Moduli spaces, instantons, monopoles and Quantum Algebras

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4d gauge theories

- What can we do beyond perturbation theory?
- Are there hidden algebraic structures?
- What are the exactly computable quantities?

2d CFT
integrable lattice models
quantum integrable systems

- conserved quantities
- holomorphic factorization
- conformal blocks, OPE
- Kac-Moody, Virasoro, W-algebra
- quantum groups, R-matrix

What of these structures appear in gauge theories?

For 4d $\mathcal{N} = 2$ theories

- vacua sector = integrable system
- holomorphic factorization
- emergence of quantum algebras
  - CFT type (Virasoro, affine Kac-Moody)
  - Spin chain type (R-matrix, quantum group)

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vacua = integrable system

Coulomb branch of vacua of 4d $\mathcal{N} = 2$ gauge theory $\Longleftrightarrow$ base of algebraic integrable system

Seiberg-Witten'94
GKMMM'95
Donagi-Witten'95
Seiberg-Witten'96

vacua = integrable system

$$\mathcal{M}_{\mathbf{R}^3\times\mathbf{S}^1} \qquad \text{phase space (dim=2r)} \quad \text{(HyperKahler)}$$

$$\Big\downarrow \; \text{Lagrangian fibration}$$

$$\mathcal{M}_{\mathbf{R}^4} \equiv \mathcal{U} \qquad \text{base (dim = r)} \quad \text{(Special Kahler)}$$

What we know?

For 4d $\mathcal{N}=2$ theories

- vacua sector = integrable system
- holomorphic factorization
- emergence of quantum algebras
    - Kac-Moody/Virasoro/W-algebra type
    - Quantum groups (R-matrix, spin chains)

holomorphic factorization

$$Z_{\mathbf{S}^4} = \int da\, Z(a)\overline{Z(a)}$$ **VP'07**

$Z(a)$ = partition function on $\mathbf{R}^4_{\varepsilon_1,\varepsilon_2}$ **Nekrasov'02**

$Z_{\varepsilon_1,\varepsilon_2}(a)$ = 2d CFT conformal block **Alday-Gaiotto-Tachikawa'08**

holomorphic factorization

700 pages review book
coming out to arXiv next week

Localization techniques in quantum field theories

Vasily Pestun, Maxim Zabzine,
Francesco Benini, Tudor Dimofte,
Thomas T. Dumitrescu,
Kazuo Hosomichi, Seok Kim,
Kimyeong Lee, Bruno Le Floch,
Marcos Marino, Joseph A. Minahan,
David R. Morrison, Sara Pasquetti,
Jian Qiu, Leonardo Rastelli,
Shlomo S. Razamat, Silvu S. Pufu,
Yuji Tachikawa, Brian Willett,
Konstantin Zarembo

$Z_{\varepsilon_1,\varepsilon_2}(a)$ is conformal block of what algebra?

Where does it come from?

emergence of quantum algebras

If 4d theory of ‘*class $S_{\mathbf{g}}(C)$*’ then

Z(a) = conformal block of W(**g**) algebra
on 2d surface C

Alday-Gaiotto-Tachikawa’08

W($\mathbf{sl_2}$) = Vir

‘*Class $S_{\mathbf{g}}(C)$*’ is the 4d theory obtained by

reduction of 6d (0,2) type **g** on C

$$\mathcal{M}_{\mathbf{R}^3\times S^1} = \mathcal{M}_{\mathrm{Hitchin}}(C, \mathbf{g})$$

emergence of quantum algebras

$$\mathcal{M}_{\mathrm{Hitchin}}(C, \mathbf{g}) \longrightarrow \mathrm{W}_{\varepsilon_1, \varepsilon_2}(\mathrm{g}) \text{ algebra}$$

Alday-Gaiotto-Tachikawa ’08
Nekrasov-Witten ‘12
Teschner ’13

emergence of quantum algebras

$$\mathcal{M}_{\mathrm{Hitchin}}(C, \mathbf{g}) \longrightarrow \mathrm{W}_{\varepsilon_1, \varepsilon_2}(\mathsf{g}) \text{ algebra}$$

Alday-Gaiotto-Tachikawa ’08
Nekrasov-Witten ‘12
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What replaces W(**g**)-algebra for generic $\mathcal{M}$?

emergence of quantum algebras

Proposition

For generic $\mathcal{M}$, W-algebra $W_{\varepsilon_1,\varepsilon_2}(\mathcal{M})$ is ε_1-quantized algebra of holomorphic functions on $\mathcal{M}$ in HyperKahler $\varepsilon_2/\varepsilon_1$-rotated complex structure

$$W_{\varepsilon_1,\varepsilon_2}(M) = \mathbb{C}_{\varepsilon_1}[M_{\varepsilon_2/\varepsilon_1}]$$

Quantization
(Planck constant)

HyperKahler
twistor space
parameter

Nekrasov-Witten, Bellinson-Drinfeld
Teschner, Kashaev
Kontsevich-Soibelman, Gaiotto-Moore-Neitzke
Fock-Goncharov,
Neitzke, Vafa, Cecotti-Neitzke-Vafa
Nekrasov-Rosly-Shatashvili, Gaiotto
Kontsevich, Gukov-Witten

Quantization on HK

(X,ω) real symplectic manifold

Quantization has **one** parameter:

$$\hbar$$

Dirac
Fedosov
Kontsevich

$$\hat{f} * \hat{g} - \hat{g} * \hat{f} = \hbar \{f, g\}_{\omega^{-1}} + \dots$$

$(X, \omega_I, \omega_J, \omega_K)$ HK manifold

Quantization has **two** complex parameters:

1) $\zeta \in \mathbb{CP}^1_{\text{twistor}}$

2) $\varepsilon \in \mathbb{C}$

Kontsevich
Cattaneo-Felder
Kapustin-Orlov
Gukov-Witten

$(X, \hbar^{-1}\Omega_\zeta)$ holomorphic symplectic

quantized by non-commutative open B-model

What we know about $W_{\varepsilon_1,\varepsilon_2}(\mathcal{M})$?

associative algebra

1. Seiberg-Witten limit

$\varepsilon_1 \to 0$

$\varepsilon_2 \to 0$

$$W_{\varepsilon_1,\varepsilon_2}(\mathcal{M}) = \mathbb{C}[\mathcal{U}]$$

classical commutative ring

'functions on u-plane'
'Coulomb chiral ring in 4d'
'classical commuting Hamiltonians'

2. Nekrasov-Shatashvili limit

quantum commutative ring

$\varepsilon_2 \to 0$

$\varepsilon_1 \to \hbar$

$$W_{\varepsilon_1,\varepsilon_2}(\mathcal{M})$$

'Ring of quantum commuting Hamiltonians'

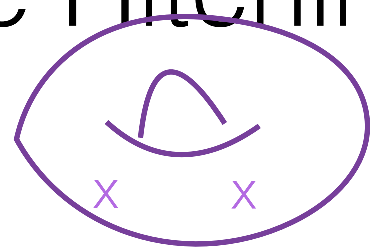
The eigenvalues of elements in $W_{\varepsilon_1,\varepsilon_2}(\mathcal{M})$ *is spectrum of quantum integrable system*

What can we say $W_{\varepsilon_1,\varepsilon_2}(\mathcal{M})$ for generic $\varepsilon_1, \varepsilon_2$?

Two large classes of 4d theories

1. class $S_{\mathbf{g}}$ 6d (0,2) g=ADE reduced on C

$\mathcal{M}$ is **g**-type Hitchin system on C



Witten'96
Gaiotto'07
AGT'08
GMN'09

2. Γ-quiver 6d quiver theories reduced on $C^{\vee}$

(2d flat)

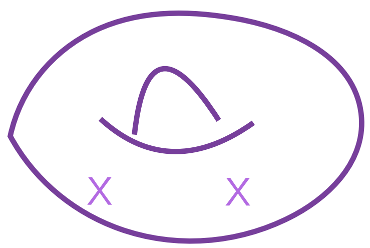
$\mathcal{M}$ is G_{Γ}-monopole system on $C \times S^1$

Douglas-Moore'96
Cherkis-Kapustin'00
Katz-Mayer-Vafa'97
Nekrasov-VP-(Shatashvili) '12,'13
Aganagic'15
Nekrasov'15, Kimura-VP'15

class $S_{\mathbf{g}}$ theories

6d (0,2) **g**=ADE reduced on C

$\mathcal{M}$ is **g**-Hitchin system on *C*



$$\mathcal{M}_{\zeta\neq 0,\infty} \simeq \mathcal{M}_{\mathrm{flat}}(C, G)$$

$W_{\varepsilon_1,\varepsilon_2}(\mathcal{M})$ *is quantum cluster algebra (algebra of Verlinde loop operators)*

Cheng's talk

Fock-Goncharov
Teschner
Gaiotto-Moore-Neitzke
Alday-Gaiotto-Tachikawa

For punctured disk $C = \mathbf{C}^{\times}$ and parabolic reduction

$W_{\varepsilon_1,\varepsilon_2}(\mathcal{M})$ *is W(**g**) algebra*

(integrals of motion in Toda 2d field theory)

Drinfeld-Sokolov
Zamolodchikov
Feigin-Frenkel
Belinson-Drinfeld

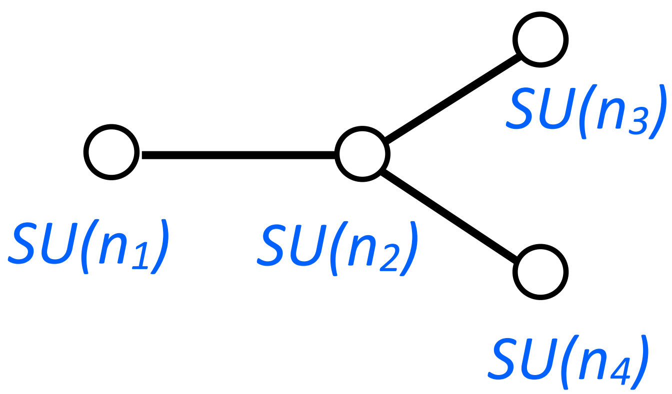
Quiver theories

6d Γ-quiver gauge theories reduced on $C^\vee$

$\mathcal{M}$ is G_Γ-monopole system on $C \times S^1$

(**group valued** analogue of **Hitchin** system)

$\Gamma = D_4$ $\quad$ G_Γ=SO(8)



$x \in C$

$\boxed{\langle \chi_i[Y] \rangle = T_{n_i}(x) \text{ -polynomial}}$

$\varepsilon_1 = 0, \varepsilon_2 = 0$

Nekrasov-VP '12

$\varepsilon_1 \neq 0, \varepsilon_2 = 0$

Nekrasov-VP-Shatashvili '13

$\varepsilon_1 \neq 0, \varepsilon_2 \neq 0$

Nekrasov '15, 'Kimura-VP '15

$$\chi_i[Y(x)] = \mathrm{tr}_{L_i}(g(x)) \qquad [g(x)] = \prod_i y(x)^{a_i^\vee}$$

character

q-character = W$_q$-algebra (g$_\Gamma$)

Frenkel-Reshetikhin'98

qq-character = W$_{qq}$-algebra (g$_\Gamma$)

Frenkel-Reshetikhin'97

UV-observables

$$Y_i(x) = \det(1 - x^{-1} e^{\Phi_i + \epsilon_1 J_1 + \epsilon_2 J_2})|_{\mathring{\mathcal{E}}_i}$$

qq-characters are proper IR-observables made from $Y_i(x)$

Nekrasov's talk

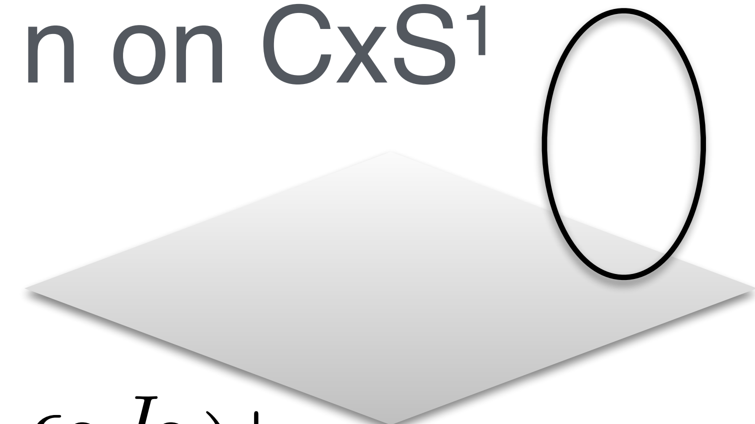
qq-characters

Nekrasov@String'14,16
VP@String-Math'15

U(n)

Example A_1 ○

SU(2)-charge monopoles of charge n on $C \times S^1$



good in UV

$$Y_i(x) = \det(1 - x^{-1} e^{\Phi_i + \epsilon_1 J_1 + \epsilon_2 J_2})|_{\mathring{\mathcal{E}}_i}$$

good in IR

$$\chi_1 = Y_1(x) + Y_1^{-1}(q^{-1}x)$$

$$q = q_1 q_2$$

$$q_i = e^{\epsilon_i}$$

characters

1. classical limit ($\varepsilon_1=\varepsilon_2=0$)

$$\chi_1[y] = \mathrm{tr}\begin{pmatrix} y(x) & 0 \\ 0 & y(x)^{-1} \end{pmatrix} = y(x) + \frac{1}{y(x)}$$

q-character =T-polynomial

is classical SW spectral curve

$$y + 1/y = T_n(x) = x^n + u_1 x^{n-1} + \cdots + u_n$$

is spectral curve of SU(2) monopoles

of charge *n* on C x S[1]

q-characters

Nekrasov-VP-Shatashvili'13

2. quantum integrable system (NS-limit ε_1=h, ε_2=0)

$$Y(x) = Q(x)/Q(q^{-1}x)$$

q-character =T-polynomial

is quantum SW spectral curve

$$Q(qx) + Q(q^{-1}x) = Q(x)T(x)$$

(aka TQ-Baxter or analytic Bethe anzats)

$$T_n(x) = x^n + u_1 x^{n-1} + \cdots + u_n$$

quantum commuting Hamiltonians

q-characters

Nekrasov-VP-Shatashvili'13

for generic quiver we find traces of R-matrices of rational, trigonometric and elliptic spin-chain types for any ADE and their affinization (toroidal)

$$\mathbf{Yang}(\mathbf{g}), U_q(\hat{\mathbf{g}}), E_{\mathbf{g},C}(\mathbf{g})$$

Gerasimov-Kharchev-Lebedev-Oblezin'04

The reason is that the phase space, the monopole moduli space M on CxS^1 is a symplectic leaf of the Poisson-Lie loop group {g(x)} with classical r-matrix Poisson bracket ! The quantization produces quantum R-matrix .

$x \in C$ is spectral parameter

c.f. Costello's & Witten's talks

q-characters=classical W-algebra

q-characters generate commutative
$\mathbf{W_q(g)}$ algebra

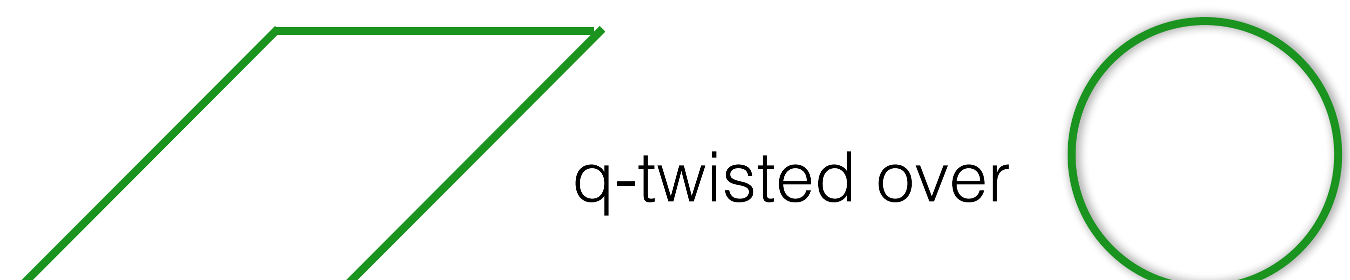
Nekrasov-VP-Shatashvili'13

Alternatively (after T-duality on the fibers of integrable system) this classical Poisson $\mathbf{W_q(g)}$ algebra can be obtained by group version of Drinfeld-Sokolov reduction on group-valued **q**-'opers' (**q**-connections)

Steinberg'65
Semenov Tian Shansky'99
Sevostiyanov'00

$$\{g(x)\}/\{g(x) \sim s(x)g(x)s^{-1}(qx)\}$$

$C \times_q S^1$



qq-characters=quantum W-algebra

qq-characters generate associative
$W_{qq}(g)$ algebra: quantization of **$W_q(g)$**

$$W_{qq}(\mathbf{g}) = \mathbb{C}_{\epsilon_1}[\mathcal{M}^{G,n}_{\text{mono}}(C \times_{\epsilon_2} S^1]$$

$$\{g(x)\}/\{g(x) \sim s(x)g(x)s^{-1}(qx)\}$$

$C \times_q S^1$

Free field realization of quiver W_{qq}-algebra

Kimura-VP'15

Consider quiver gauge theory
extended Nekrasov partition function

Nekrasov-Marshakov

$$Z(t) = \int_{\mathfrak{M}_\Gamma} \exp\left(\sum_i \sum_{p=1}^{\infty} t_{i,p} \mathbf{ch} Y_i^{[p]} \right)$$

which depends on infinite set of parameters

$$\{t\} = \{t_{i,p}\} \qquad \begin{array}{l} p = 1, 2, \ldots, \infty \\ i \in \Gamma_0 \end{array}$$

(generating function of intersection of all Chern characters of Y-bundle over instanton moduli space)

Free field realisation of quiver W$_{\mathbf{qq}}$-algebra

Interpret Z{t} is a state in Fock space C[[t]]

of Heisenberg algebra generated by $\{t_{i,p}, \frac{\partial}{\partial t_{i,p}}\}$

Kimura-VP'15:
c.f. Aganagic talk

$$|Z\rangle = \prod_{x\in\mathcal{X}} \hat{S}(x)|1\rangle$$

$$\hat{S}(x) = \int d_{q_2}x S(x)$$

$$S_{i,x} =: \exp\Big(\sum_{p>0} s_{i,-p}x^p + s_{i,0}\log x + \tilde{s}_{i,0} + \sum_{p>0} s_{i,p}x^{-p}\Big):$$

Here the screening current

$$S_{i,x} =: \exp\Big(\sum_{p>0} s_{i,-p} x^p + s_{i,0}\log x + \tilde{s}_{i,0} + \sum_{p>0} s_{i,p} x^{-p}\Big) :$$

is realised in terms of the free fields:

$$s_{i,-p} \overset{p>0}{=} (1-q_1^p) t_{i,p}, \qquad s_{i,0} = t_{i,0}, \quad s_{i,p} \overset{p>0}{=} -\frac{1}{p}\frac{1}{1-q_2^{-p}} c_{ij}^{[p]} \partial_{i,p}$$

c_{ij} is deformed Cartan matrix of quiver

$$c_{ij} = (\delta_{ji} - \sum_{e:j\to i} \mu_e^{-1}) + (q^{-1}\delta_{ji} - \sum_{e:i\to j} q^{-1}\mu_e)$$

Elliptic version and non-simply laced version are defined similarly for generic quiver

A_1 : Kozcaz's talk

qq-characters = free field realisation of $W_{q1,q2}(\mathbf{g})$

U(n)

Example A_1 $\circ$

$$T_1(x) = Y_1(x) + Y_1^{-1}(q^{-1}x)$$

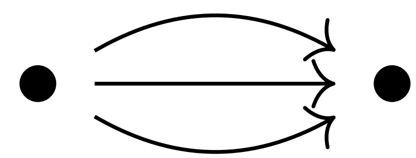
where

$$\mathsf{Y}_{i,x} =: \exp\Big(\sum_{p>0} y_{i,-p}x^p + y_{i,0} + \sum_{p>0} y_{i,p}x^{-p}\Big):$$

$$y_{i,-p} = (1-q_1^p)(1-q_2^p)(\tilde{c}^{[-p]})_{ij}t_{j,p}, \qquad y_{i,0} = -\tilde{c}_{ij}^{[0]}t_{i,0}\log q_2 \qquad y_{i,p} = -\frac{1}{p}\partial_{i,p}$$

$\langle 1|T_1(x)|Z\rangle =$ polynomial deg n

The construction of **$W_{q1,q2}(\Gamma)$** in **Kimura-VP'15** is more general then **$W_{q1,q2}(g)$** in **Frenkel-Reshetikhin'97** and applicable to any quiver, not necessarily of finite Dynkin type. It depends on parameters $\mu \in H^1(\Gamma, \mathbb{C}^\times)$

Hyperbolic example  $c = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$

$$\mathcal{S}(w) = \frac{(1-q_1 w)(1-q_2 w)}{(1-qw)(1-w)}$$

$$\begin{aligned} T_{1,x} \;=\; & \mathsf{Y}_{1,x} + q_1^{-1} : \frac{\mathsf{Y}_{2,\mu_1^{-1}x}\mathsf{Y}_{2,\mu_2^{-1}x}\mathsf{Y}_{2,\mu_3^{-1}x}}{\mathsf{Y}_{1,q^{-1}x}} : \\ & + q_1^{-2}\Bigg[\mathcal{S}(\mu_1\mu_2^{-2})\mathcal{S}(\mu_1\mu_3^{-1}) : \frac{\mathsf{Y}_{1,\mu_1^{-1}\mu_2 q^{-1}}\mathsf{Y}_{1,\mu_1^{-1}\mu_3 q^{-1}}\mathsf{Y}_{2,\mu_2^{-1}x}\mathsf{Y}_{2,\mu_3^{-1}x}}{\mathsf{Y}_{2,\mu_1^{-1}q^{-1}x}} : \\ & \qquad + \mathcal{S}(\mu_2\mu_1^{-2})\mathcal{S}(\mu_2\mu_3^{-1}) : \frac{\mathsf{Y}_{1,\mu_2^{-1}\mu_1 q^{-1}}\mathsf{Y}_{1,\mu_2^{-1}\mu_3 q^{-1}}\mathsf{Y}_{2,\mu_1^{-1}x}\mathsf{Y}_{2,\mu_3^{-1}x}}{\mathsf{Y}_{2,\mu_2^{-1}q^{-1}x}} : \\ & \qquad + \mathcal{S}(\mu_3\mu_1^{-2})\mathcal{S}(\mu_3\mu_1^{-1}) : \frac{\mathsf{Y}_{1,\mu_3^{-1}\mu_1 q^{-1}}\mathsf{Y}_{1,\mu_3^{-1}\mu_2 q^{-1}}\mathsf{Y}_{2,\mu_1^{-1}x}\mathsf{Y}_{2,\mu_2^{-1}x}}{\mathsf{Y}_{2,\mu_3^{-1}q^{-1}x}} : \Bigg] \\ & \qquad + \cdots \end{aligned}$$

$Z_{gauge}=Z_{top}$

W-algebra of (`refined' topological strings)

What are higher times {t} ?

Elliptic version and non-simply laced version are defined similarly for generic quiver

A_1 Kozcaz's talk

Generalization of Nakajima's results on equivariant K-theory of quiver variety and representation theory of $\mathbf{U_q(Lg)}$ to affine Kac-Moody **g** with non-zero central charge

Quantum q-geometric Langlands is transparent and completely geometric: exchange of ε_1 vs ε_2

Rich algebraic structures coming from quantization of HyperKahler moduli spaces

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N. Nekrasov
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T. Kimura

Thank you