AN $\mathbb{R}^3$ INDEX FOR $\mathcal{N} = 2$ THEORIES IN $d = 4$

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PREFACE

The aim of this talk is to describe an interesting protected quantity $\mathcal{I}$ in four-dimensional $\mathcal{N} = 2$ supersymmetric field theory.

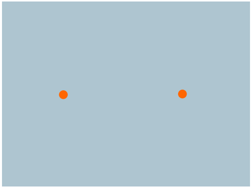
$\mathcal{I}$ is a generating function which counts BPS states in the Hilbert space of the theory on spatial $\mathbb{R}^3$, and which has various nice geometric properties.

We studied $\mathcal{I}$ in joint work with Sergei Alexandrov, Greg Moore and Boris Pioline.

$\mathcal{N} = 2$ THEORIES

Fix an $\mathcal{N} = 2$ SUSY QFT in $d = 4$.

Such a theory has a moduli space of vacua. We work on the Coulomb branch. At generic points u, the IR physics is abelian gauge theory. At discriminant locus, this description can break down. [Seiberg-Witten]



$\mathcal{N} = 2$ THEORIES

Particles of electromagnetic/flavor charge γ obey a BPS bound

$$M \geq |Z_\gamma|$$

where $Z_\gamma(u)$ is the central charge, depending on point u of Coulomb branch.

Those with

$$M = |Z_\gamma|$$

are called BPS.

BPS COUNTS IN $\mathcal{N} = 2$

BPS particles of charge γ are "counted" by second helicity supertrace

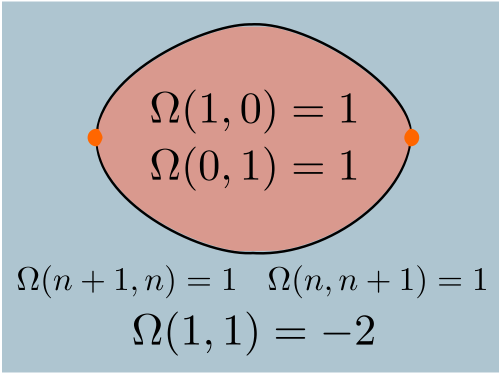
$$\Omega(\gamma; u) = -\frac{1}{2}\,\mathrm{Tr}_{\mathcal{H}^1_{\mathbb{R}^3,\gamma}}(-1)^F J_3^2$$

e.g. BPS hypermultiplet of charge γ contributes $\Omega(\gamma; u) = 1$, BPS vector multiplet $\Omega(\gamma; u) = -2$, and so on.

The $\Omega(\gamma; u)$ are protected by supersymmetry, but nevertheless can jump at certain walls in the Coulomb branch, where BPS particles are only marginally stable.

BPS counts in $\mathcal{N} = 2$

A fundamental example: $\mathcal{N} = 2$ pure $SU(2)$ super Yang-Mills. [Seiberg-Witten]



A "simple" answer (fits on this slide).

BPS COUNTS IN $\mathcal{N} = 2$

In the last few years there has been a lot of progress in methods for computing $\Omega(\gamma; u)$:

- Wall crossing [Denef-Moore, Kontsevich-Soibelman, Gaiotto-Moore-AN, Cecotti-Vafa, Manschot-Pioline-Sen, ...]
- Quivers [Denef, Alim-Cecotti-Cordova-Espahbodi-Rastogi-Vafa, Cecotti-del Zotto, ...]
- Spectral networks [Gaiotto-Moore-AN, Maruyoshi-Park-Yan, ...]

BPS counts in $\mathcal{N} = 2$

One thing we've learned: field theory BPS spectra are more intricate than we thought!
There can be exponential towers of BPS threshold bound states,

$$\Omega(n\gamma) \sim n^a e^{cn}$$

(e.g. this happens already in pure $SU(3)$ Yang-Mills; similar growth seems to occur in the Minahan-Nemeschansky E_6 theory).
[Galakhov-Longhi-Mainiero-Moore-AN, Hollands-AN in progress]
Moreover, the pattern of walls where $\Omega(\gamma)$ jump can be extremely complicated.

Generating function

Another way to study the $\Omega(\gamma; u)$: try to organize them into a generating function with some physical meaning.

Simplest try would be to introduce potentials θ_i and write

$$F(u, \theta_i) = \sum_{\gamma} \Omega(\gamma; u) e^{\mathrm{i}\theta_i \gamma^i}$$

But then F would jump at walls of marginal stability. Since the theory has no phase transition (we think), physical observables should be continuous.

CFIV INDEX

In two-dimensional massive $\mathcal{N} = (2,2)$ theories, such a generating function does exist: CFIV index [Cecotti-Fendley-Intriligator-Vafa]

$$Q_{ij} = \lim_{L\to\infty} \frac{\beta}{L} \mathrm{Tr}_{\mathcal{H}_{ij}} (-1)^F F e^{-\beta H}$$

Expanding around $\beta \to \infty$,

$$Q_{ij} \sim \mu(i,j) \sqrt{\beta |Z_{ij}|} e^{-\beta |Z_{ij}|}$$

where $\mu(i,j)$ is an index counting BPS solitons between vacua i and j, and $|Z_{ij}|$ is their mass.

CFIV INDEX

Around $\beta \to \infty$,

$$Q_{ij} \sim \mu(i,j)\sqrt{\beta|Z_{ij}|}e^{-\beta|Z_{ij}|}$$

As we vary parameters, $\mu(i,j)$ can jump. So the asymptotics of Q_{ij} as $\beta \to \infty$ are not smooth.

Nevertheless, Q_{ij} is a nice smooth function of parameters!

Key is contribution from 2-particle states: there is a jump in this contribution too, which cancels the jump in the 1-particle sector.

$\mathbb{R}^3$ INDEX

There is a quantity in four-dimensional $\mathcal{N} = 2$ theories which seems to be an analogue of the two-dimensional CFIV index.
[Alexandrov-Moore-AN-Pioline]

$$\mathcal{I} = \mathcal{I}(u, \beta, \theta_i)$$

a single function, depending on:

- Coulomb branch modulus u,
- "temperature" β,
- potentials θ_i dual to components γ^i of EM charge γ.

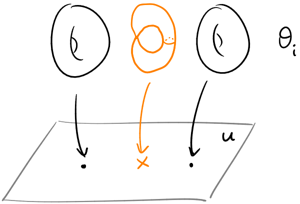
Line defect vevs

Next, a formula for $\mathcal{I}$. To state it, we need some geometric preliminaries.

Circle compactification

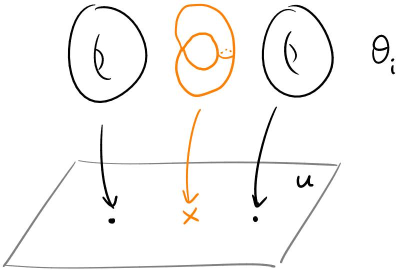
Compactify 4d to 3d on S^1_β, dualize gauge fields to scalars. Get 3d sigma model in IR.

Fields: Coulomb branch scalars from 4d, plus e/m Wilson lines θ_i of the abelian gauge fields around S^1_β.



Thus sigma model target is a torus fibration $\mathcal{M}$ over the Coulomb branch of the 4d theory.

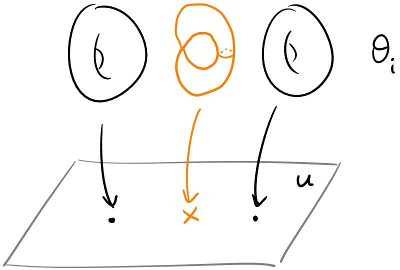
Circle compactification



$\mathcal{M}$ has singular fibers, above the loci in the 4d Coulomb branch where the abelian gauge theory breaks down.

SUSY of the 3d theory says $\mathcal{M}$ is hyperkähler.

Circle compactification



The index $\mathcal{I}$ is best thought of as a function on $\mathcal{M}$; in particular, it extends even to the singular fibers.

The torus fiber coordinates θ_i will play the role of the potentials which enter into $\mathcal{I}$.

LINE DEFECT VEVS

In the theory on $\mathbb{R}^3 \times S^1_\beta$, vevs of SUSY line defects wrapped on S^1_β admit a universal expansion, of the form [Gaiotto-Moore-AN]

$$\langle L(\zeta) \rangle = \sum_\gamma \underline{\overline{\Omega}}(L, \gamma; u) \mathcal{X}_\gamma(\zeta)$$

The coefficients $\underline{\overline{\Omega}}(L, \gamma; u) \in \mathbb{Z}$ count framed BPS states of charge γ. The parameter $\zeta \in \mathbb{C}^\times$ keeps track of which SUSY the defect preserves.

The universal functions $\mathcal{X}_\gamma(\zeta)$ are a local coordinate system on $\mathcal{M}$; vevs of "IR line defects"; we will build $\mathcal{I}$ out of these.

A formula for the index

We define

$$\mathcal{I} = -4\pi^2\beta^2 \mathrm{i}\langle Z, \bar{Z}\rangle - \sum_{\gamma} \Omega(\gamma)|Z_\gamma|\mathcal{I}_\gamma,$$

where

$$\mathcal{I}_\gamma = \int_{-\infty}^{\infty} \mathrm{d}t \, \cosh t \, \log\left(1 - \mathcal{X}_\gamma(-e^{t+\mathrm{i}\arg Z_\gamma})\right).$$

Looks mysterious, but engineered to have the properties we want!

$\mathbb{R}^3$ INDEX

We can compute the asymptotics of $\mathcal{I}$ as $\beta \to \infty$, using known properties of the functions $\mathcal{X}_\gamma(\zeta)$.

More precisely consider coefficient of Fourier mode $e^{\mathrm{i}\theta_i \gamma^i}$. As $\beta \to \infty$ it counts 1-particle BPS states of charge γ,

$$\mathcal{I}^1(\gamma) \sim \Omega(\gamma) \times e^{\mathrm{i}\theta_i \gamma^i} \sqrt{\beta |Z_\gamma|} e^{-\beta |Z_\gamma|}$$

CONTINUITY

As $\beta \to \infty$

$$\mathcal{I}^1(\gamma) \sim \Omega(\gamma) \times e^{\mathrm{i}\theta_i \gamma^i} \sqrt{\beta |Z_\gamma|} e^{-\beta |Z_\gamma|}$$

Nevertheless, $\mathcal{I}$ is smooth across walls where $\Omega(\gamma)$ jumps.

A direct proof of this uses dilogarithm identities, arising in "semiclassical limit" of the refined wall-crossing formula obeyed by the BPS spectrum. [Kontsevich-Soibelman, Dimofte-Gukov-Soibelman, Gaiotto-Moore-AN, Cecotti-Vafa, Alexandrov-Persson-Pioline]

Geometric interpretation, II

Suppose the 4d theory which we consider is conformal.

Then $\mathcal{I}$ is a Kähler potential on the space $\mathcal{M}$.
[Alexandrov-Roche]

(More precisely, since $\mathcal{M}$ is hyperkähler, it has an S^2 worth of complex structures; $\mathcal{I}$ is a Kähler potential for a circle's worth of these complex structures.)

GEOMETRIC INTERPRETATION, III

Suppose the 4d theory which we consider is of class S, associated to Riemann surface C and Lie algebra $\mathfrak{g}$. Then $\mathcal{M}$ is space of vacua of twisted 5d SYM on $C \times \mathbb{R}^3$ (Hitchin system).

In this language $\mathcal{I}$ becomes very simple:

$$\mathcal{I} = \mathrm{i} \int_C \mathrm{Tr}(\varphi \varphi^\dagger)$$

where φ is twisted adjoint scalar of 5d SYM.

GEOMETRIC INTERPRETATION, III

$$\mathcal{I} = \mathrm{i} \int_C \mathrm{Tr}(\varphi \varphi^\dagger)$$

Thus, in theories of class S, the quantum observable $\mathcal{I}$ (summing up the whole BPS spectrum) can be computed by a purely classical formula in 5d SYM.

We proved this in a rather roundabout way; there should be a simple and direct argument.

QUESTIONS

- What is a more conceptual definition of $\mathcal{I}$? Can we prove that it is

$$\mathcal{I} = \lim_{V \to \infty} \frac{1}{V} \mathrm{Tr}_{\mathcal{H}_{\mathbb{R}^3}} (-1)^F J_3^2 e^{\mathrm{i}\theta_i \gamma^i - \beta H}$$

  (at least the 1-particle contribution matches, with an appropriate regulator)? cf. [Cecotti-Fendley-Intriligator-Vafa]

- How is $\mathcal{I}$ related to more familiar protected quantities in $\mathcal{N} = 2$ theories, such as instanton partition functions? [Nekrasov]

QUESTIONS

- Recently [Gerchkovitz-Gomis-Komargodski] showed that for conformal $\mathcal{N} = 2$ theories the S^4 partition function is a Kähler potential for the Zamolodchikov metric on the conformal manifold.

  The index $\mathcal{I}$ is something like an $\mathbb{R}^3 \times S^1_\beta$ partition function and is also a Kähler potential — but on the IR moduli space $\mathcal{M}$ instead of the conformal manifold. Are these two stories somehow related?

QUESTIONS

- The $\mathcal{X}_\gamma(\zeta)$, which entered our formula for $\mathcal{I}$, are solutions of integral equations which look like 2-d thermodynamic Bethe ansatz.

$$\mathcal{X}_\gamma(\zeta) = \mathcal{X}_\gamma^{sf}(\zeta) \exp\left[\sum_{\gamma'} \langle \gamma, \gamma' \rangle \Omega(\gamma') \frac{1}{4\pi \mathrm{i}} \int_{Z_\gamma \mathbb{R}_-} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log(1 - \mathcal{X}_{\gamma'}(\zeta'))\right]$$

$$\mathcal{X}_\gamma^{sf}(\zeta) = \exp\left(\frac{\beta Z_\gamma}{\zeta} + \mathrm{i}\theta_i \gamma^i + \beta \bar{Z}_\gamma \zeta\right)$$

Why 2-d? We are studying a 4-d system!

$\mathcal{I}$ is the TBA free energy. Can this help us understand why the TBA is there?

Thank you!

Spectral networks

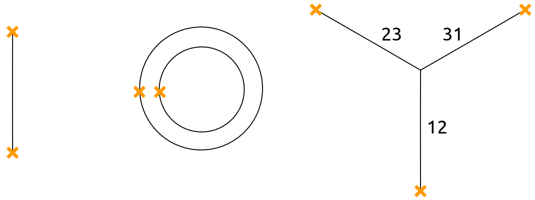
The idea of spectral networks is to study BPS states indirectly, through their interaction with surface defects.

In principle it can be done in any theory, if we have enough surface defects and understand them well enough.

SPECTRAL NETWORKS

In theories of class S, spectral networks count webs of BPS strings of the $(2,0)$ theory on C.

For simple webs, the Ω turn out to be simple:



$\Omega = 1$ $\Omega = -2$ $\Omega = 1$

For A_1 theory this recovers results of [Klemm-Lerche-Mayr-Vafa-Warner]

BPS COUNTS IN E_6 SCFT

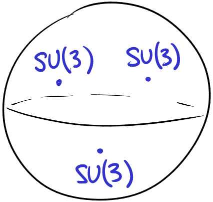
A recent example [Hollands-AN, in progress]: computation of part of the BPS spectrum of $\mathcal{N} = 2$ SCFT with E_6 global symmetry [Minahan-Nemeschansky]. ("Part" means we consider only some directions in the charge lattice.)

This theory is non-Lagrangian (today).

Coulomb branch is 1-dimensional, so superconformal invariance implies the spectrum at any point is the same as at any other.

BPS COUNTS IN E_6 SCFT

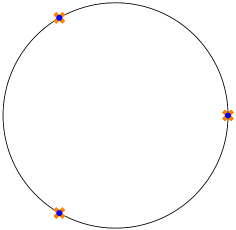
We use spectral networks and the class S realization of the E_6 theory: $\mathfrak{g} = \mathfrak{su}(3)$, $C = \mathbb{CP}^1$ with 3 punctures. [Gaiotto]



The construction makes manifest only $SU(3) \times SU(3) \times SU(3) \subset E_6$ but the spectrum comes out "miraculously" organized into E_6 representations!

BPS COUNTS IN E_6 SCFT

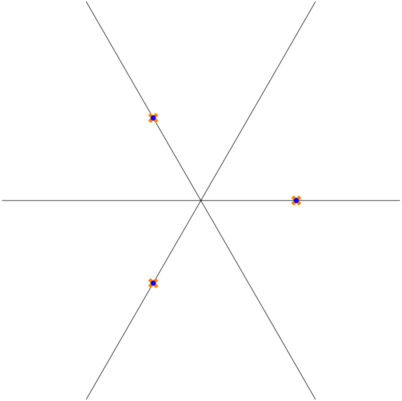
For example, along one ray in charge lattice, the degeneracies are controlled by this network:



$$\Omega(\gamma) = \overline{\mathbf{27}}$$
$$\Omega(2\gamma) = 2 \times \mathbf{27}$$
$$\Omega(3\gamma) = 3 \times (\mathbf{78} \oplus \mathbf{1} \oplus \mathbf{1})$$
$$\Omega(4\gamma) = 4 \times (\overline{\mathbf{351}} \oplus \overline{\mathbf{27}} \oplus \overline{\mathbf{27}}) \qquad \cdots$$

BPS counts in E_6 SCFT

But there are infinitely many such networks contributing; and so far we have to deal with them one by one!



BPS COUNTS IN E_6 SCFT

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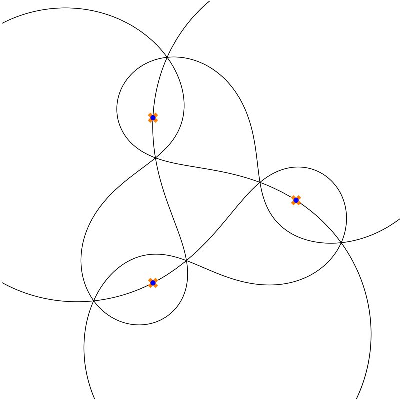

BPS COUNTS IN E_6 SCFT

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