Gravitational Positive Energy Theorems from Information Inequalities

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Swampland Question

Given an effective theory of gravity, how can one judge whether it is realized as a low energy appropximation to a consistent quantum theory with **ultra-violet completion**, such as string theory?

Constraints on Symmetry

Conjectures:

- ☆ There are no global symmetry.
- ☆ All continuous gauge symmetries are compact.
- ☆ The spectrum of electric and magnetic charges forms a complete set consistent with the Dirac quantization condition.

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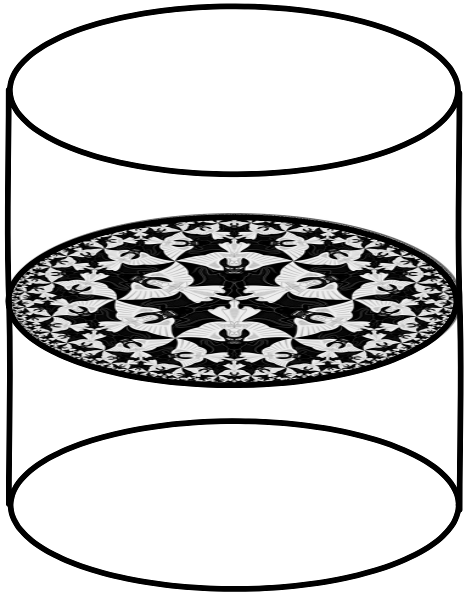

Holographic understanding:

Harlow, arXiv: 1510.07911
Harlow + H.O., to appear

Constraints on Moduli Space

Conjectures:

☆ The moduli space is **non-compact, complete, and has finite volume**.

☆ If we move a large distance *T* from a reference point, **a tower of light particles emerges** with mass of the order exp(- *a* *T*) for some *a*. The number of such light particles becomes infinite at *T* tends to the infinity.

☆ There is **no non-trivial one-cycle with minimal length** within a given homotopy class in the moduli space.

as formulated by Vafa + H.O., arXiv:0605264

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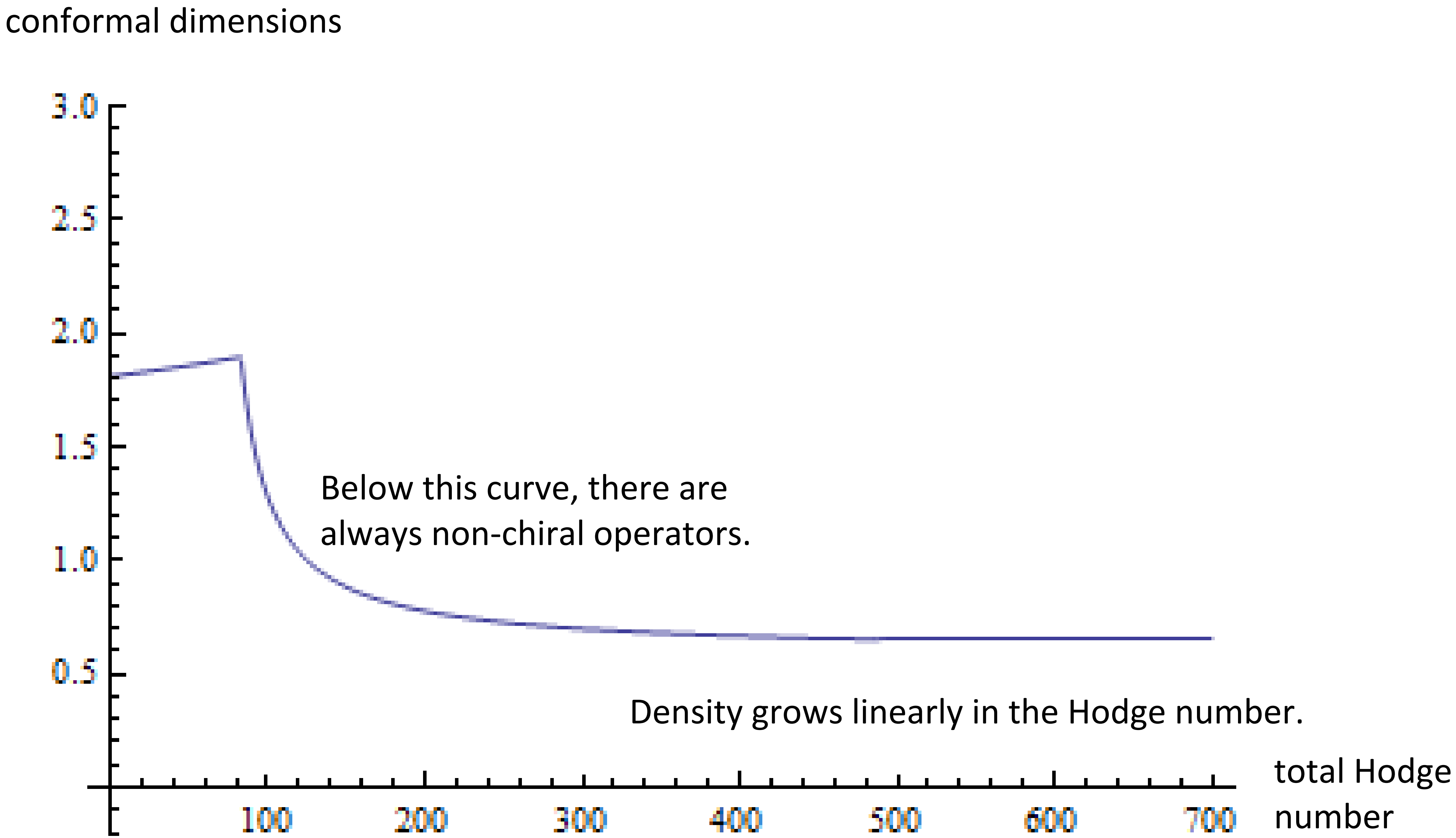
These moduli space constraints have been proven for theories with N=3 or higher supersymmetry.

Cecotti, "Supersymmetric Field Theories," section 4.9.1

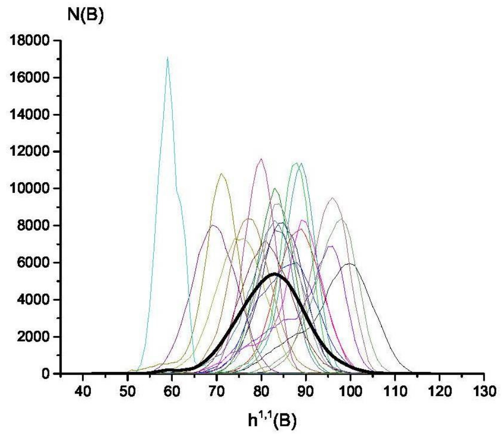
Constraints on Calabi-Yau Topology

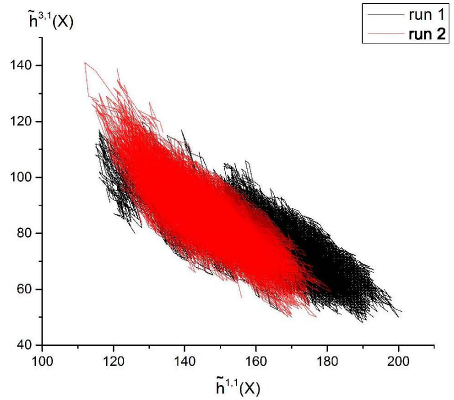
Modular invariance constraints

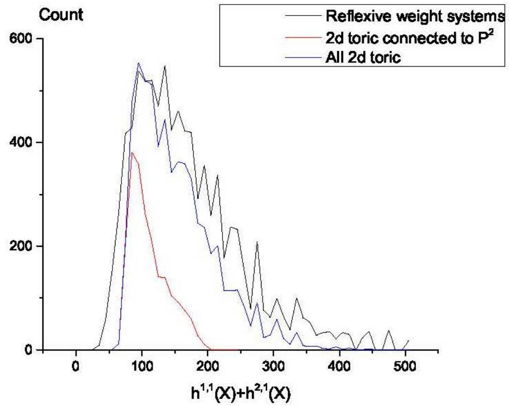
Keller + H.O., arXiv: 1209.4649

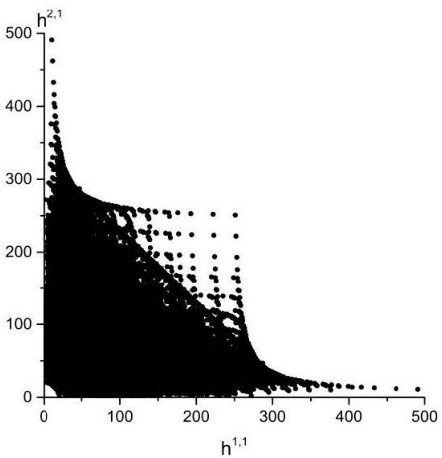


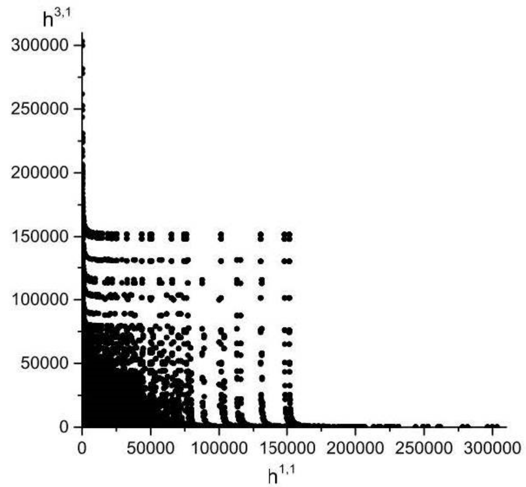
Recent experimental data on Calabi-Yau 3 and 4 folds







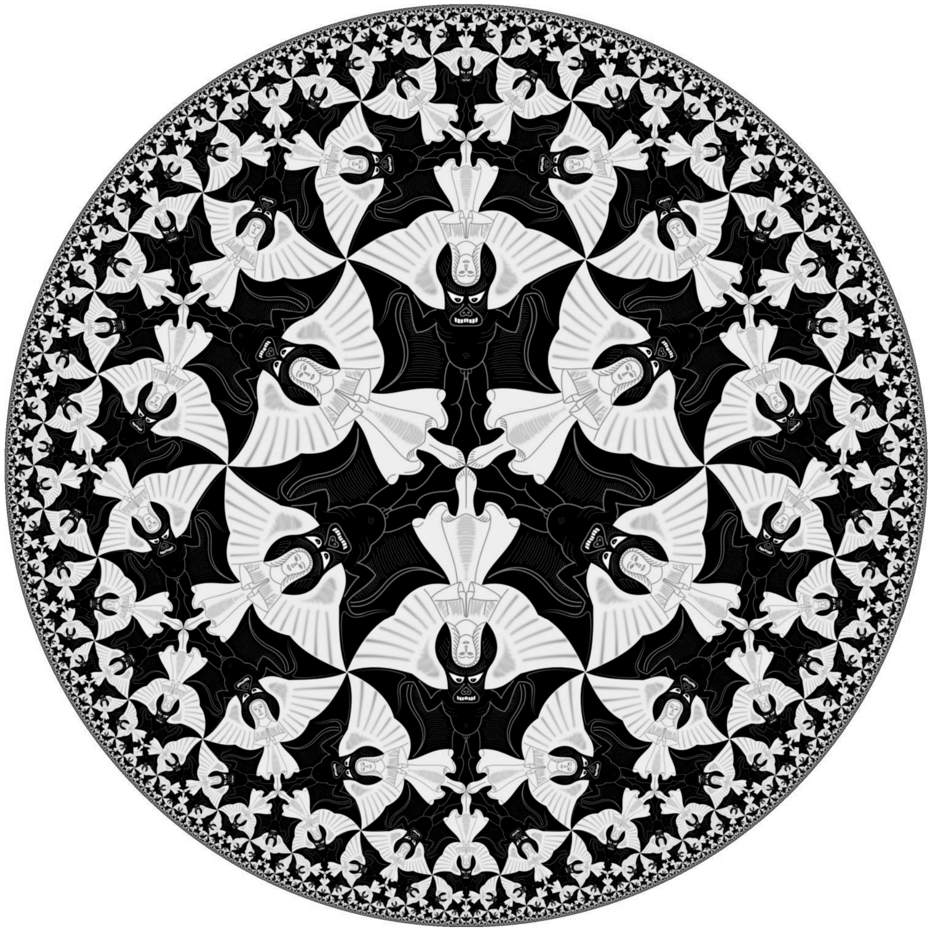




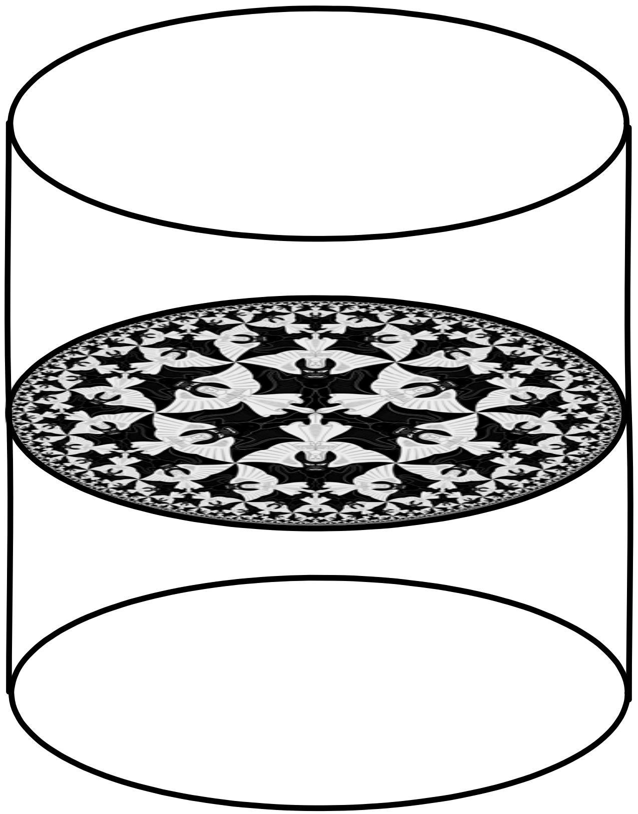
Taylor + Wang,
arXiv: 1510.04978, 1511.03209

Holographic Constraints

Suppose there is a low energy effective field theory whose gravity solutions asymptote to the anti-de Sitter space at the infinity.



Suppose there is a low energy effective field theory whose gravity solutions asymptote to the anti-de Sitter space at the infinity.

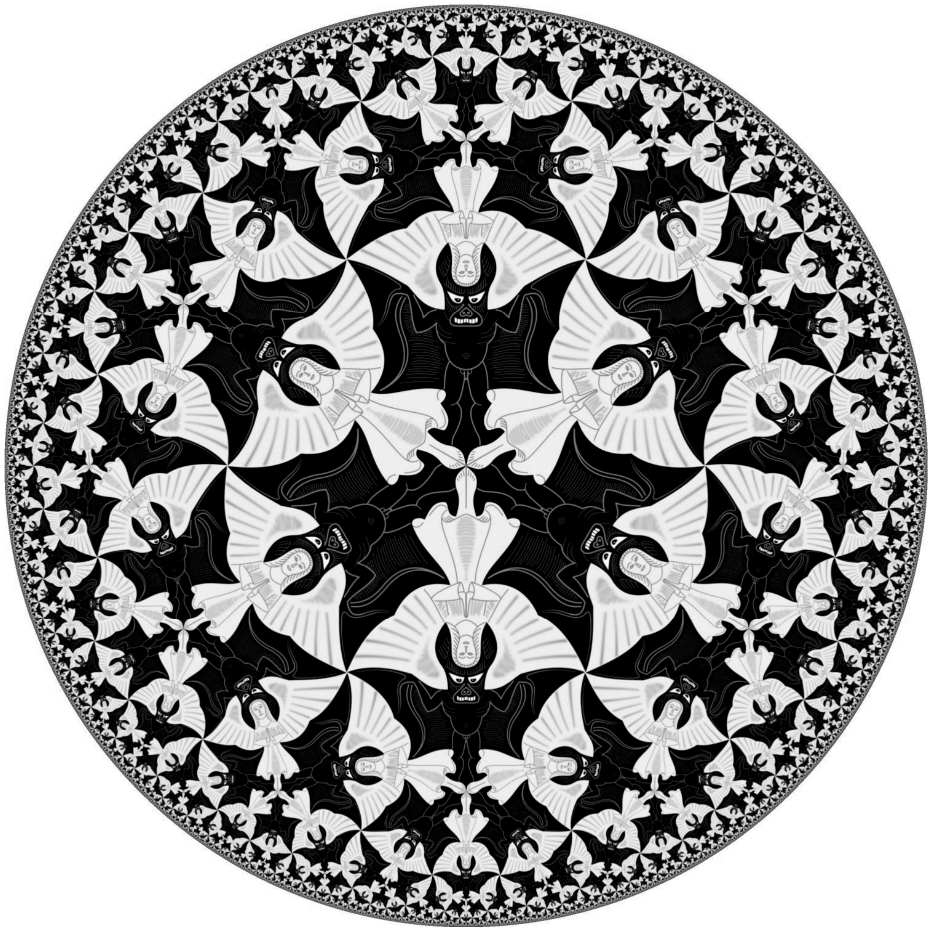


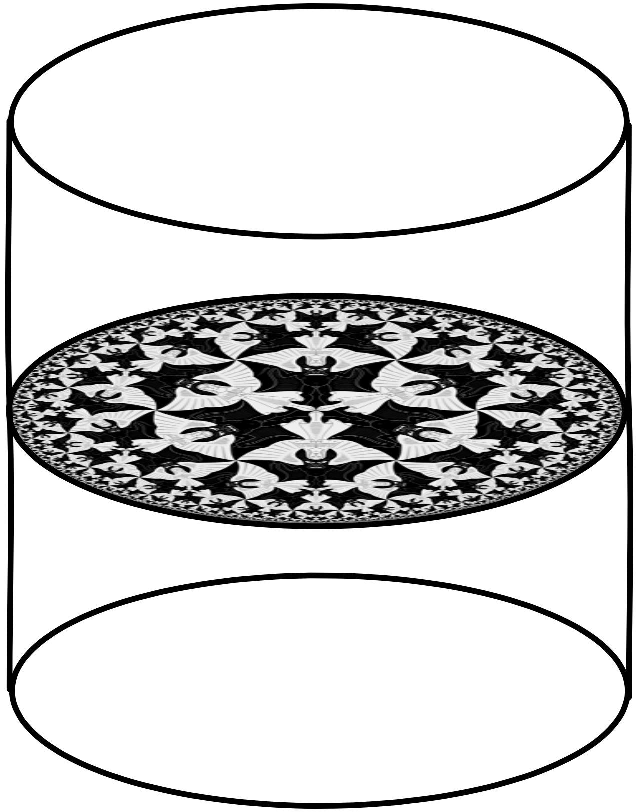
Holography of Quantum Gravity:

Consistent quantum gravity in AdS is equivalent to a conformal field theory on the boundary.

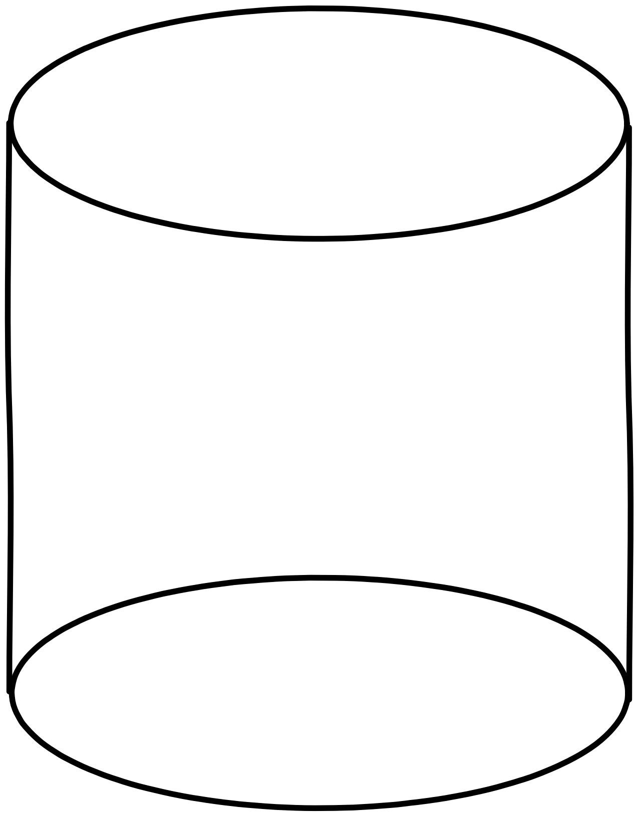
AdS/CFT Correspondence

Question: What does consistency of the conformal field theory mean for the gravity theory?

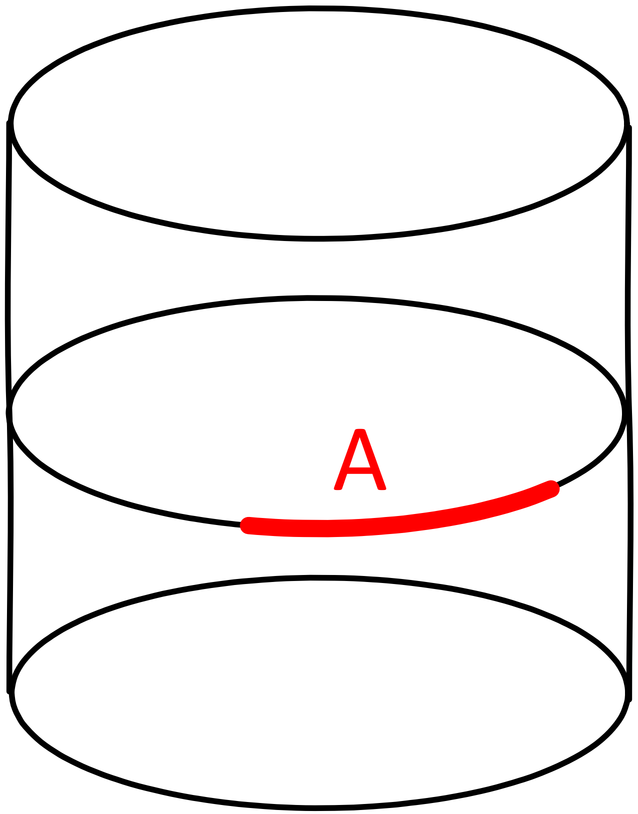




Gravity theory in (d+1)-dim AdS



Gravity theory in (d+1)-dim AdS is equivalent to d-dim CFT.



Gravity theory in (d+1)-dim AdS
is equivalent to d-dim CFT.

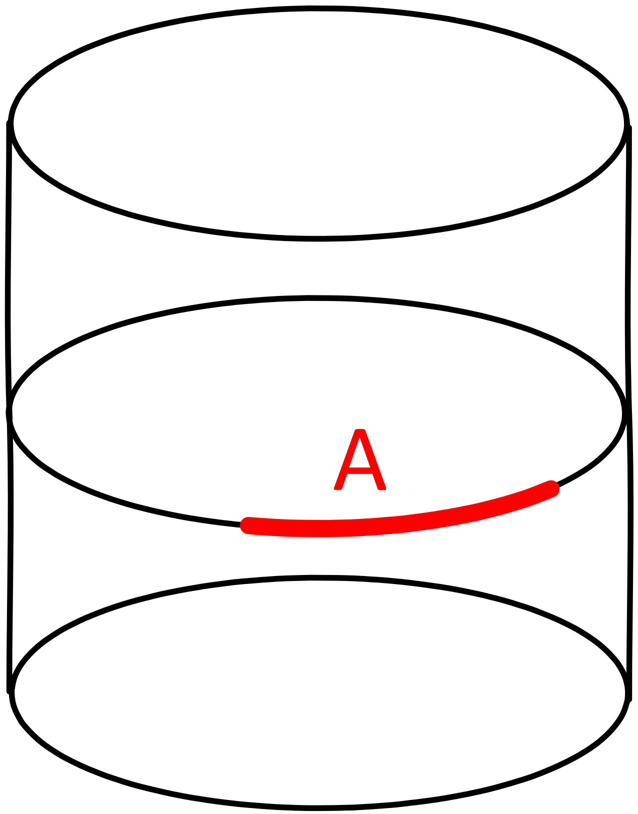
Entanglement Density Matrix ρ

For any state $|\psi\rangle$ in CFT,
choose a spacelike region A.

$$\rho = tr_{\bar{A}} \, |\psi\rangle\langle\psi|$$

☆ The trace is on the Hilbert space over the complement of A.

☆ It is an operator acting on the Hilbert space over A.



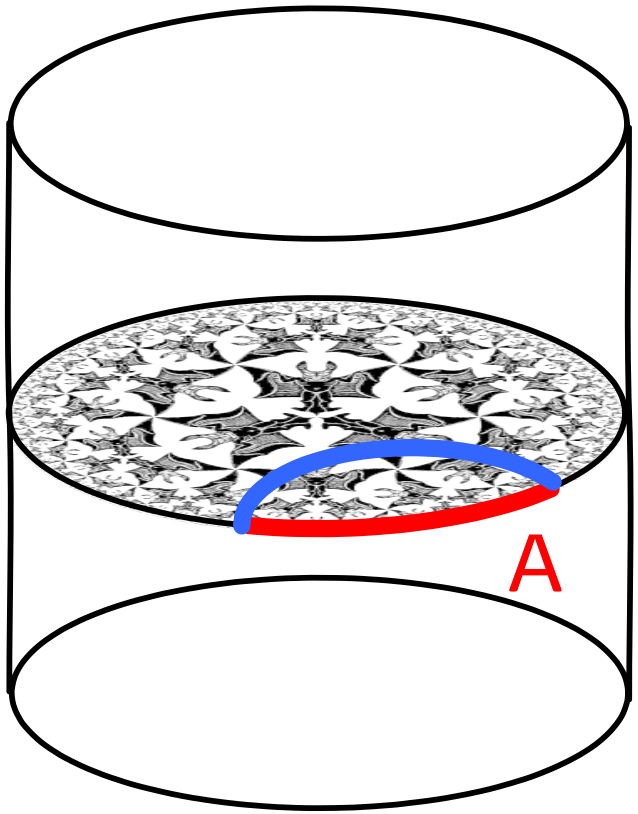
Entanglement Density Matrix ρ

$$\rho = tr_{\bar{A}} \, |\psi\rangle\langle\psi|$$

Entanglement Entropy *S*

$$S = -\mathrm{tr}\, \rho \log \rho$$

S measures the amount of entanglement between the region A and its complement.



Entanglement Entropy S

$$S = -\operatorname{tr} \rho \log \rho$$

When the bulk gravity theory is described with smooth geometry, the entanglement entropy ***S*** is proportional to the area of the minimum surface ending of the boundary of A.

$$S = \frac{1}{4G_N} Area(\Sigma)$$

Ryu-Takayanagi (2006)

Entanglement Entropy
satisfies inequalities:

$$S = -\operatorname{tr} \rho \log \rho$$

☆ Some inequalities are satisfied
both by any CFT and by AdS gravity.

☆ Some inequalities are satisfied
by any CFT but not always by AdS gravity.

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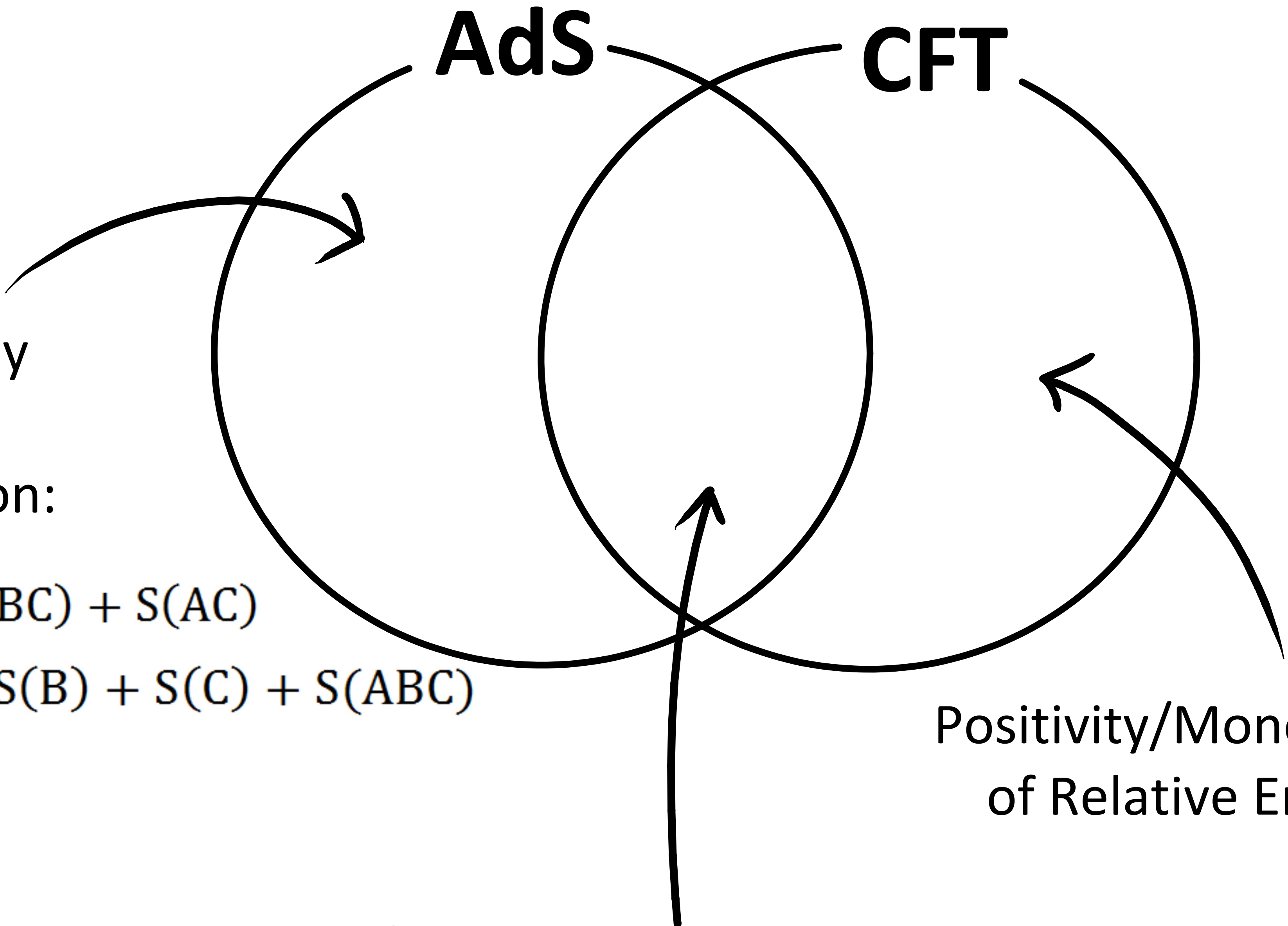
by any AdS gravity but not always by CFT.



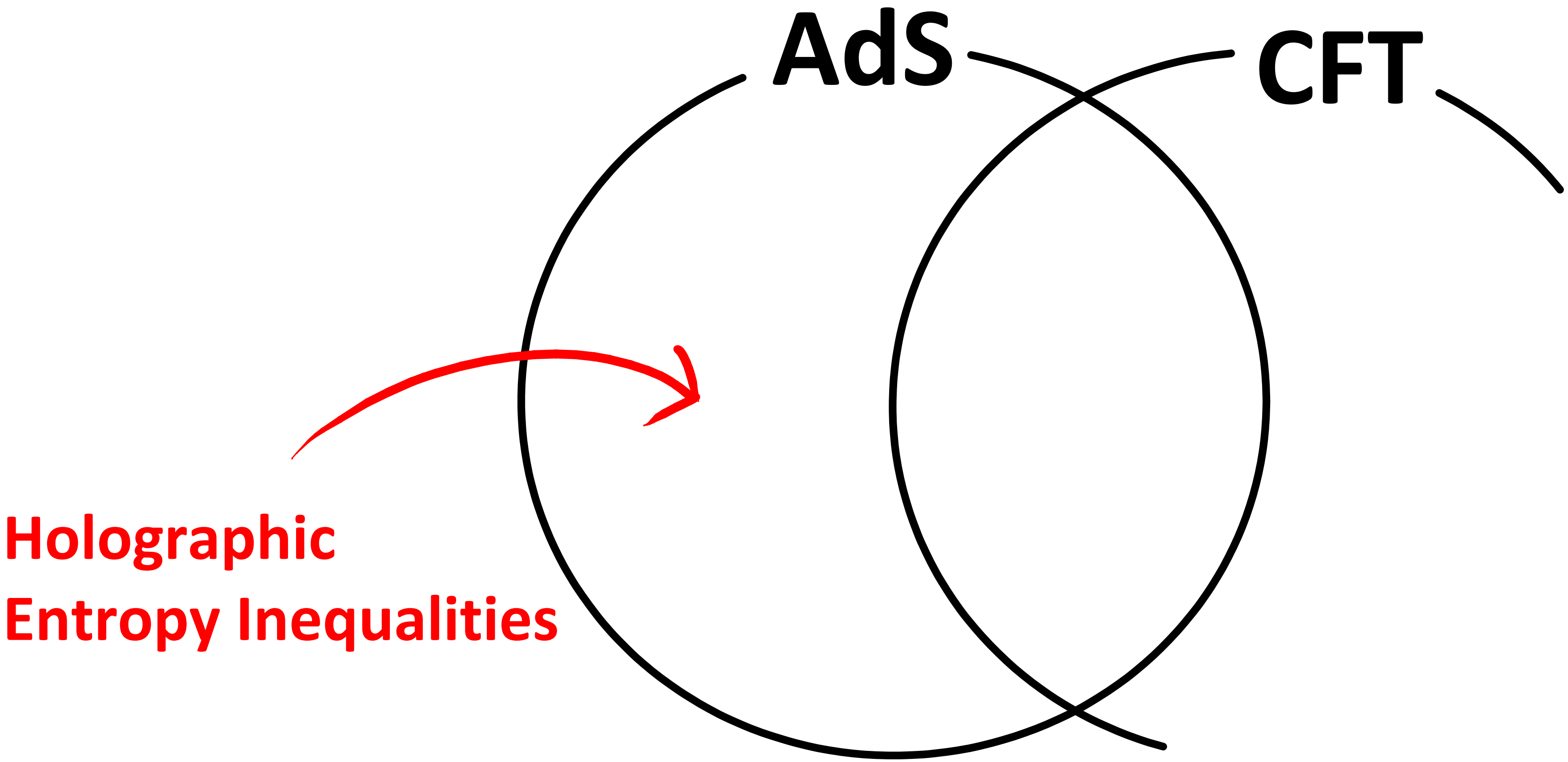
Monogamy
of Mutual
Information:

$$S(AB) + S(BC) + S(AC)$$
$$\geq S(A) + S(B) + S(C) + S(ABC)$$

Positivity/Monotonicity
of Relative Entropy

Strong Subadditivity:

$$S(AB) + S(BC) \geq S(B) + S(ABC)$$



CFT states with gravitational duals have interesting entanglement properties.

Entropy Inequalities

(Classical) Shannon Entropy:

There are ***infinite number*** of independent entropy inequalities for more than 3 regions.

$\Rightarrow$ Asymptotic performance for information processing tasks

Matus (2007)

(Quantum) von Neumann Entropy:

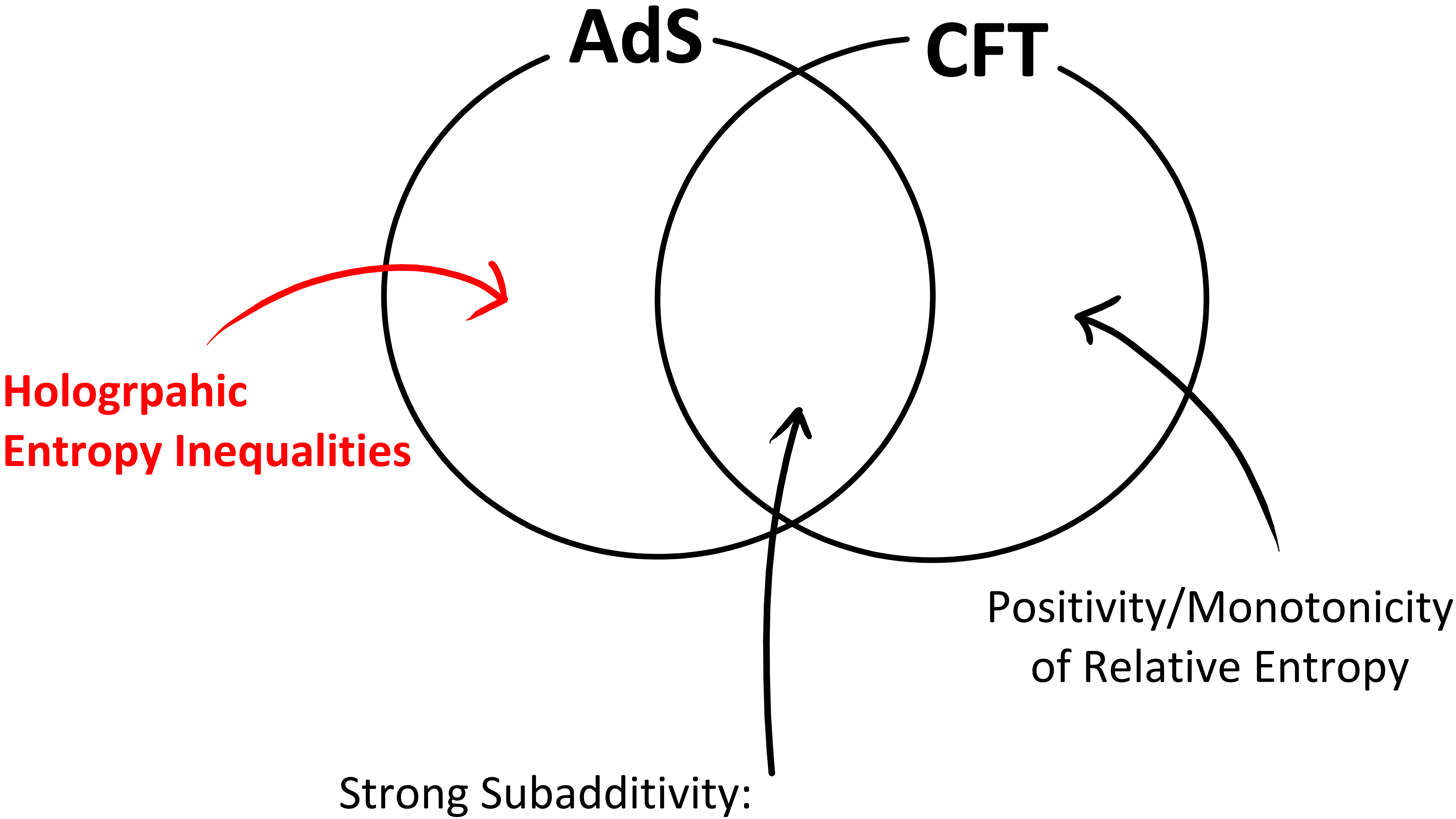
For more than 3 regions, the complete set of independent inequalities is ***not known***.

$\Rightarrow$ Numerical evidences that the number is infinite.

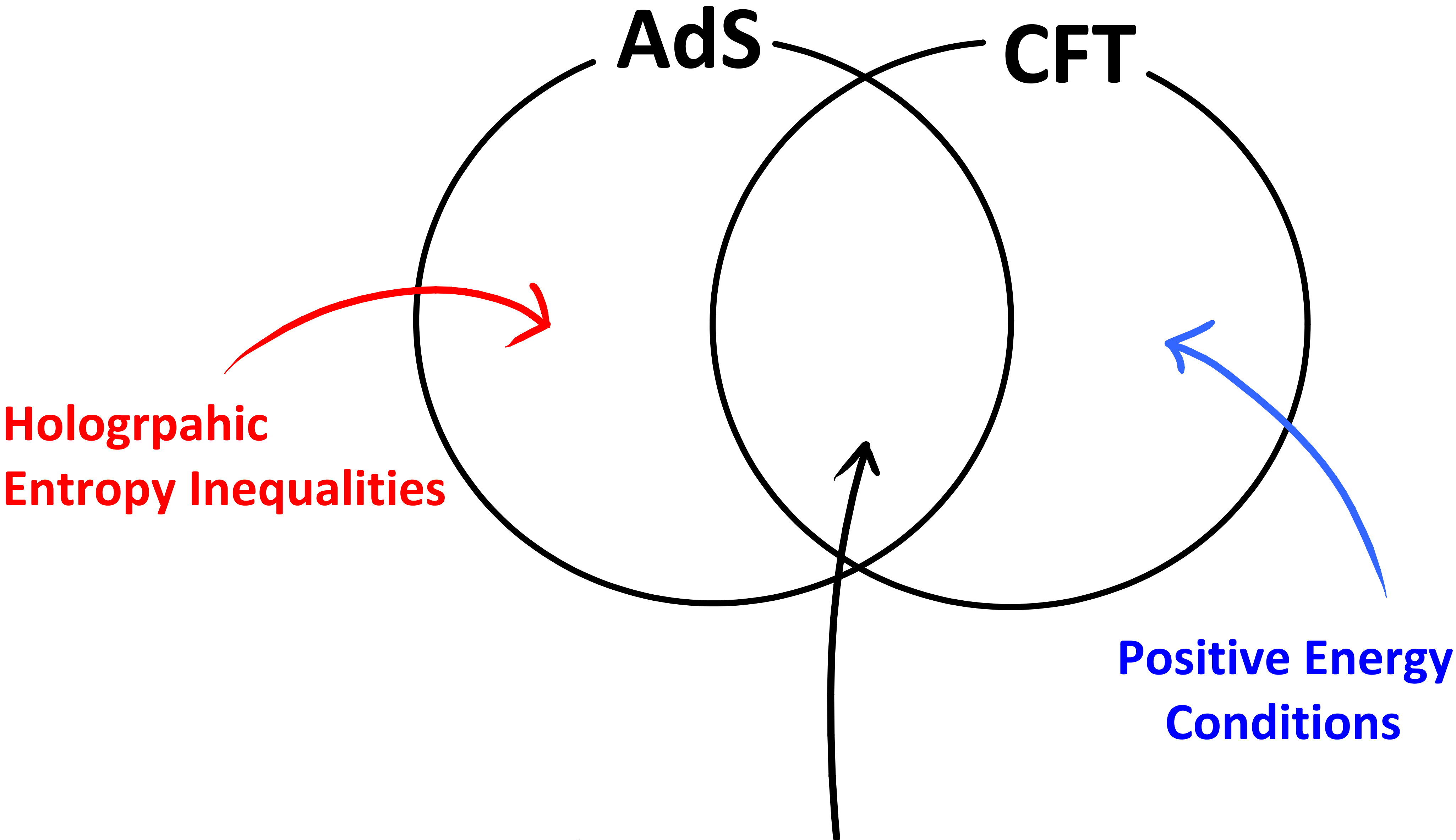
For holographic states:

- ☆ **Finite algorithm** to classify all inequalities.
- ☆ There are **finitely many independent inequalities** for a fixed number of regions.
- ☆ Complete classification for 2, 3, 4 regions.
- ☆ A new family of inequalities for 5 and more regions.

Bao, Nezami, Stoica, Sully, Walter + H.O., arXiv:1505.07839



$$S(AB) + S(BC) \geq S(B) + S(ABC)$$



Strong Subadditivity:

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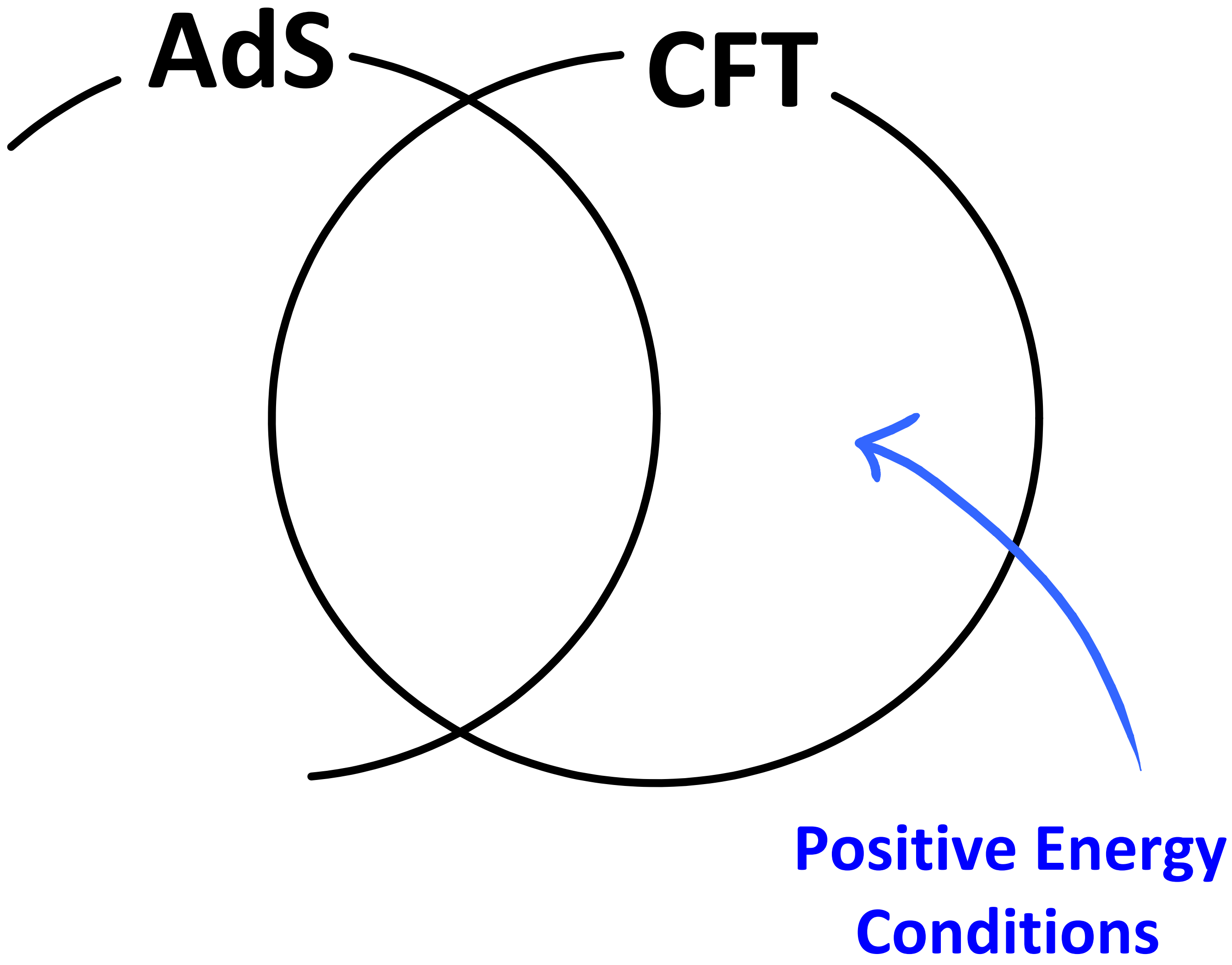

Information theoretical constraints on low energy effective theories

Positivitive Energy Conditions

Energy and Entropy

based on formalism
developed by Wald & collaborators

Σ : subregion of a Cauchy surface

We will choose Σ = entanglement wedge $\cap$ Cauchy surface,

i.e. a subregion bounded by

a Ryu-Takayanagi surface

(or HRT surface for

a time-dependent case)

and the AdS boundary.

$\Sigma \subset$ Cauchy surface , g : metric + matter on Σ.

$L(g)$: Lagrangian density

$$\delta L(g) = d\,\theta(\delta g) + \text{e.o.m.}$$

$$\Downarrow$$

$$\int_\Sigma \delta_1 \theta(\delta_2 g) - \delta_2 \theta(\delta_1 g)$$

$$= \Omega(\delta_1 g, \delta_2 g)$$

symplectic form

Analogy :

$$L(Q) = \frac{1}{2}\left(\frac{dQ}{dt}\right)^2 - V(Q)$$

$$\delta L(Q) = \frac{d}{dt}\left(\frac{dQ}{dt}\delta Q\right) + \text{e.o.m.}$$

$$= \frac{d}{dt}\theta(\delta Q) + \text{e.o.m.}$$

$$\theta(\delta Q) = P\,\delta Q$$

$$\delta\theta = \delta P \wedge \delta Q$$

Hamiltonian H_ξ for a vector field ξ on Σ to generate $\mathcal{L}_\xi g$

$$\delta H_\xi = \Omega(\delta g, \mathcal{L}_\xi g)$$

$$= \int_\Sigma \delta\theta(\mathcal{L}_\xi g) - \mathcal{L}_\xi \theta(\delta g)$$

$$\left(\mathcal{L}_\xi \theta = \xi \cdot \underbrace{d\theta}_{= \, \delta L + \text{e.o.m.}} + d(\xi \cdot \theta) \right)$$

$$= \int_\Sigma \delta\left(\theta(\mathcal{L}_\xi g) - \xi \cdot L\right)$$

$$- \oint_{\partial\Sigma} \xi \cdot \theta(\delta g)$$

Analogy:

$$\delta H = \delta p \frac{dQ}{dt} - \delta Q \frac{dP}{dt}$$

$$= \delta\left(p \frac{dQ}{dt}\right)$$

$$\underbrace{- \frac{d}{dt}(p\,\delta Q)}_{= \, \delta L + \text{e.o.m.}}$$

$$= \delta\left(p \frac{dQ}{dt} - L\right)$$

Hamiltonian H_ξ for a vector field ξ on Σ to generate $\mathcal{L}_\xi g$

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boundary terms are important in gravity

Analogy :

$$\delta H = \delta p \frac{dQ}{dt} - \delta Q \frac{dP}{dt}$$

$$= \delta\left(p \frac{dQ}{dt}\right) \underbrace{- \frac{d}{dt}(p\,\delta Q)}_{= \, \delta L + \text{e.o.m.}}$$

$$= \delta\left(p \frac{dQ}{dt} - L\right)$$

For a vector field ξ on Σ,

$$\delta H_\xi = \int_\Sigma \delta\left(\theta(\mathcal{L}_\xi g) - \xi \cdot L \right) - \oint_{\partial\Sigma} \xi \cdot \theta(\delta g).$$

If $\exists\, B$ on $\partial\Sigma$ such that $\xi \cdot \theta(\delta g) = \delta(\xi \cdot B)$,

$$H_\xi = \int_\Sigma J_\xi - \oint_{\partial\Sigma} \xi \cdot B$$

where

$$J_\xi = \theta(\mathcal{L}_\xi g) - \xi \cdot L.$$

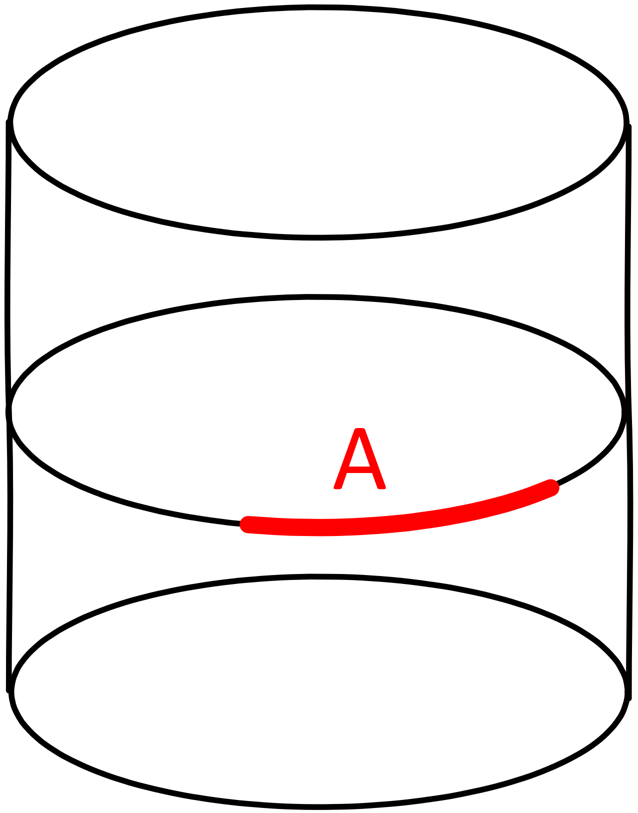
e.g. pure Einstein gravity,

$L = \frac{1}{2}(R - \Lambda)\, e$, $\quad e$: spacetime volume form

$\theta(\delta g) = \frac{1}{2}\left(g^{\mu\nu} D^\rho - g^{\nu\rho} D^\mu\right) \delta g_{\nu\rho}\, e_\mu$, $\quad e_\mu$: volume form on Σ

$B \propto$ extrinsic curvature (Gibbons - Hawking term)

Relative Entropy



$|\psi_0\rangle$: vacuum in CFT

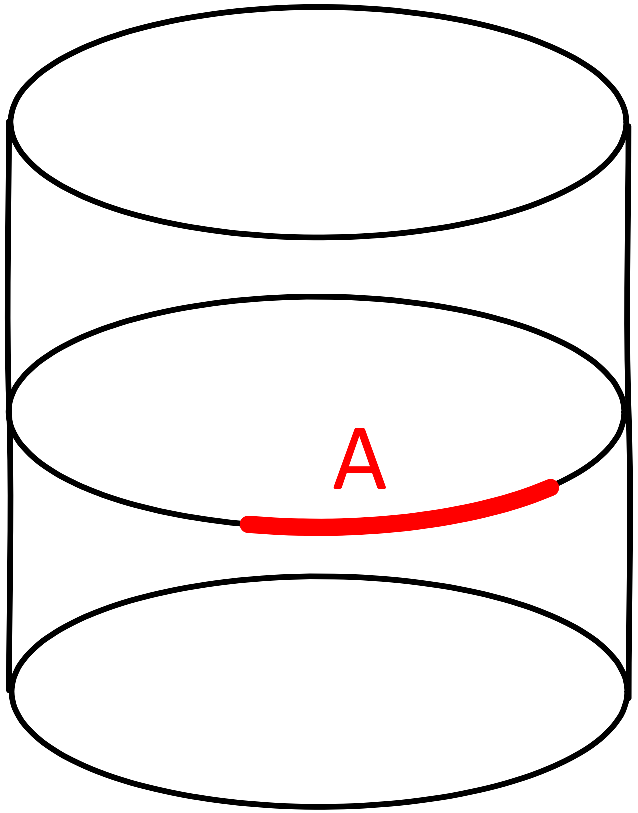
$\Leftrightarrow$ pure AdS geometry

$|\psi\rangle$: any CFT state

$\Leftrightarrow$ gravity solution

$$\rho_0 = \mathrm{tr}_{\bar{A}} |\psi_0\rangle\langle\psi_0|$$

$$\rho = \mathrm{tr}_{\bar{A}} |\psi\rangle\langle\psi|$$



Relative entropy :

$$S(\rho \,|\, \rho_0) = -\,\mathrm{tr}\,[\rho \log \rho_0] + \mathrm{tr}\,[\rho \log \rho]$$

measures the distance between

$$\rho_0 = \mathrm{tr}_{\bar{A}}\, |\psi_0\rangle\langle\psi_0|$$

$$\rho = \mathrm{tr}_{\bar{A}}\, |\psi\rangle\langle\psi|$$

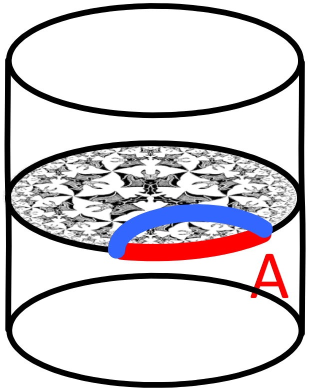
When A is a ball,

the modular Hamiltonian $= -\log \rho_0$ is simplified,

and $S(\rho \,|\, \rho_0)$ has a holographic expression.

Relative Entropy

$$S(\rho | \rho_0) = \underbrace{-\mathrm{tr}[\rho \log \rho_0]}_{\langle \text{modular Hamiltonian} \rangle_\rho} + \underbrace{\mathrm{tr}[\rho \log \rho]}_{-(\text{Entanglement Entropy})}$$



Metric asymptotics on A

Minimum surface area

$\exists\, \xi$, such that $S(\rho | \rho_0) = H_\xi(\rho) - H_\xi(\rho_0)$.

Hamiltonian $H_\xi = \int_\Sigma J_\xi - \oint_{\partial\Sigma} \xi \cdot B$

Relative Entropy = Energy in Entanglement Wedge

$$S(\rho \mid \rho_0) = H_\xi(\rho) - H_\xi(\rho_0)$$

Lashkari, Lin, Stoica, van Raamsdonk + H.O. arXiv:1605.01075

For linear variation, $\rho = \rho_0 + \delta\rho$

$$S(\rho_0 + \delta\rho, \rho_0) = 0$$

implies the linearized Einstein equation in the bulk.

Faulkner, Guica, Hartman, Myers + Van Raamsdonk, arXiv: 1312.7856

In the quadratic order, including backreaction to geometry,

$$S(\rho \,|\, \rho_0) \geq 0 \quad , \quad \frac{d}{dR} S(\rho \,|\, \rho_0) \geq 0$$

(R : radius of A)

$$\Downarrow$$

Integrated positivity of the bulk energy-momentum tensor,

$$\int_\Sigma \xi^\mu \left(T_{\mu\nu}^{\text{matter}} + T_{\mu\nu}^{\text{gravity}} \right) e^\nu_\Sigma \geq 0$$

Lin, Marcolli, Stoica + H.O. arXiv: 1412.1879

Lashkari, Rabideau, Sabella-Garnier, Van Raamsdonk, arXiv: 1412.3514

Lashkari, Van Raamsdonk, arXiv: 1508.0089

More on the quadratic perturbation:

Relative Entropy = Energy in Entanglement Wedge

implies

Fisher Information = Canonical Energy of Hollands and Wald

The positivity of Fisher information guarantees linear stability of AdS-Rindler wedge.

Relative Entropy = Energy in Entanglement Wedge

Positivity and monotonicity of the relative entropy

⇒

- Linearlized Einstein equations. arXiv:1312.7856
- Integrated positivity of $T_{\mu\nu}$ arXiv:1412.1879, 1412.3514
- Positivity of quasi-local energy arXiv:1605.01075

Any low energy effective theory of a consistent ultraviolet complete quantum theory of gravity must satisfy these positive energy conditions.

How strong are these positive energy conditions?

Which low energy theories are ruled out by them?

Note: $S(\rho | \sigma)_{CFT} = S(\tilde{\rho} | \tilde{\sigma})_{bulk}$

Jafferis, Lewkowycz, Maldacena, Suh: 1512.06431

Dong, Harlow, Wall: 1601.05416

Harlow: 1607.03901

Or, can we prove a new type of positivity theorems for quasi-local energies?

c.f. Bekenstein bound Casini: 0804.2182

Swampland Question:

How to characterize an effective gravity theory that can emerge in a low energy approximation to a consistent quantum theory, such as string theory.

Constraints on Symmetry

Constraints on Moduli Space

Constraints on Calabi-Yau Topology

New Type of Positive Energy Theorems

String-Math 2018

Tohoku University, Sendai

18 - 22 June 2018

Strings 2018

OIST, Okinawa

25 - 29 June 2018

We look forward to welcoming you
in Japan in 2018.