A}] = \sum_{B\neq A}\frac{\langle\gamma_A,\gamma_B\rangle/2}{|z^A - z^B|}$$

equivariant index on $\mathcal{M}$ computes

Kim+Park+Wang+P.Y. 2011

Q^A_α

$2_J \otimes 2_I$

$SU(2)_J \times SU(2)_I$

$$\Omega = -\frac{1}{2}\mathrm{tr}\left[(-1)^{2J_3}(2J_3)^2 y^{2(J_3+I_3)}\right]$$

$$H = H_{\text{center of mass}} \otimes H_{\text{reduced}}$$

~~$3_{\mathcal{J}}$~~ $+ 1_{\mathcal{J}}$

$\mathcal{J} = J + I$

$$\Omega = \mathrm{tr}_{H_{\text{reduced}}}\left[(-1)^{2L_3+2(S_3-I_3)}(-1)^{2I_3}y^{2(J_3+I_3)}\right]$$

reduction to $\mathcal{M}$

$\sum_{A=\alpha} Q^A_\alpha$

$SO(3)_{\mathcal{J}}$

$$\Omega = (-1)^{\sum_{A>B}\langle\gamma_A,\gamma_B\rangle+n-1}\mathrm{tr}((-1)^F y^{2\mathcal{J}_3})$$

protected spin character for field theory BPS states

$$SO(4) = SU(2)_{\text{rotation}} \times SU(2)_{\text{R-symmetry}}$$

$$J \qquad\qquad\qquad I$$

$$\Omega = -\frac{1}{2}\mathrm{tr}\,(-1)^{2J_3}(2J_3)^2 \quad \underset{y=1}{\Leftarrow} \quad \Omega(y) = -\frac{1}{2}\mathrm{tr}\,(-1)^{2J_3}(2J_3)^2 y^{2I_3+2J_3}$$

2nd helicity trace

protected spin character

Gaiotto, Moore, Neitzke 2010 / Maldacena

step (5) equivalence to KS with two assumptions

Ashoke Sen 2011

1) all relevant charges γ_A on a single plane of the charge lattice

~ always true, anyhow, near marginal stability walls

2) in the absence of scaling regimes

~ in the absence of quiver invariants

~ more likely to hold for field theory BPS states

→ scaling regime is more typical of black hole examples

is KS true physically ?

how to see from BPS state building/counting ?

why rational invariants ?

$$\bar{\Omega}(\Gamma) = \sum_{p|\Gamma} \Omega(\Gamma/p)/p^2$$

input data ?

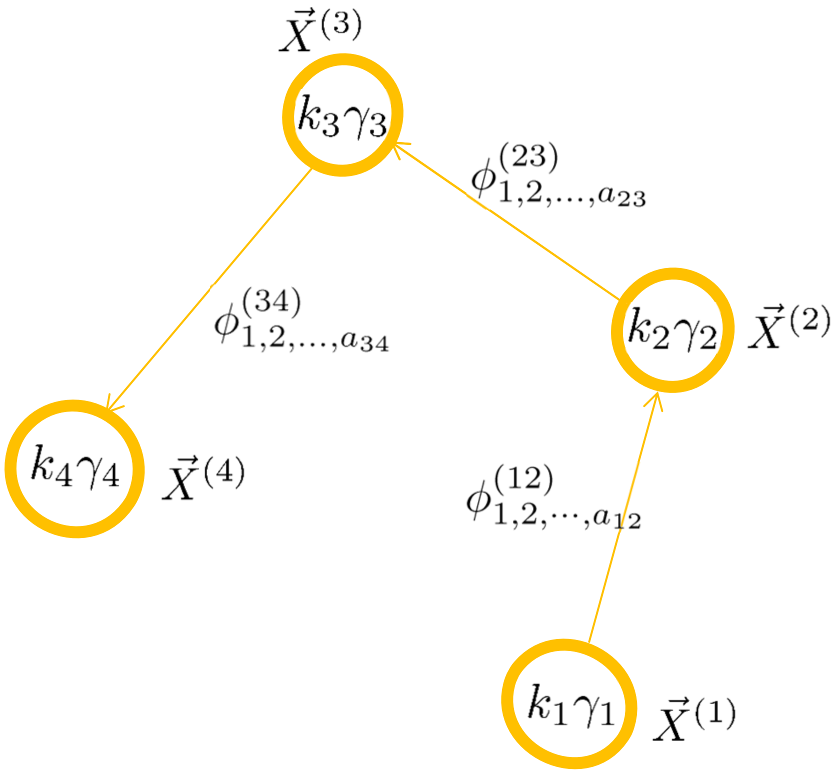
$$\Omega^+(\gamma) \neq 0 \neq \Omega^-(\gamma)$$

quivers & quiver invariants

Denef 2002
Denef+Moore 2007
Bena+Berkooz+de Boer+El-Showk+Van den Bleeken 2012
Lee+Wang+P.Y. 2012
Manschot+Pioline+Sen 2012/2013

Denef 2002

D3 wrapped on 3-cycles in CY3 → BPS quiver quantum mechanics

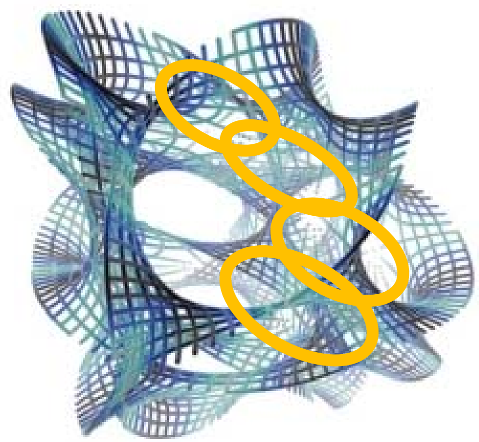


$$\vec{X}^{(1)} \qquad \vec{X}^{(2)} \qquad \vec{X}^{(3)} \qquad \vec{X}^{(4)}$$

$$U(k_1) \times U(k_2) \times U(k_3) \times U(k_4)$$

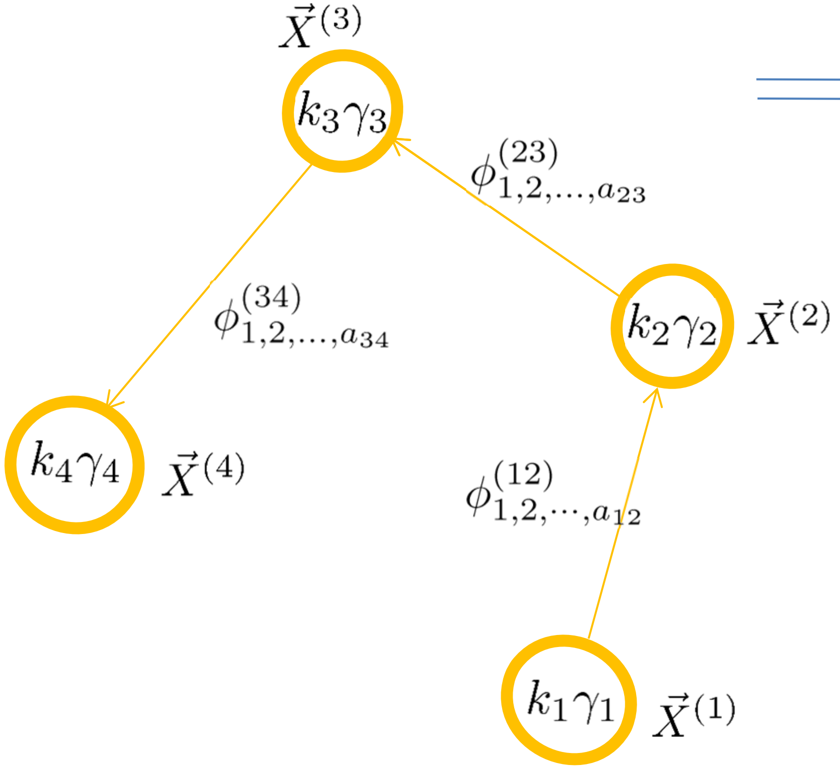
$$\phi^{(12)}_{1,2,\cdots,a_{12}} \qquad \phi^{(23)}_{1,2,\ldots,a_{23}} \qquad \phi^{(34)}_{1,2,\ldots,a_{34}}$$

$$a_{ik} = \langle \gamma_i, \gamma_k \rangle$$



Coulomb "phase" → multi-center picture of BPS states

small & "positive"
FI constants

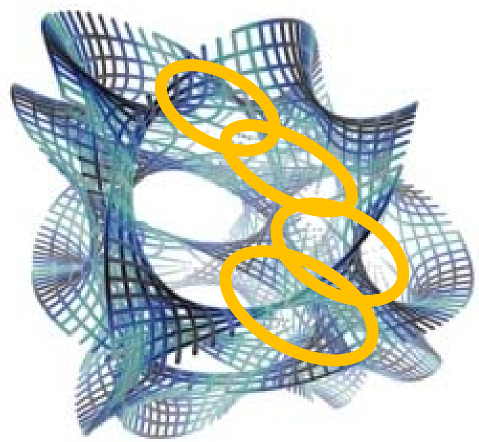


$$\vec{X}^{(1)} \qquad \vec{X}^{(2)} \qquad \vec{X}^{(3)} \qquad \vec{X}^{(4)}$$

$$U(k_1)\times U(k_2)\times U(k_3)\times U(k_4)$$

$$\phi^{(12)}_{1,2,\cdots,a_{12}} \qquad \phi^{(23)}_{1,2,\ldots,a_{23}} \qquad \phi^{(34)}_{1,2,\ldots,a_{34}}$$

$$a_{ik} = \langle \gamma_i, \gamma_k \rangle$$

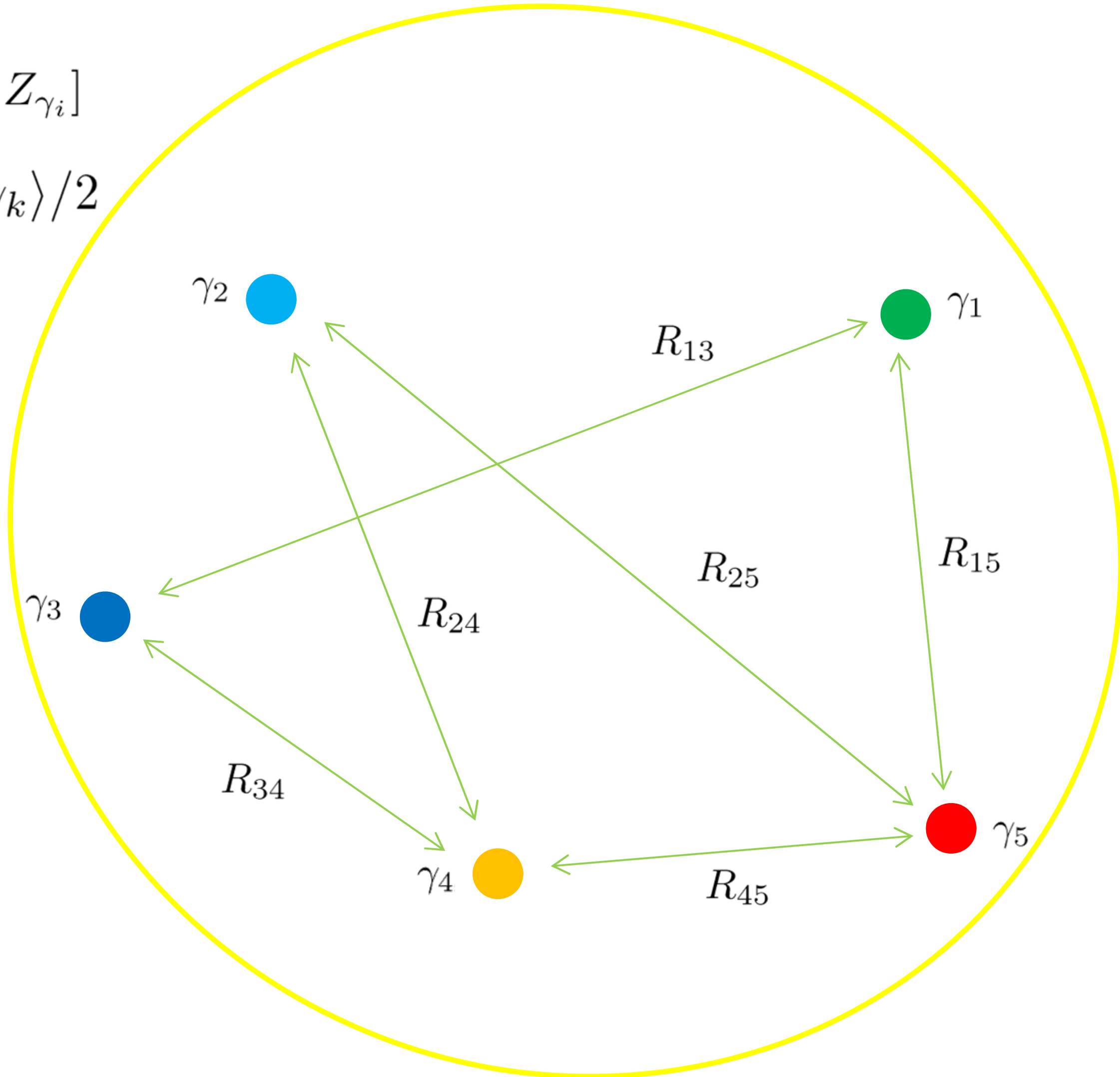


thus, index thm above for real-space wall-crossing also
also address general BPS states including black holes as well ?

$\xi^i \sim \mathrm{Im}[\zeta^{-1} Z_{\gamma_i}]$

$a_{ik} = \langle \gamma_i, \gamma_k \rangle / 2$

$R^3 = \{\vec{X}\}$



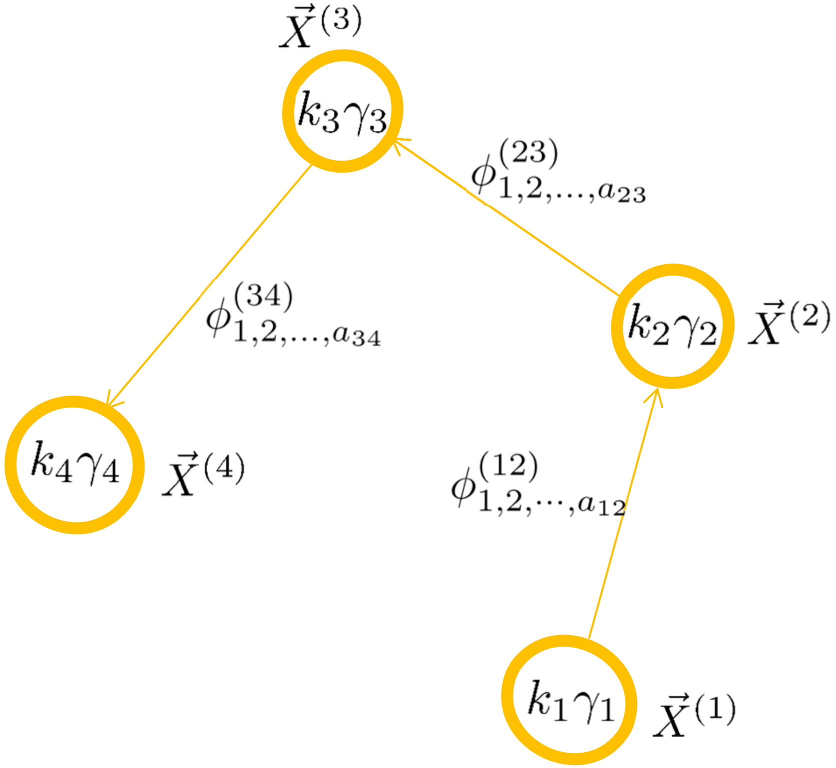
Higgs phase : $\mathcal{M}_{\rm H} = \{\phi^{(12)}, \ldots | D^{(i)} = \xi^{(i)} 1_{k_i \times k_i}\} / \prod_i U(k_i)$

large & "positive"
FI constants

$\vec X^{(1)} \quad \vec X^{(2)} \quad \vec X^{(3)} \quad \vec X^{(4)}$

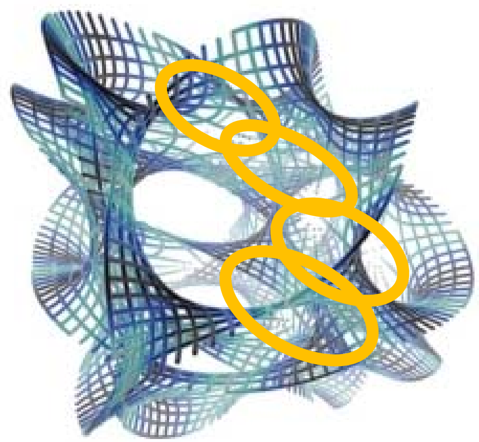
$$U(k_1) \times U(k_2) \times U(k_3) \times U(k_4)$$

$\phi^{(12)}_{1,2,\cdots,a_{12}} \quad \phi^{(23)}_{1,2,\ldots,a_{23}} \quad \phi^{(34)}_{1,2,\ldots,a_{34}}$

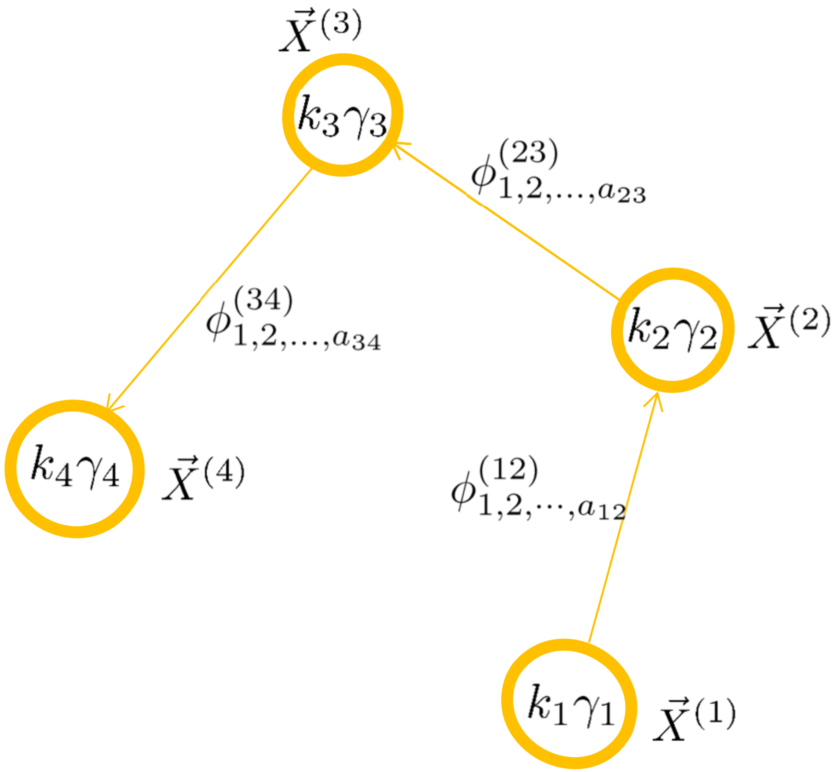


$\xi^i \sim {\rm Im}[\zeta^{-1} Z_{\gamma_i}]$

$a_{ik} = \langle \gamma_i, \gamma_k \rangle / 2$



Higgs phase : $\mathcal{M}_{\rm H} = \{\phi^{(12)}, \ldots | D^{(i)} = \xi^{(i)} 1_{k_i \times k_i}\} / \prod_i U(k_i)$



$$\Omega_{\rm Higgs}\left(\sum_i k_i\gamma_i; \xi^{(i)}\right)$$

$$\sim \chi\left(\mathcal{M}_{\rm H}\right)$$

$$= \sum_l (-1)^l \dim\left[H^l\left(\mathcal{M}_{\rm H}\right)\right]$$

large FI constants

small FI constants

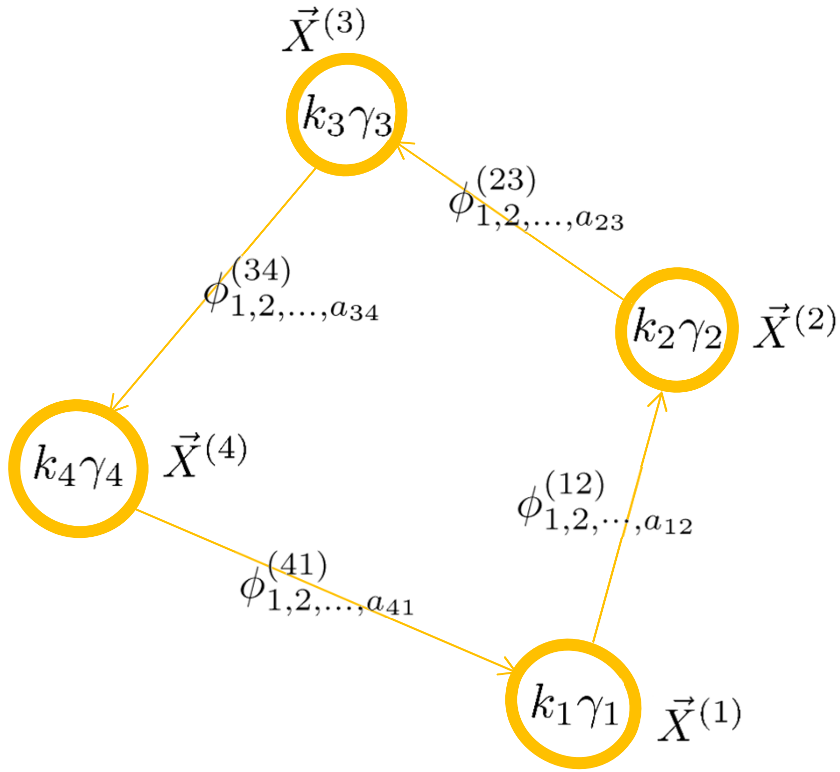
$$\Omega_{\text{Higgs}} = \Omega_{\text{Coulomb}}$$

F. Denef 2002 + A. Sen 2011

?

the equality is known to fail for quivers with loops

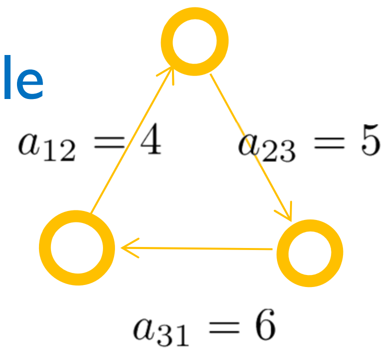
$$\Omega_{\text{Higgs}} \neq \Omega_{\text{Coulomb}}$$

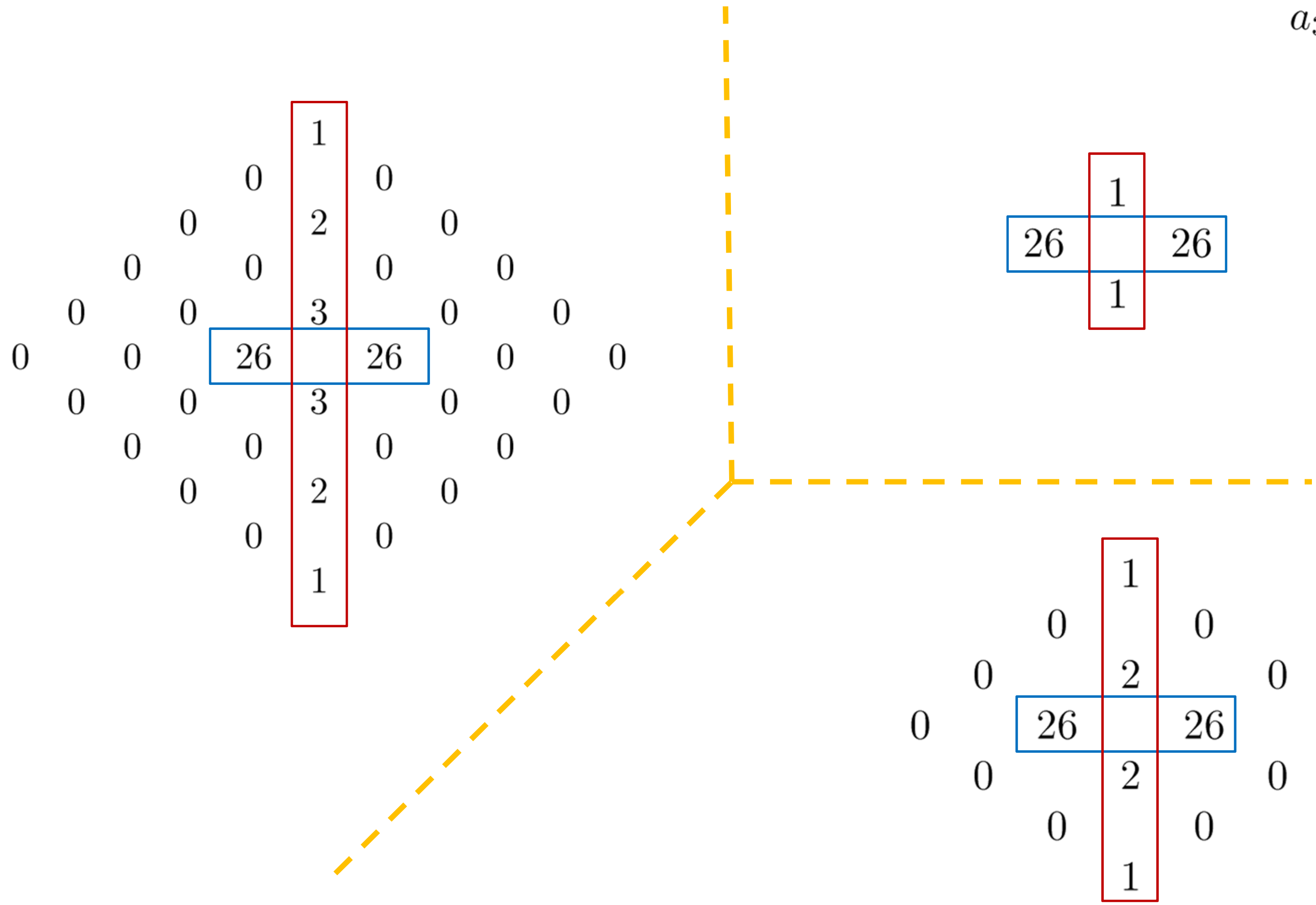


$$W(\phi) = \mathrm{tr}\left[\phi^{(12)}\phi^{(23)}\phi^{(34)}\phi^{(41)}\right]$$

a simple but illuminating example

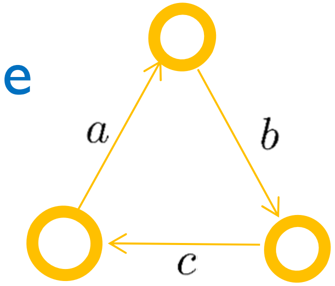
$$\dim\, H^{(p,q)}(\mathcal{M}_H)$$





a simple but illuminating example

Denef + Moore 2007

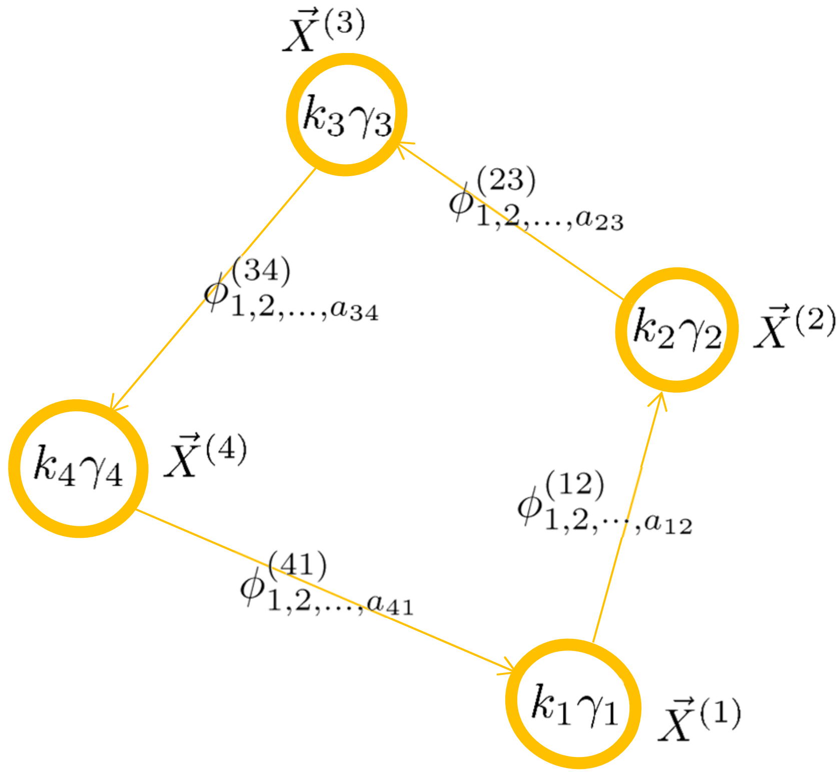


$$= \Omega_{\text{Coulomb}}$$

$$\Omega_{\text{Higgs}} = \begin{pmatrix} a\cdot(c-b) \\ b\cdot(a-c) \\ c\cdot(b-a) \end{pmatrix} + \# \cdot 2^{(a+b+c)/2} + \cdots$$

what physical & mathematical properties characterize these Higgs-only wall-crossing-safe BPS states ?

$$\Omega^{\pm}_{\mathrm{Higg}} = \Omega_{\mathrm{Invariant}} + \Omega^{\pm}_{\mathrm{Coulomb}}$$



$$W(\phi) = \mathrm{tr}\left[\phi^{(12)}\phi^{(23)}\phi^{(34)}\phi^{(41)}\right]$$

also, some of wall-crossing formulae need input data when

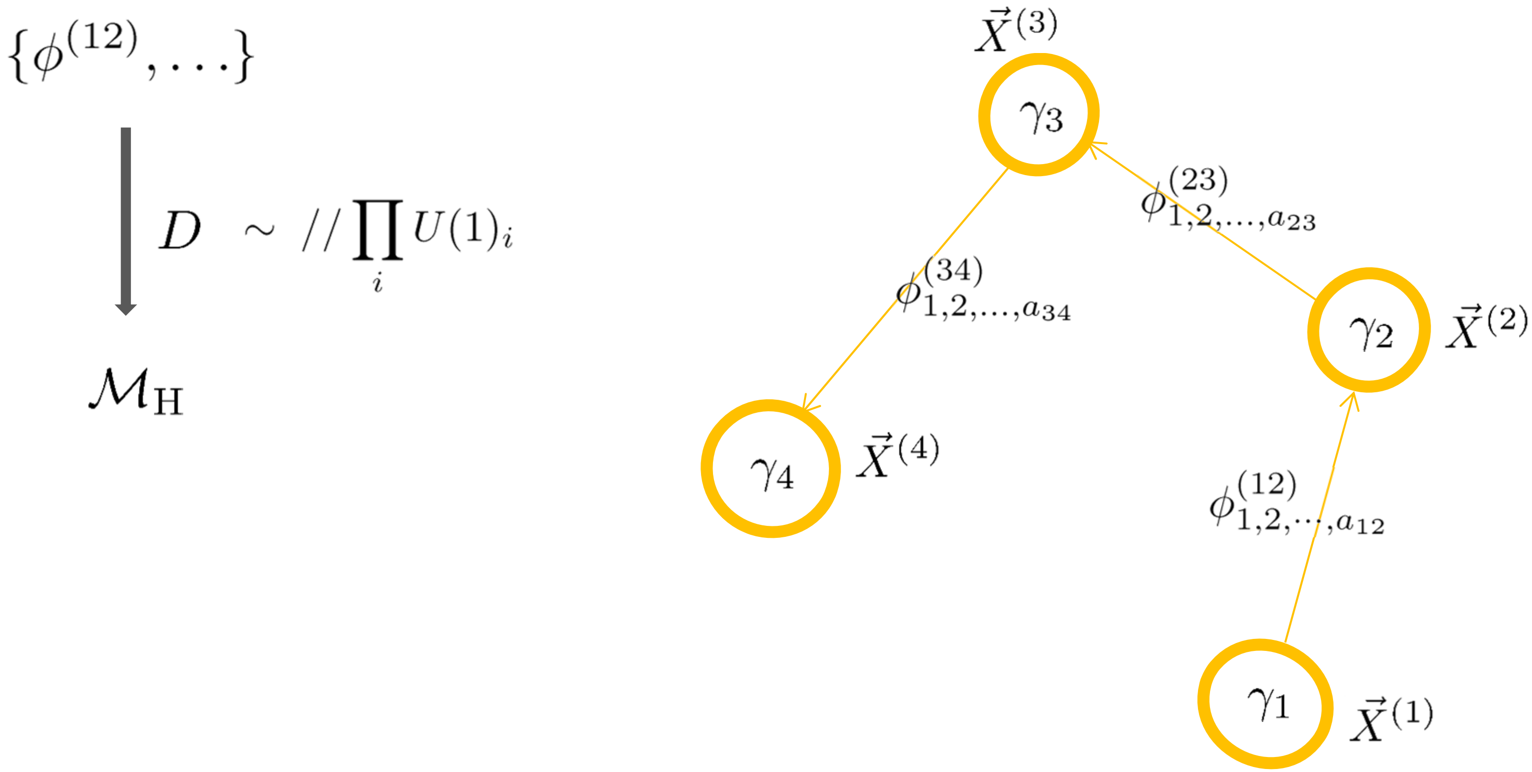
$$\Omega^+(\gamma) \neq 0 \neq \Omega^-(\gamma)$$

how to isolate and count wall-crossing-safe states in such cases ?

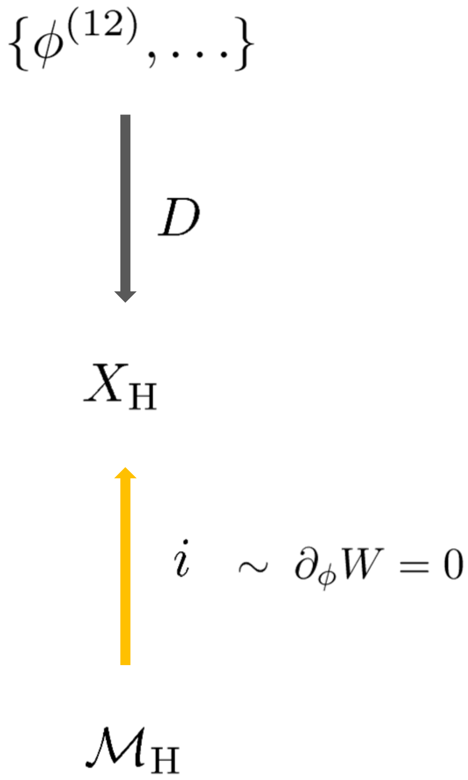
quiver invariants

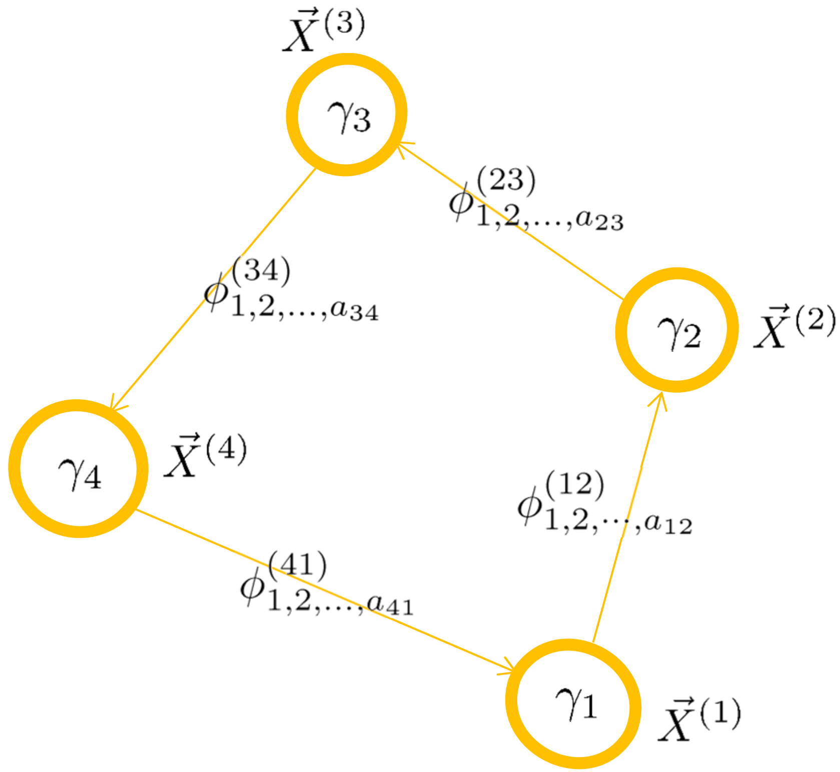
for cyclic Abelian quivers

$$\Omega_{\text{Higgs}} = \Omega_{\text{Coulomb}}$$



$$\Omega_{\text{Higgs}} \neq \Omega_{\text{Coulomb}}$$





$$W(\phi) \sim \phi^{(12)}\phi^{(23)}\phi^{(34)}\phi^{(41)}$$

wall-crossing vs. wall-crossing-safe

$\{\phi^{(12)}, \ldots\}$ $\quad H^*(\mathcal{M}_H)$

D

$$= \; i^*\left[H^*(X_H)\right] \;\oplus\; H^*(\mathcal{M}_H)_{\text{Intrinsic}}$$

X_{H}

complete intersection $\quad i \; \sim \; \partial_\phi W = 0$

$\mathcal{M}_{\text{H}}$

wall-crossing vs. wall-crossing-safe

$$\{\phi^{(12)}, \ldots\} \qquad H^*(\mathcal{M}_H) = \sum H^{(p,q)}(\mathcal{M}_H)$$

$$\Big\downarrow D \qquad = i^*\left[H^*(X_H)\right] \oplus H^*(\mathcal{M}_H)_{\text{Intrinsic}}$$

$$X_{\text{H}} \qquad \text{tr}_{i^*(H(X))}(-1)^{p+q-d} y^{2p-d} \qquad \text{tr}_{\text{Intrinsic}}(-1)^{p+q-d} y^{2p-d}$$

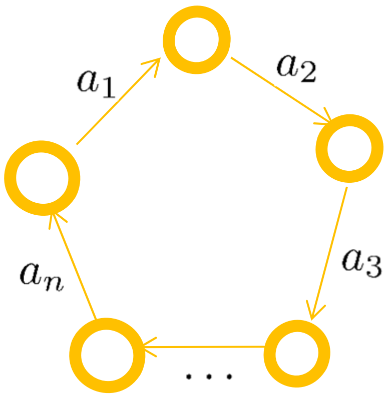
complete intersection $\Big\uparrow i$ $\qquad \Updownarrow \qquad \Updownarrow$

$$\mathcal{M}_{\text{H}} \qquad \Omega_{\text{Coulomb}} \qquad \Omega_{\text{Invariant}}$$

S.J. Lee + Z.L. Wang + P.Y., 2012

Bena + Berkooz + de Boer + El-Showk + d. Bleeken, 2012

general proof & explicit counting !



$$H^*(\mathcal{M}_H) = \sum H^{(p,q)}(\mathcal{M}_H)$$

$$= i^*\left[H^*(X_H)\right] \oplus H^*(\mathcal{M}_H)_{\text{Intrinsic}}$$

$$\mathrm{tr}_{i^*(H(X))}(-1)^{p+q-d}y^{2p-d} \qquad \mathrm{tr}_{\text{Intrinsic}}(-1)^{p+q-d}y^{2p-d}$$

$$\Updownarrow \qquad \Updownarrow$$

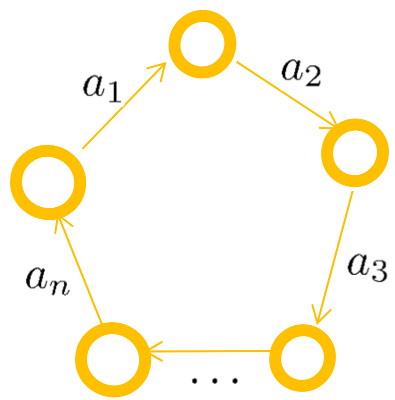
$$\Omega_{\text{Coulomb}} \qquad \Omega_{\text{Invariant}}$$

S.L. Lee + Z.L. Wang + P.Y., 2012
Manschot + Pioline + Sen, 2012

the total equivariant index ~ Hirzebruch character

$$\Omega^{(k)}_{\text{Higgs}}(y) = \text{tr}_{H^*(\mathcal{M}^{(k)}_H)}(-1)^{2J_3} y^{2J_3+2I} = \sum (-1)^{p+q-d} y^{2p-d} h^{(p,q)}(\mathcal{M}^{(k)}_H)$$

$$= (-y)^{-d_k} \chi_{t=-y^2}(\mathcal{M}^{(k)}_H)$$



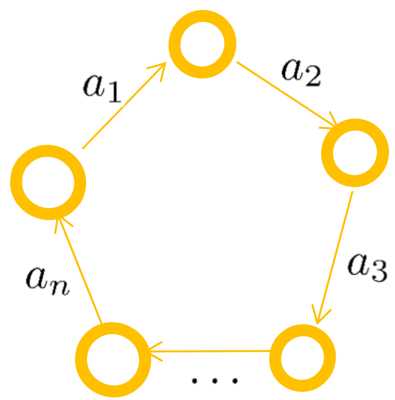
which is easily computable here, via Riemann-Roch theorem

$$\chi_t(\mathcal{M}_H^{(k)}) = \frac{1}{(1+t)^n} \int_{X_H^{(k)}} \left[\prod_{i \neq k} \left(J_i \frac{1+te^{-J_i}}{1-e^{-J_i}} \right)^{a_i} \right] \cdot \left(\frac{1-e^{-\sum_{i\neq k} J_i}}{1+te^{-\sum_{i\neq k} J_i}} \right)^{a_k}$$

$$X_{\mathrm{H}}^{(k)} = \prod_{i \neq k} CP^{a_i - 1}$$

embedding map ↑ i

$$\mathcal{M}_{\mathrm{H}}^{(k)}$$



and decomposed into two parts

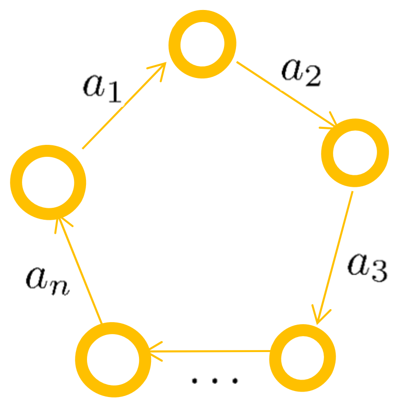
$$\Omega^{(k)}_{\text{Higgs}}(y) = \boxed{(-1)^{d_k} y^{a_k} \prod_{i \neq k} \frac{y^{a_i} - y^{-a_i}}{y - y^{-1}} + \Delta\Omega_{\{a_i\}}(y)}$$

$$+ \boxed{\frac{(-y)^{n+2-\sum_i a_i}}{(y^2-1)^n} \prod_i \oint_{\omega_i = 1} \frac{d\omega_i}{2\pi i} \left[\prod_i \left(\frac{1 - y^2 \omega_i}{1 - \omega_i} \right)^{a_i} \right] \cdot \frac{1}{1 - y^2 \prod_i \omega_i} - \Delta\Omega_{\{a_i\}}(y)}$$

$$X^{(k)}_{\text{H}} = \prod_{i \neq k} CP^{a_i - 1}$$

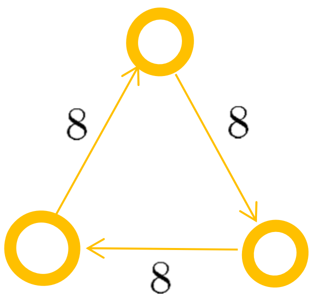
embedding map $\Big\uparrow\, i$

$$\mathcal{M}^{(k)}_{\text{H}}$$



wall-crossing states vs. wall-crossing-safe states

$$H^*(\mathcal{M}_H) \;=\; i^*\left[H^*(X_H)\right] \;\oplus\; H^*(\mathcal{M}_H)_{\text{Intrinsic}}$$



$\dim H^{(p,q)}(\mathcal{M}_H)$

						1						
					0		0					
				0		2		0				
			0		0		0		0			
		0		0		3		0		0		
	0		0		0		0		0		0	
1		322		6803		4 + 18212		6803		322		1
	0		0		0		0		0		0	
		0		0		3		0		0		
			0		0		0		0			
				0		2		0				
					0		0					
						1						

wall-crossing states vs. wall-crossing-safe states

$$H^*(\mathcal{M}_H) = i^*\left[H^*(X_H)\right] \oplus H^*(\mathcal{M}_H)_{\mathrm{Intrinsic}}$$

$\dim H^{(p,q)}(\mathcal{M}_H)$

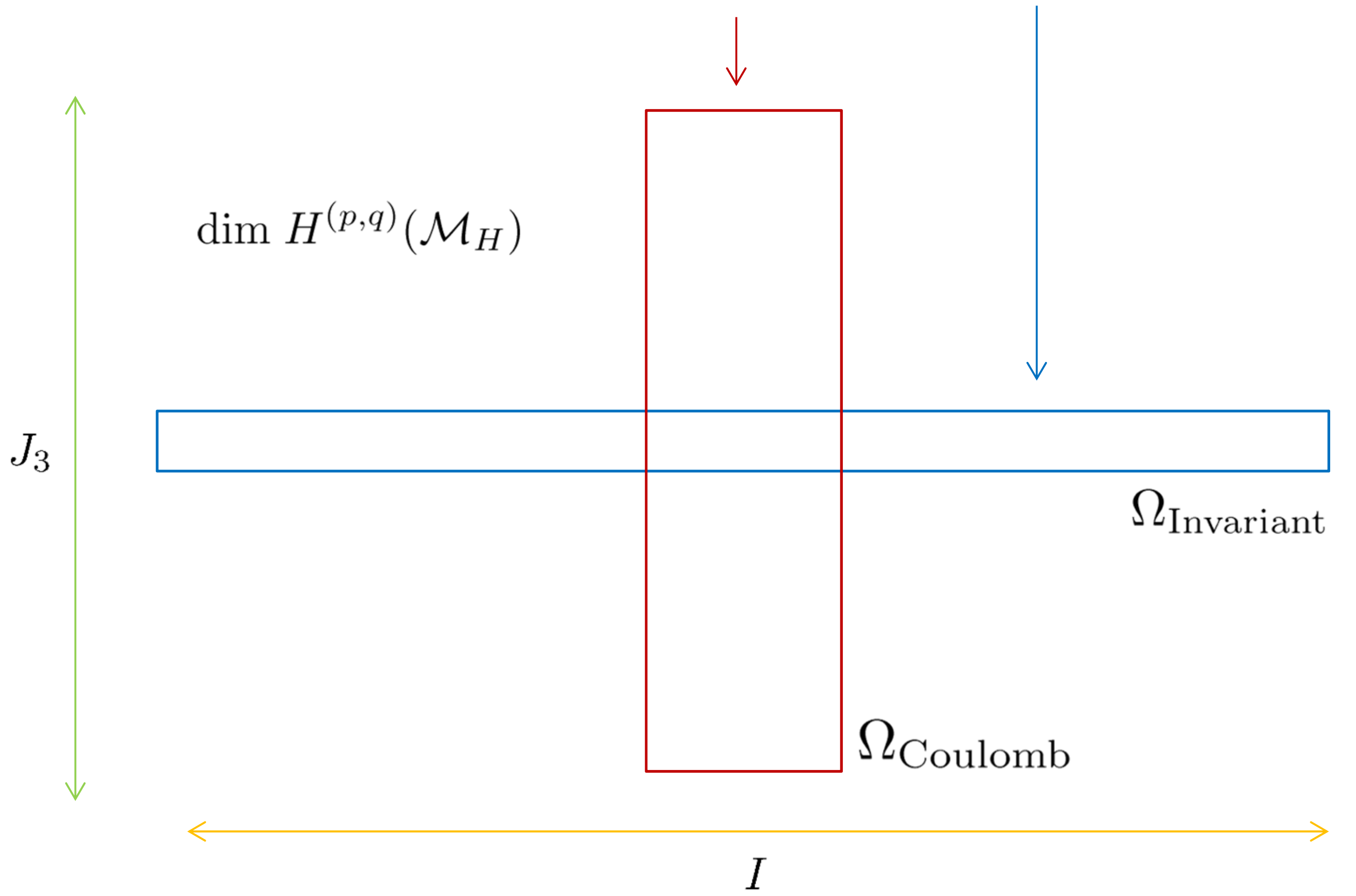
						1						
					0		0					
				0		2		0				
			0		0		0		0			
		0		0		3		0		0		
	0		0		0		0		0		0	
1		322		6803		$4 + 18212$		6803		322		1
	0		0		0		0		0		0	
		0		0		3		0		0		
			0		0		0		0			
				0		2		0				
					0		0					
						1						

J_3

I

wall-crossing states vs. wall-crossing-safe states

$$H^*(\mathcal{M}_H) = i^*\left[H^*(X_H)\right] \oplus H^*(\mathcal{M}_H)_{\text{Intrinsic}}$$



wall-crossing states vs. wall-crossing-safe states

$$H^*(\mathcal{M}_H) = i^*\left[H^*(X_H)\right] \oplus H^*(\mathcal{M}_H)_{\text{Intrinsic}}$$

$$\Updownarrow \quad \mathrm{tr}(-1)^{p+q-d} y^{2p-d} \quad \Updownarrow$$

$$\Omega_{\text{Coulomb}} \qquad \Omega_{\text{Invariant}} = \Omega_{\text{Higgs}} - \Omega_{\text{Coulomb}}$$

many-body bound states
wall-crossing

angular momentum
multiplets

single-center states
wall-crossing-safe

angular momentum
singlets

wall-crossing states vs. wall-crossing-safe states

$$H^*(\mathcal{M}_H) = i^*[H^*(X_H)] \oplus H^*(\mathcal{M}_H)_{\mathrm{Intrinsic}}$$

$$\Updownarrow \quad \mathrm{tr}(-1)^{p+q-d} y^{2p-d} \quad \Updownarrow$$

$$\Omega_{\mathrm{Coulomb}} \qquad \Omega_{\mathrm{Invariant}} = \Omega_{\mathrm{Higgs}} - \Omega_{\mathrm{Coulomb}}$$

many-body bound states
wall-crossing

single-center states
wall-crossing-safe

polynomial degeneracy:
most of familiar BPS states in field theories belong here

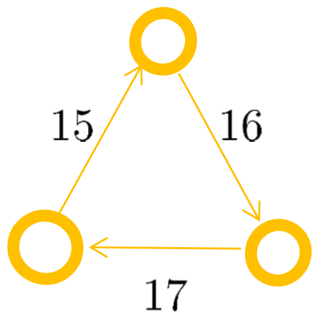
exponential degeneracy:
single-center BH's belong here

more examples of quiver invariants

$$H^*(\mathcal{M}_H) = i^*[H^*(X_H)] \oplus H^*(\mathcal{M}_H)_{\text{Intrinsic}}$$

$$\Omega(y)\Big|^{\{a_{i,i+1}\}=(15,16,17)}_{\text{Intrinsic}} = \text{tr}_{\text{Intrinsic}}(-1)^{2J_3}y^{2J_3+2I} = \sum(-1)^{p+q-d}y^{2p-d}h^{(p.q)}_{\text{Intrinsic}}$$

$$\begin{aligned}
&= 1665y^{-12}\\
&+ 724674y^{-10}\\
&+ 60686563y^{-8}\\
&+ 1523273844y^{-6}\\
&+ 13886938949y^{-4}\\
&+ 50685934038y^{-2}\\
&+ 77668453887\\
&+ 50685934038y^{2}\\
&+ 13886938949y^{4}\\
&+ 1523273844y^{6}\\
&+ 60686563y^{8}\\
&+ 724674y^{10}\\
&+ 1665y^{12}
\end{aligned}$$

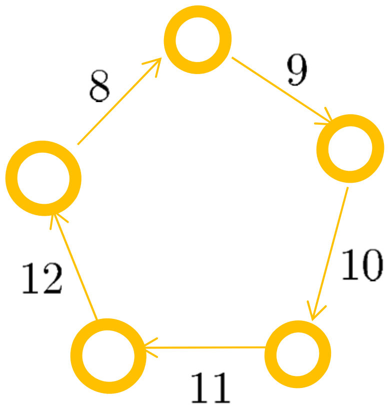


more examples of quiver invariants

$$H^*(\mathcal{M}_H) = i^*\left[H^*(X_H)\right] \oplus H^*(\mathcal{M}_H)_{\text{Intrinsic}}$$

$$\Omega(y)\Big|_{\text{Intrinsic}}^{\{a_{i,i+1}\}=(8,9,10,11,12)} = \text{tr}_{\text{Intrinsic}}(-1)^{2J_3}y^{2J_3+2I} = \sum(-1)^{p+q-d}y^{2p-d}h^{(p,q)}_{\text{Intrinsic}}$$

$$\begin{aligned} = \; & 32294250/y^{22} + 58872952926/y^{20} + 23086762587054/y^{18} \\ & +3146301650299568/y^{16} + 186529800766285403/y^{14} \\ & +5480846262397291070/y^{12} + 86780383421802203555/y^{10} \\ & +783408269154731872224/y^{8} + 4192271239441338802849/y^{6} \\ & +13657486692285216220742/y^{4} + 27560691162972524163666/y^{2} \\ & +34791235315880411958041 + 27560691162972524163666y^{2} \\ & +13657486692285216220742y^{4} + 4192271239441338802849y^{6} \\ & +783408269154731872224y^{8} + 86780383421802203555y^{10} \\ & +5480846262397291070y^{12} + 186529800766285403y^{14} \\ & +3146301650299568y^{16} + 23086762587054y^{18} \\ & +58872952926y^{20} + 32294250y^{22} \end{aligned}$$



the real-space wall-crossing formulae must be revised recursively such that

Manschot+Pioline+Sen 2012/2013

$$\bar{\Omega}(\gamma) \to \bar{\Omega}_{\rm Intrinsic}(\gamma) + \cdots$$

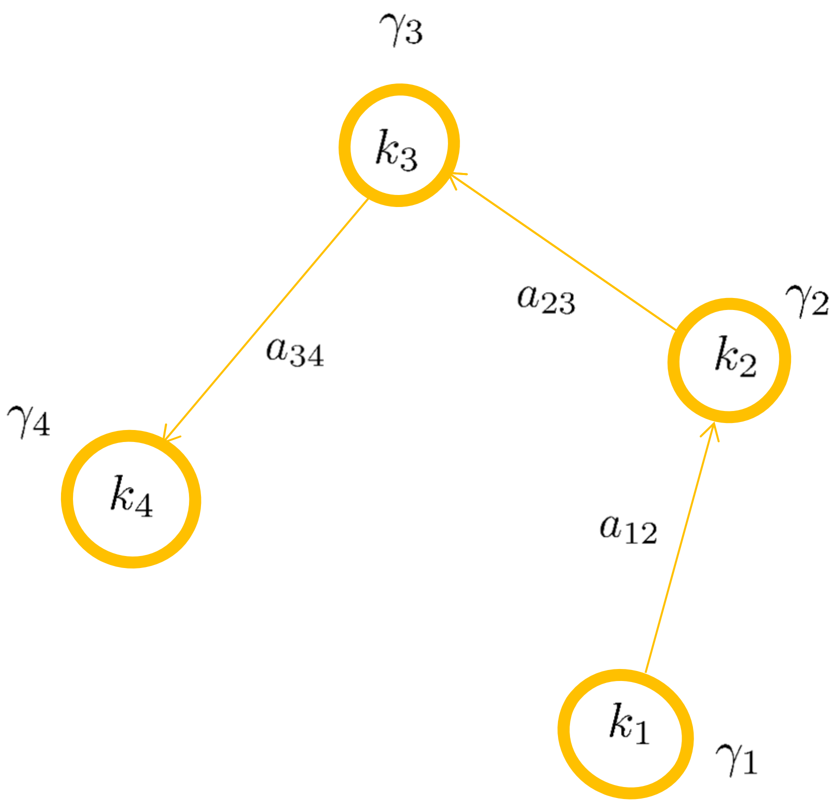
$$\begin{aligned}
\Omega^-\left(\sum \gamma_A\right) - \Omega^+\left(\sum \gamma_A\right) &= (-1)^{\sum_{A>B}\langle\gamma_A,\gamma_B\rangle+n-1}\frac{\prod_A \bar{\Omega}^-(\gamma_A)}{|\Gamma|}\int_{\mathcal{M}} ch(\mathcal{F})\wedge\mathcal{A}(M)\\
&\quad\vdots\\
&\quad+(-1)^{\sum_{A'>B'}\langle\gamma'_{A'},\gamma'_{B'}\rangle+n'-1}\frac{\prod_{A'} \bar{\Omega}(\gamma'_{A'})}{|\Gamma'|}\int_{\mathcal{M}'} ch(\mathcal{F}')\wedge\mathcal{A}(M')\\
&\quad\vdots\\
&\quad+(-1)^{\sum_{A''>B''}\langle\gamma''_{A''},\gamma''_{B''}\rangle+n''-1}\frac{\prod_{A''} \bar{\Omega}(\gamma''_{A''})}{|\Gamma''|}\int_{\mathcal{M}''} ch(\mathcal{F}'')\wedge\mathcal{A}(M'')\\
&\quad\vdots
\end{aligned}$$

quiver mutations

is there a more intelligent way to count BPS states ?

Derksen + Weyman + Zelevinsky 2007/2009
Keller 2011
Alim+Cecotti+Cordova+Espahbodi+Rastogi+Vafa 2011

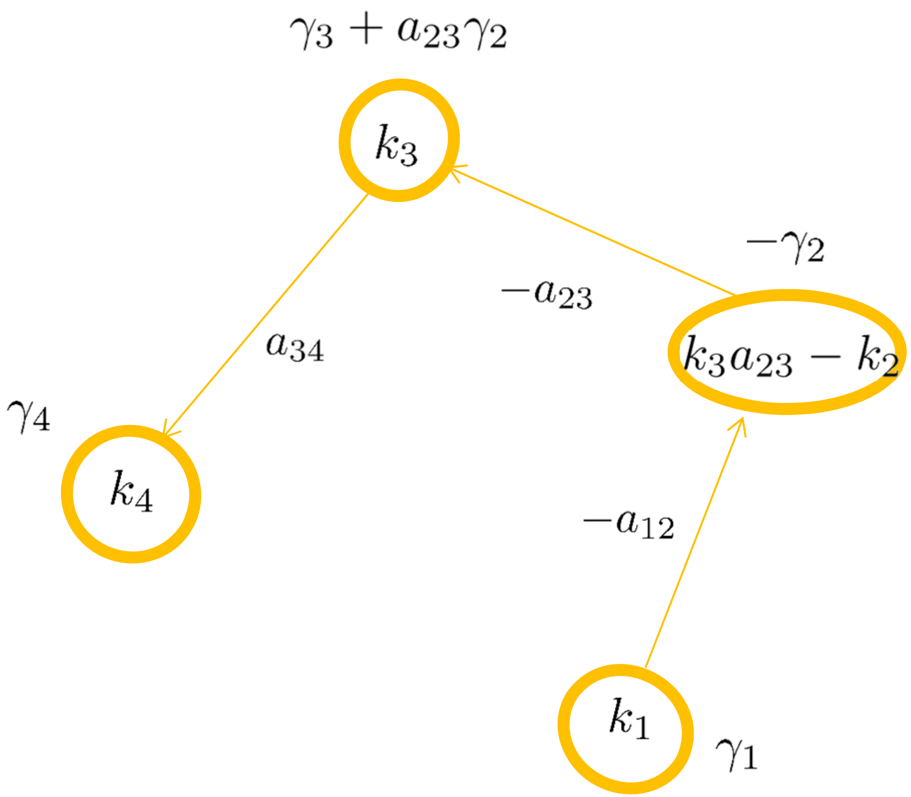
quiver mutation ~ Seiberg duality for quiver rep. theory



$$\gamma_{\text{total}} = \sum k_i \gamma_i$$

Derksen + Weyman + Zelevinsky 2007/2009
Keller 2011

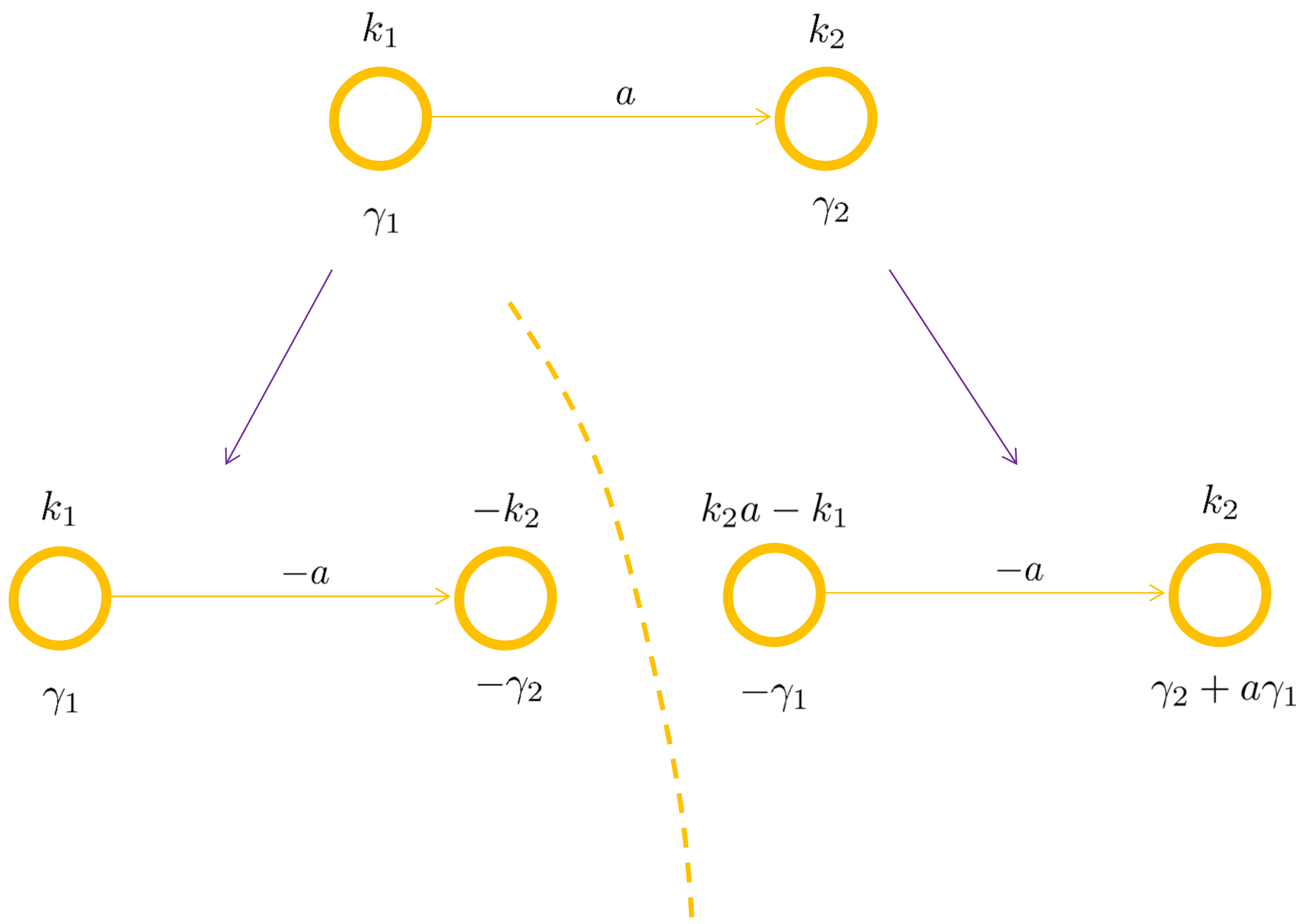
quiver mutation ~ Seiberg duality



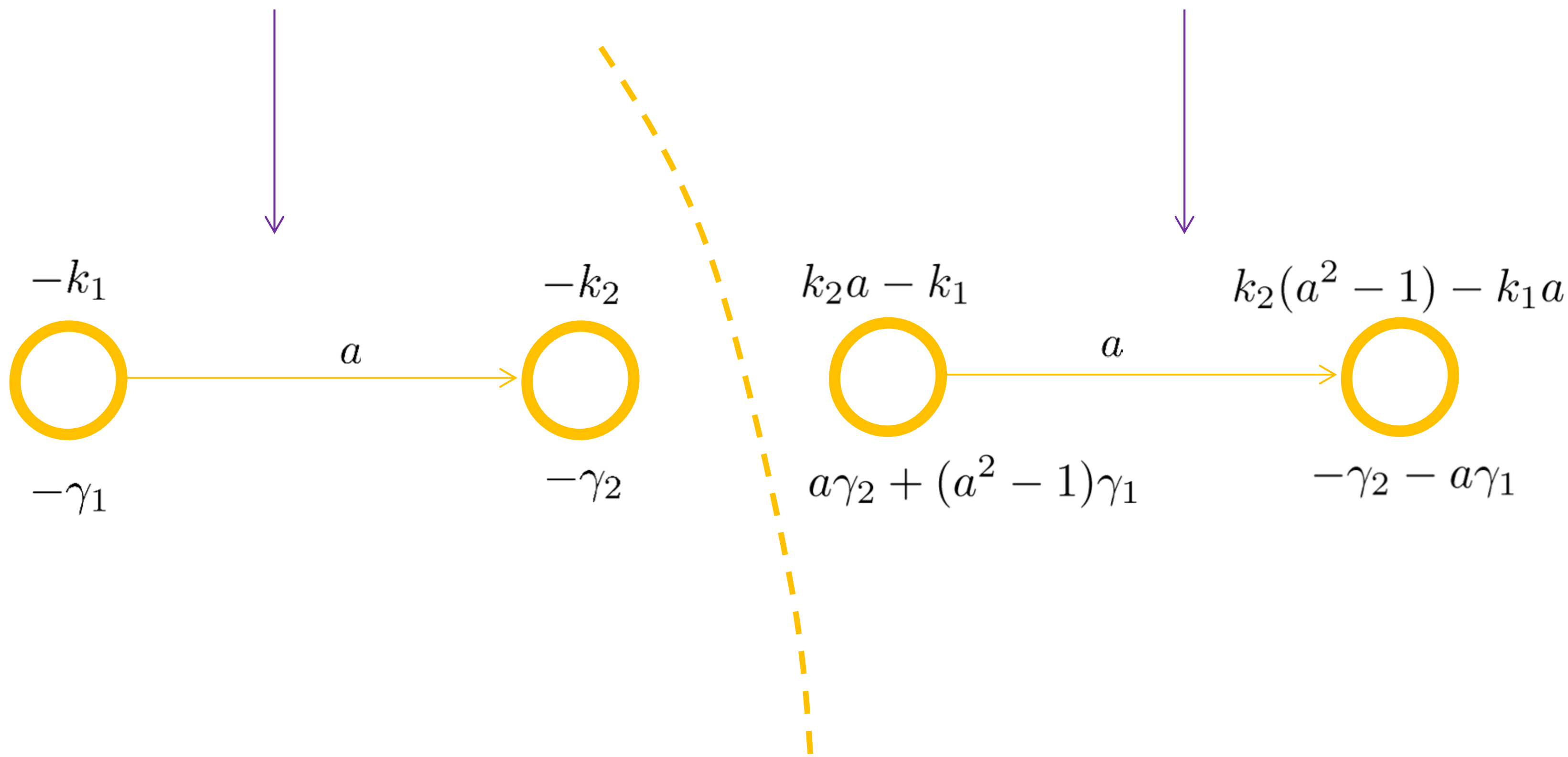
$$\gamma_{\text{total}} = \sum k_i \gamma_i$$
$$= \sum k'_i \gamma'_i$$

Derksen + Weyman + Zelevinsky 2007/2009
Keller 2011

quiver mutation ~ Seiberg duality



quiver mutation ~ Seiberg duality

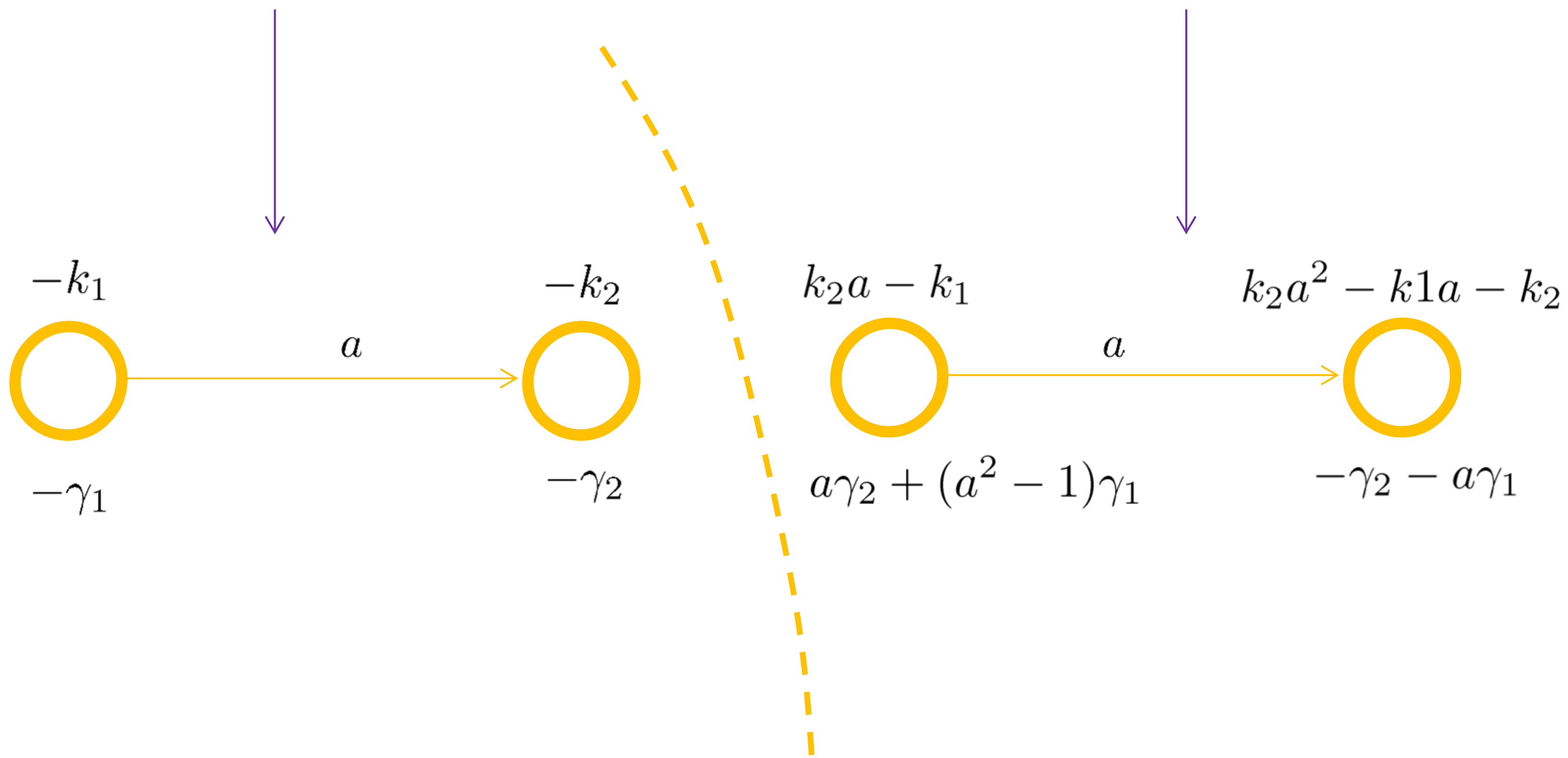


BPS quiver ~ quiver of 2r+f basis hypermultiplets

$$\Omega\left(\sum k_i\gamma_i\right) \neq 0 \qquad k_i \geq 0 \text{ for all } i \text{ or } k_i \leq 0 \text{ for all } i$$

Andriyash+Denef+Jafferis+Moore 2012

Alim+Cecotti+Cordova+Espahbodi+Rastogi+Vafa 2011



BPS quiver ~ quiver of 2r+f basis hypermultiplets

$$\Omega\left(\sum k_i \gamma_i\right) \neq 0 \ \text{ if and only if } k_i \geq 0 \text{ for all } i \text{ or } k_i \leq 0 \text{ for all } i$$

Alim+Cecotti+Cordova+Espahbodi+Rastogi+Vafa 2011

γ_1 $\quad -\gamma_1$ $\quad \sum k_i \gamma_i$

γ_2 $\quad -\gamma_2$

BPS quiver ~ quiver of 2r+f basis hypermultiplets

$$\Omega\left(\sum k_i\gamma_i\right) \neq 0 \quad \text{if and only if } k_i \geq 0 \text{ for all } i \text{ or } k_i \leq 0 \text{ for all } i$$

Alim+Cecotti+Cordova+Espahbodi+Rastogi+Vafa 2011
Xie 2012

γ_1 $\quad -\gamma_1$

γ_2 $\quad -\gamma_2$

for hypers, one can mutate enough to find eventually

$$\sum k_i\gamma_i$$

$$\to \sum k_i'\gamma_i'$$

$$= 0 + \cdots + \gamma_p' + \cdots + 0$$

which by itself can determine entire hyper content in a given chamber; finite chambers offer ideal test-bed

quiver mutations in its most general form may prove to be
a very efficient particle#-reducing algorithm
if we can unclutter the intricate chamber dependences

summary

KS & GMN represent giant leaps over what we knew before, bits and pieces, mostly for Seiberg-Witten theories; various technical difficulties for rank > 1 field theories & for BH's still remain

real-space-based, constructive approach to wall-crossing has grown very competitive, last 2~3 years, with partial equivalence to KS shown

the intuitive Coulomb picture with recursive wall-crossing augmented by the comprehesive Higgs picture with the extra wall-crossing-safe states

quiver invariants = wall-crossing-safe states (input data for KS, e.g.) are essential ingredient to the wall-crossing beyond simple examples
→ Jan's talk, tomorrow, for how to do this……