Rényi Entropy and Spectral Geometry

Alexander Patrushev
in collaboration with
Dmitri Fursaev

Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research

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Entanglement and Rényi entropy

- Take a system H with ρ and divide it into two subsystem H_A and H_B separated by a boundary (not necessarily physical)
- $$S_n = \frac{1}{1-n} \ln tr\rho_A^n \quad (1)$$

  ρ_A with integrated out DOF's of B
- We measure correlation between A and B, measured on the entangling surface Σ
- Gentle coarse-graining $\Rightarrow$ Universality
- Information theory (e.g.$n \to \infty$ randomness extractors)

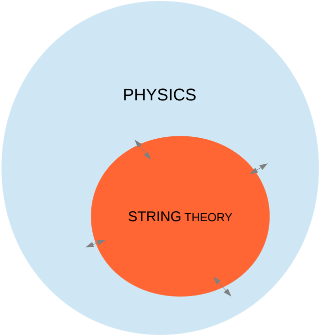

Figure: String theory traced away

People working on the border are most vulnerable.

A replica method is based on calculation of the traces of density matrix powers $\mathrm{Tr}_R \hat{\rho}_R^n$. Hard!

- Direct calculations using regularized metric Maldacena,Lewkowicz; Fursaev,A.P.,Solodukhin

$$ds^2 = f(r,b)dr^2 + r^2 d\tau^2 + [h_{ij} + r^n \cos(\tau) K_{ij} + r^n \sin(\tau)\, K_{ij}] dx^i dx^j \quad (2)$$
$$ds^2 = f(r,n)dr^2 + r^2 d\tau^2 + [h_{ij} + r \cos(\tau) K_{ij} + r \sin(\tau)\, K_{ij}] dx^i dx^j \quad (3)$$

give the Weyl tensor up to O(n-1) $\Rightarrow$ not good for Renyi entropy

- Conformal transformation maps entanglement entropy on a flat space-time to thermal entropy on $S^1 \times H^3$ Casini,Huerta,Myers (Good for Renyi entropy for a spherical entangling surface, possible corrections Rosenhaus,Smolkin)
- Numerical methods
- Holography?

General expression for the logarithmic term in the Rényi entropy in $(3+1)$-dimensional CFT Fursaev

$$S^n_\Sigma = \frac{f_a(n)}{180}\int_\Sigma E_2 + \frac{f_b(n)}{240\pi}\int_\Sigma \left(\mathrm{Tr}k^2 - \frac{1}{2}k^2\right) - \frac{f_c(n)}{240\pi}\int_\Sigma W^{ab}{}_{ab}\log\epsilon\,, \quad (4)$$

where $f_{a,b,c}(n)$ only depend on n

Numerics $f_b(q) = f_c(q)$ Lee,McGough,Safdi

Rényi entropy for excited states Takayanagi et al

$$Tr\rho^n = \int \mathcal{D}\phi \exp\left(-I_n[\phi, J]\right) \tag{5}$$

$$S_n(J) = \frac{1}{1-n}(\ln Z_n - n \ln Z_1) \tag{6}$$

Finite entropy

$$\Delta S_n = \frac{1}{2(n-1)} \int_{\mathcal{M}_\backslash} J_n(G_n - (n-1)G_1)J_n \tag{7}$$

G_n is a Green function for $\mathcal{M}_n$

$$G_\alpha = -\int ds K_\alpha(s) \tag{8}$$

Sommerfeld formula

$$K_\alpha(t) = \frac{1}{2\alpha} \int_B \cot \frac{\pi}{\alpha}(z-t) K\left(x(z), x'(0)|t\right) dz \tag{9}$$

Generalization for squashed geometries.

Generalization of the Sommerfeld formula for squashed geometries

$$K_\alpha(t) = \frac{1}{2\alpha}\int_B \cot\frac{\pi}{\alpha}(w-\Delta\tau)F\left(x(z),x'(0)|t\right)dw, \quad (10)$$

where $F(x,x',\omega|t)$ is a new kernel

$$F(x,x',\omega|t) = K(x(\omega),x'(\omega)|t)\mathrm{e}^{-s(x,x',\omega|t)/(4t)}(1+A(x,x',\omega|t)) \quad (11)$$

with a consistency condition

$$F(x,x',\omega|t)|_{\omega=\tau-\tau'} = K(x(\omega),x'(\omega)|t)|_{\alpha=2\pi} \quad (12)$$

Heat kernel coefficient a_4 gives a logarithmic term

$$K_\beta(t) \sim t^{-1} a_2 + a_4 + \ldots (t - dependent) \quad (13)$$

$$a_4 \to a(\gamma_n) f_a(n) + b(\gamma_n) f_b(n) + c(\gamma_n) f_c(n), \quad (14)$$

$\gamma_n = \frac{2\pi}{\alpha n}$

Figure: Thank you for your attention. Your Questions, Please.