Spin-cobordisms, surgeries, and fermionic modular bootstrap

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Goal

For a given symmetry group G, evaluate explicitly the homomorphisms of abealian groups:

$$\{\text{representations}\} \xrightarrow{\substack{\text{free}\\\text{fermions}}} \{\text{anomalies}\} \xrightarrow{\text{“}\int_X\text{”}} \left\{ \begin{array}{c} \text{anomalous phases} \\ \text{of large} \\ \text{diffeomorphisms} \\ \text{\& gauge transf.} \\ \text{on spacetime } X \end{array} \right\}$$

Motivation:

- Anomaly cancellation/matching
- Constraints on partition function/spectrum from anomalies (e.g. modular bootstrap)

Naturality

For a homorphism between symmetry groups

$$G \longrightarrow G'$$

we have pullbacks

$$\begin{tikzcd}
\{\text{reps}\} \arrow[d] & \{\text{reps}\}' \arrow[l] \arrow[d] \\
\{\text{anomalies}\} \arrow[d] & \{\text{anomalies}\}' \arrow[l] \arrow[d] \\
\left\{\begin{array}{c}\text{anomalous}\\ \text{phases}\end{array}\right\} & \left\{\begin{array}{c}\text{anomalous}\\ \text{phases}\end{array}\right\}' \arrow[l]
\end{tikzcd}$$

Commutativity of the diagram is very constraining!

Remark: a version of this has been used before when G' is continuous [Witten '83, Elitzur-Nair '84, Ibanez-Ross '91, Davighi-Lohitsiri '20,...]

Setup for this talk

For this talk we consider fermionic 2-dimensional theories, free of perturbative anomalies, on $X = \mathbb{T}^2$ with symmetry $\mathbb{Z}_2^f \times G$, where G is finite and unitary.

$$\{\text{reps}\} \overset{\rho_2 - \rho_1}{=\!=\!=\!=\!=} \underbrace{RO(G)}_{\text{ring of real representations}}$$

[Kapustin-Thorngren-Turzillo-Wang '14, Freed-Hopkins '16, Yonekura '18]

$\downarrow$ left fermions in ρ_1 $\otimes$ right fermions in ρ_2 $\otimes$ $\pm(\dim\rho_2 - \dim\rho_1)$ left/right fermions in triv. rep.

$$\{\text{anomalies}\} =\!=\!=\!=\!= \mathrm{Hom}(\Omega_3^{\mathrm{Spin}}(BG), U(1)) \equiv \mathcal{A}(G)$$

bordism classes of spin 3-manifods with G-bundles

$\downarrow$ action of symmetries on Hilbert spaces of 3d bulk invertible TQFT

$$\widetilde{\mathrm{SL}}(2,\mathbb{Z})\times G \curvearrowright \mathrm{Spin}_G(\mathbb{T}^2) =\!=\!= \mathrm{Spin}(\mathbb{T}^2) \times \mathrm{Hom}(\mathbb{Z}^2, G)$$

action on the coefficients

$$\mathbb{Z}_2^f \longrightarrow \widetilde{\mathrm{SL}}(2,\mathbb{Z}) \longrightarrow \mathrm{SL}(2,\mathbb{Z})$$

$$\left\{\begin{matrix}\text{anomalous}\\ \text{phases}\end{matrix}\right\} =\!=\!=\!=\!= H^1(\widetilde{\mathrm{SL}}(2,\mathbb{Z}) \times G, U(1)^{\mathrm{Spin}_G(\mathbb{T}^2)}) \equiv \Phi(G)$$

$$|||$$

$$H^1(\underbrace{(\mathrm{SL}(2,\mathbb{Z}) \times G) \ltimes \mathrm{Spin}_G(\mathbb{T}^2)}_{\text{action groupoid}}, U(1))$$

A well known example: $G = \mathbb{Z}_2$

$$
\begin{array}{ccc}
RO(\mathbb{Z}_2) = \langle \mathsf{triv}, \mathsf{sign} \rangle \cong \mathbb{Z}^2 & & (a, b) \\
\big\downarrow & & \big\downarrow \\
\mathcal{A}(\mathbb{Z}_2) \cong \mathbb{Z}_8 & & b \mod 8
\end{array}
$$

[Ryu-Zahng '11, Gu-Levin '13]

General finite group G

$$\mathfrak{ch}_2 : RO(G) \longrightarrow \mathcal{A}(G) \equiv \mathrm{Hom}(\Omega_3^{\mathrm{Spin}}(BG), U(1))$$

Certain components of the map (w.r.t. supercohomology presentation) were considered in [Chen-Kapustin-Turzillo-You '19]

Claim: can be defined axiomatically by the following properties

1. Linearity:

$$\mathfrak{ch}_2(\rho_1 \oplus \rho_2) = \mathfrak{ch}_2(\rho_1) + \mathfrak{ch}_2(\rho_2).$$

2. Naturality: for any group homomorphism $f : G \to G'$

$$f^* \mathfrak{ch}_2(\rho) = \mathfrak{ch}_2(f^* \rho)$$

3. Normalization: for all abelian G the values of $\mathfrak{ch}_2$ are given (tabulated in the paper)

Remark: similar to axiomatic definitions of Chern and Stiefel-Whitney classes of representations $c_i : RU(G) \to H^{2i}(BG, \mathbb{Z})$, $w_i : RO(G) \to H^i(BG, \mathbb{Z}_2)$.

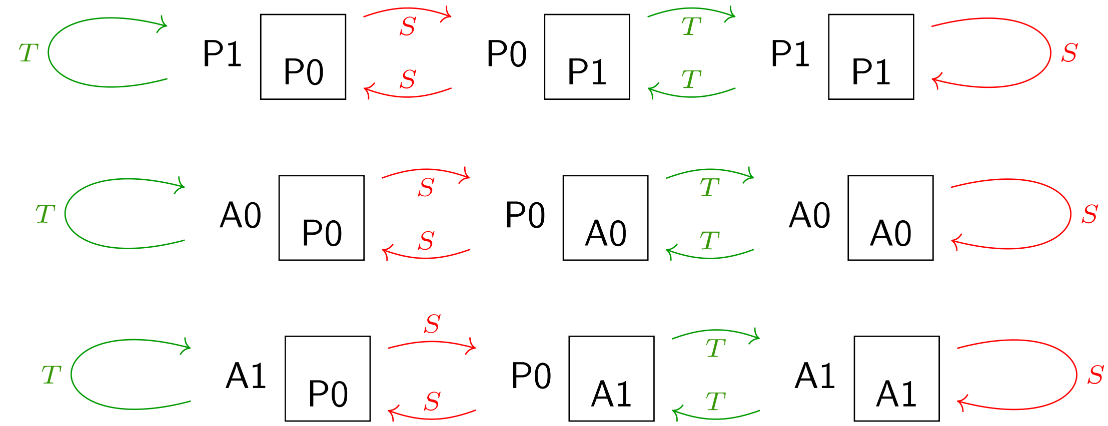
A simple non-abelian example: $G = S_3$

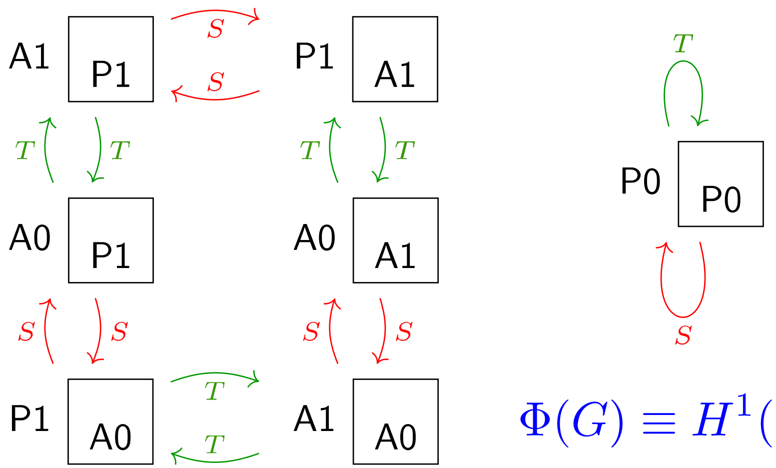
$$\begin{tikzcd}
\mathbb{Z}_2 \arrow[r, "\text{transposition}"] \arrow[rr, bend left, dashed, "\text{id}"] & S_3 \arrow[r, "\text{parity}"] & \mathbb{Z}_2 \\
\mathbb{Z}_3 \arrow[ur, "\text{cyclic}"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{A}(\mathbb{Z}_2) \cong \mathbb{Z}_8 & \mathcal{A}(S_3) \cong \mathbb{Z}_8 \times \mathbb{Z}_3 \arrow[l] \arrow[dl] & \mathcal{A}(\mathbb{Z}_2) \cong \mathbb{Z}_8 \arrow[l] \arrow[ll, bend right, dashed, "\text{id}"'] \\
\mathcal{A}(\mathbb{Z}_3) \cong \mathbb{Z}_3 & &
\end{tikzcd}$$

$$\begin{array}{rcl}
RO(S_3) \cong \mathbb{Z}^3 & \xrightarrow{\mathfrak{ch}_2} & \mathcal{A}(G) \cong \mathbb{Z}_8 \times \mathbb{Z}_3, \\
\text{1d: trivial } \rho_0 & \longmapsto & (0,0), \\
\text{1d: parity sign } \rho_1 & \longmapsto & (1,0), \\
\text{2d: isometries of equilateral triangle } \rho_2 & \longmapsto & (1,1).
\end{array}$$

Anomalous phases





Example: $G = \mathbb{Z}_2$

$$\Phi(G) \equiv H^1(\widetilde{\mathrm{SL}}(2,\mathbb{Z}) \times G, U(1)^{\mathrm{Spin}_G(\mathbb{T}^2)}) \cong U(1)^5 \times \mathbb{Z}_8^4 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_2^5$$

"Anomalous phases" data is an assignment of 1d complex vector spaces to the elements of $\mathrm{Spin}_G(\mathbb{T}^2) \equiv \mathrm{Spin}(\mathbb{T}^2) \times \mathrm{Hom}(\mathbb{Z}^2, G)$ and linear maps between them to S, T, and generators of G, consistent with the group law in $\widetilde{\mathrm{SL}}(2,\mathbb{Z}) \times G$. This is a representaiton of the action groupoid $(\widetilde{\mathrm{SL}}(2,\mathbb{Z}) \times G) \ltimes \mathrm{Spin}_G(\mathbb{T}^2)$. Up to a change of basis they are classified by $\Phi(G) \equiv H^1(\widetilde{\mathrm{SL}}(2,\mathbb{Z}) \times G, U(1)^{\mathrm{Spin}_G(\mathbb{T}^2)})$.

The bulk 3d TQFT corresponding to the given anomaly by inflow gives such an assignment. Up to change of basis, such assignments fixed by partition functions on some closed manifolds (one can always take mapping tori). This is done by fixing a "path" from each object in the groupoid to the "basepoints" in connected components. This gives $\mathcal{A}(G) \to \Phi(G)$.

Remark: in bosonic case the analogous map was considered in [Dijkgraaf-Witten '90, Freed-Quinn '93, Ganter '09, Yu '21]

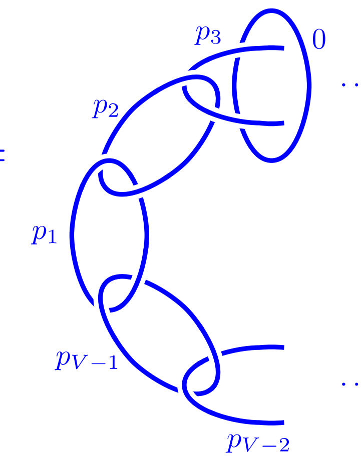
Reducing to abelian case

Moreover, since $\mathrm{Hom}(\mathbb{Z}^2, G)$ contains commuting pairs of elements of G it is enough to consider abelian subgroups $G' \hookrightarrow G$ with at most two independent generators.

Thus it is enough to calculate the values of bordism invariants in $\mathrm{Hom}(\Omega_3^{\mathrm{Spin}}(BG), U(1))$ on mapping tori in the case of abelian groups with at most two independent generators. In fact, we show how to easily calculate such invariants on any closed spin 3-manifolds using their surgery representation.

Surgery realization of 3-manifolds

Example: Y is mapping torus of $ST^{p_{V-1}} \dots ST^{p_2} ST^{p_1} \in \mathrm{SL}(2,\mathbb{Z})$ (here $p_V = 0$)

$\mathcal{L}$ $=$ 

$\mathcal{L} = \cup_{i=1}^{V} \mathcal{L}_i$ is V-component link in S^3 with framings (self-linking numbers) $p_i \in \mathbb{Z}$.

A closed 3-manifold $Y := S^3(\mathcal{L})$ is obtained by removing tubular neighborhoods for each link component and then gluing back solid tori so that their meridians (contractible cycles) are mapped to the curves traced by the framing vectors on the boundaries of tubular neighborhoods. Let $B_{ij} = \ell k(\mathcal{L}_i, \mathcal{L}_j)$ be its $V \times V$ linking matrix.

$$H^1(Y, \mathbb{Z}_n) = \{a \in \mathbb{Z}_n^V \mid Ba = 0 \mod n\}$$

$$\mathrm{Spin}(Y) = \{s \in \mathbb{Z}_2^V \mid Bs = \mathrm{diag}\, B \mod 2\}$$

$\mathcal{C}_s := \cup_{s_i=1} \mathcal{L}_i$ is a *characteristic sublink*

Abelian invariants

In the case of abelian G with at most two independent generators all spin-bordism invariants can be expressed (the relations can be found using naturality of $RO \to \mathcal{A}$ transformation) via the following two:

$$\begin{array}{rcl} \gamma_s: \quad H^1(Y,\mathbb{Z}_n) & \longrightarrow & \mathbb{Z}_{2n}, \\ a & \longmapsto & \frac{a^TBa}{n} + a^TBs \mod 2n. \end{array}$$

[Kirby-Taylor '91]+[Kirby-Melvin '91]

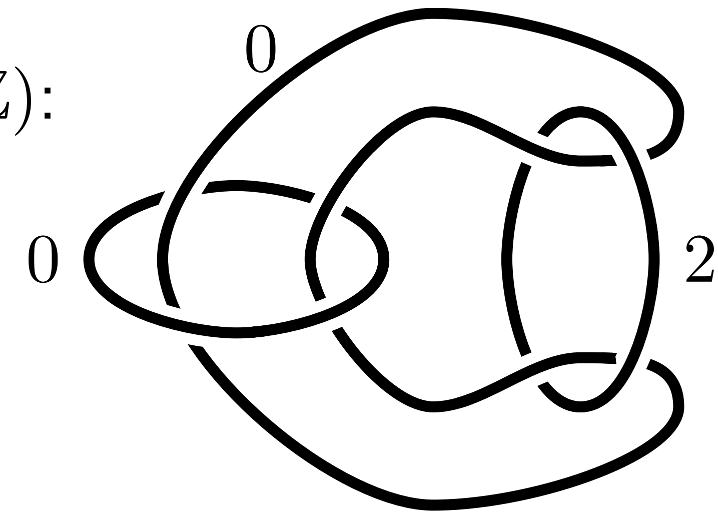
$$\begin{array}{rcl} \beta_s: \; H^1(Y,\mathbb{Z}_2) & \longrightarrow & \mathbb{Z}_8, \\ a & \longmapsto & -\frac{s^TBs}{2} - \frac{(a+s)^TB(a+s)}{2} + 4(\mathrm{Arf}(\mathcal{C}_{s+a}) - \mathrm{Arf}(\mathcal{C}_s)) \mod 8. \end{array}$$

mod 2 valued Arf
invariant of a link in S^3
(well known how to calculate)

Remarks: 1) for most general abelian G we need to just add $\int_Y a\cup b \cup c$ "bosonic" invariant.
2) the invariants also have geometric interpretation in terms of the (in general "folded") surface representing Poincaré dual to a.

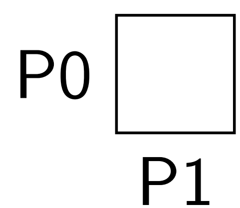
An example: $G = \mathbb{Z}_2$

surgery for mapping torus of $T^2 \in \mathrm{SL}(2,\mathbb{Z})$:



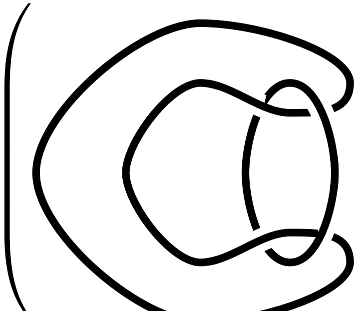
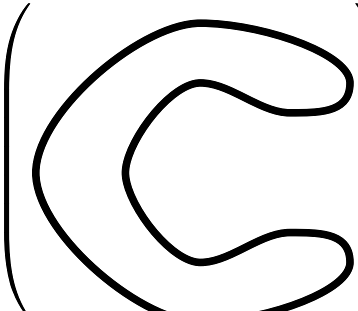
$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Fix $\nu \in \mathcal{A}(G) \cong \mathbb{Z}_8$

$T^2\Big|$ 

$= e^{\frac{\pi i \nu}{4}\beta_{(1,1,0)}((0,1,0))} = e^{-\frac{\pi i \nu}{4}}$

$$\beta_{(1,1,0)}((0,1,0)) = -\frac{1}{2}s^T B s + \frac{1}{2}(a+s)^T B(a+s) +$$

$+4\mathrm{Arf}\Big($  $\Big) - 4\mathrm{Arf}\Big($  $\Big) = -1 \mod 8$

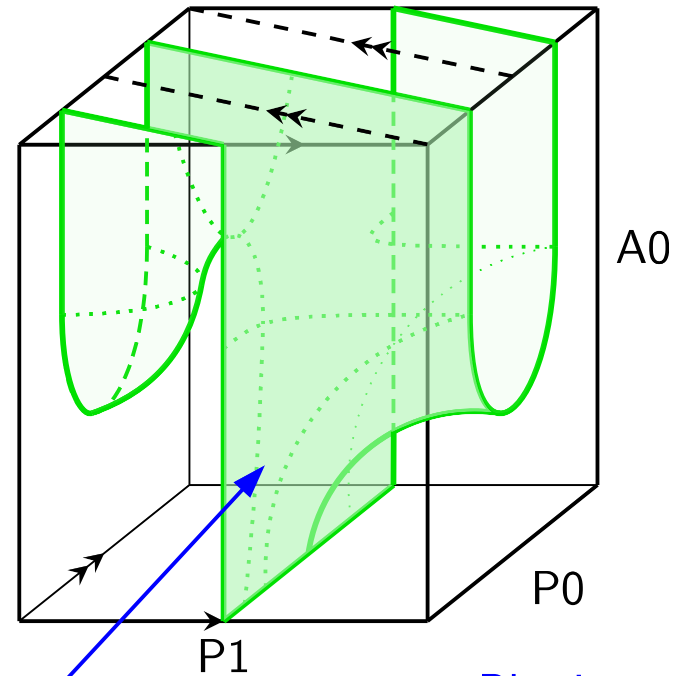
Geometric calculation for $G = \mathbb{Z}_2$

$$\beta_s(a) = \mathrm{ABK}(\mathsf{PD}(a)) \in \mathbb{Z}_8$$

Arf-Brown-Kervaire invariant

smooth surface representing
Poncaré dual to $a \in H^1(Y, \mathbb{Z}_2)$
with induced pin$^-$ structure

example: mapping torus
of $T^2 \in SL(2,\mathbb{Z})$



Physics:
1+1d SPT – Kitaev spin chain
supported on charge
operator of $1 \in \mathbb{Z}_2$

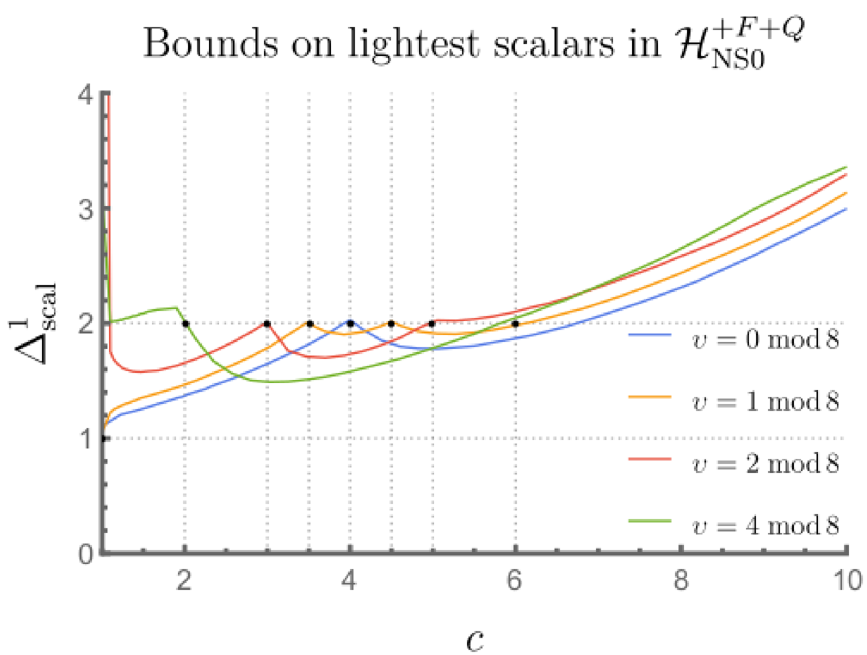
$$T^2\Big|_{\mathsf{P0}\,\square\,\mathsf{P1}} = e^{-\frac{\pi i \nu}{4}}$$

$$\cong \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$$

$$\mathrm{ABK} = -1 \mod 8$$

Fermionic modular bootstrap for $G = \mathbb{Z}_2$

Can use modular transformation for bootstrap, genralizing bosonic case [Collier-Lin-Yin '16, Lin-Shao '19, '21] and using SDPB [Simmons-Duffin '15]. Some of the results:



$\nu \in \mathcal{A}(\mathbb{Z}_2) \cong \mathbb{Z}_8$

Kinks: free $c/2$ Majorana fermions with generator of $G = \mathbb{Z}_2$ acting as $(-1)^{F_L}$ or $(-1)^{F_R}$.

Some future directions

• Perform numerical boostrap for some non-abelian symmetries

• Higher dimensional modular bootstrap?

• Generalized symmetries?

• E.g. higher form symmetries: similar classification by bordism invariants (but groups are only abealian), need to go to higher dimensions to get something interesting. The correct analogue of $RO(G) \to \mathcal{A}(G)$ map?

• Classification of 2d theories with G globabl symmetry and $\mathcal{N} = (0, 1)$ supersymmetry via G-equivariant TMF ($\mathcal{A}(G)$ is an extra grading)