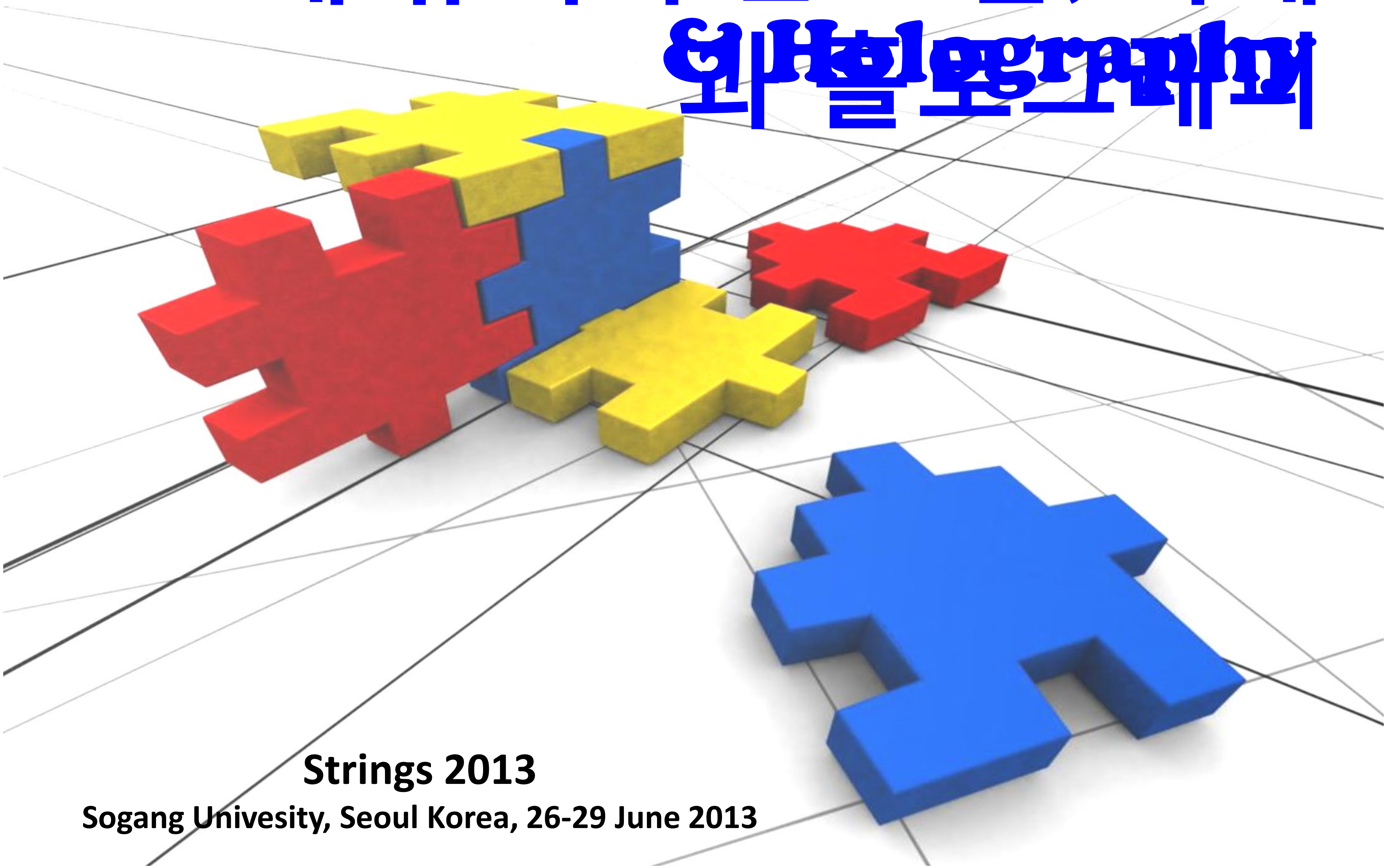
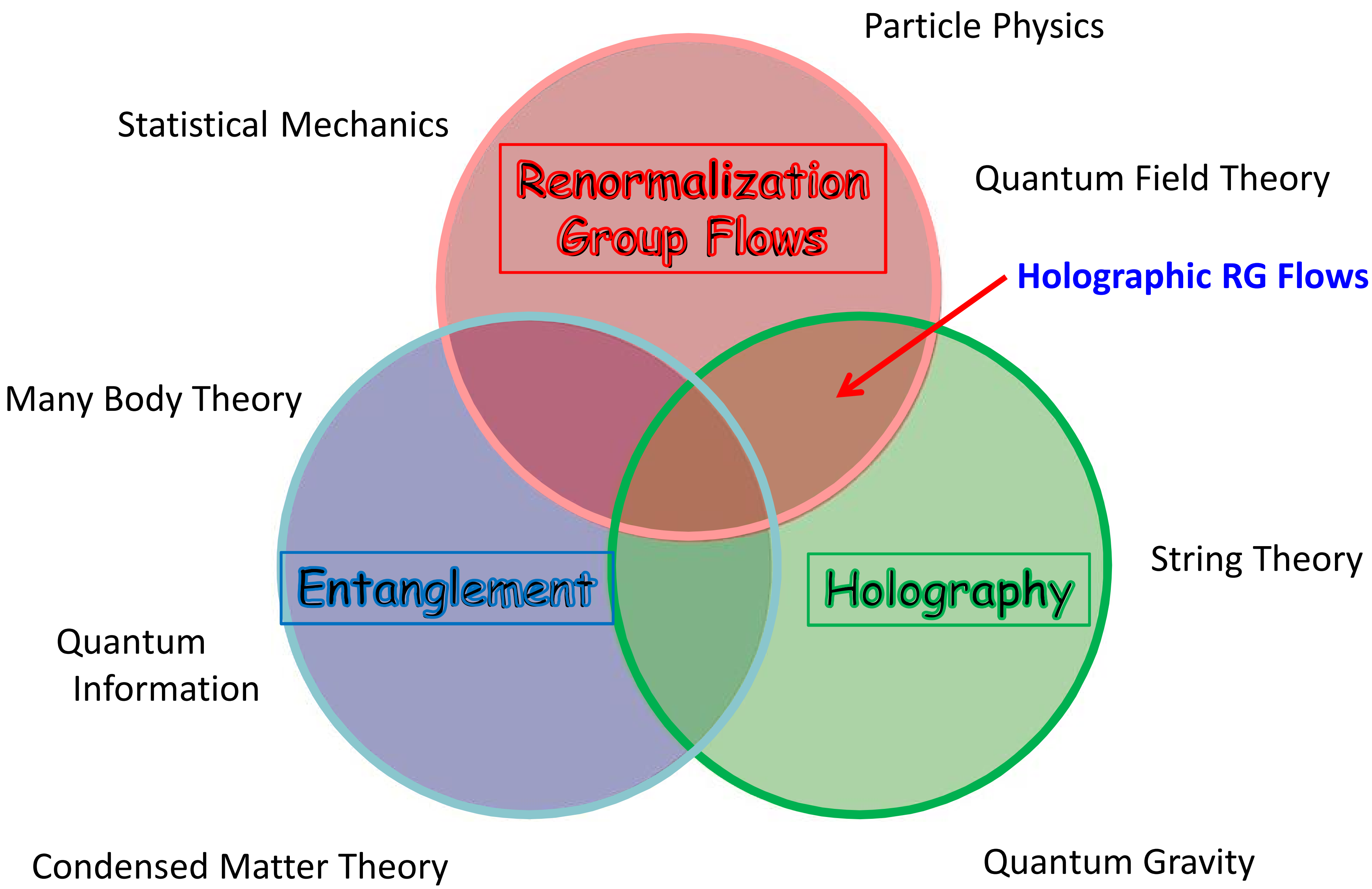
RG Flows & Entanglement & Holography



Strings 2013

Sogang Univesity, Seoul Korea, 26-29 June 2013

New Dialogues in Theoretical Physics:

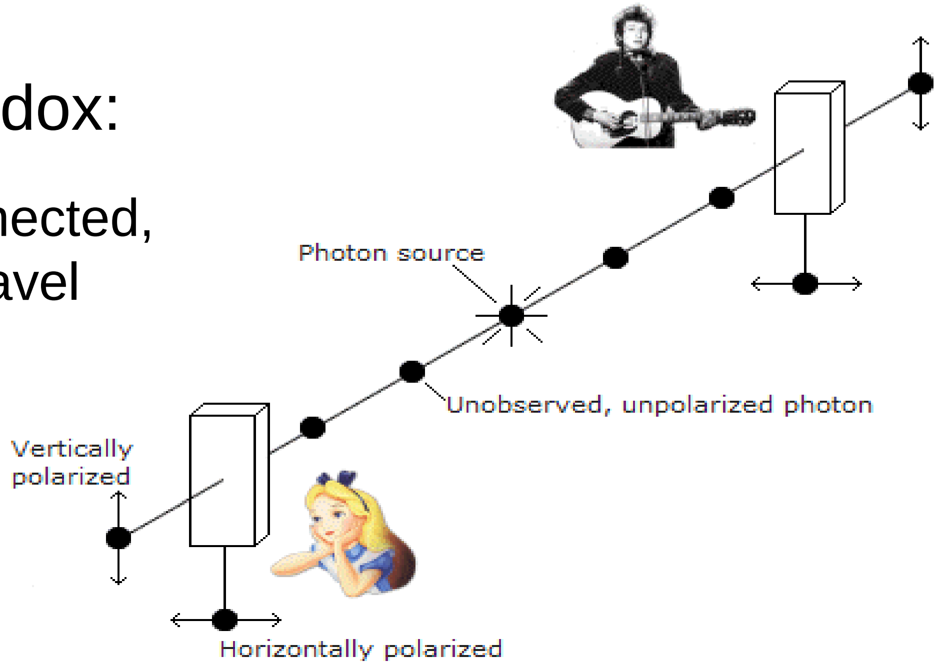


Quantum Entanglement

Einstein-Podolsky-Rosen Paradox:

Åproperties of pair of photons connected,
no matter how far apart they travel

ñ*spukhafte Fernwirkung”* =
spooky action at a distance



$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(\,|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\,\Big)$$

Quantum Information: entanglement becomes a resource for (ultra)fast computations and (ultra)secure communications

Condensed Matter: key to ñexoticò phases and phenomena, e.g., quantum Hall fluids, unconventional superconductors, quantum spin fluids, ...

Quantum Entanglement

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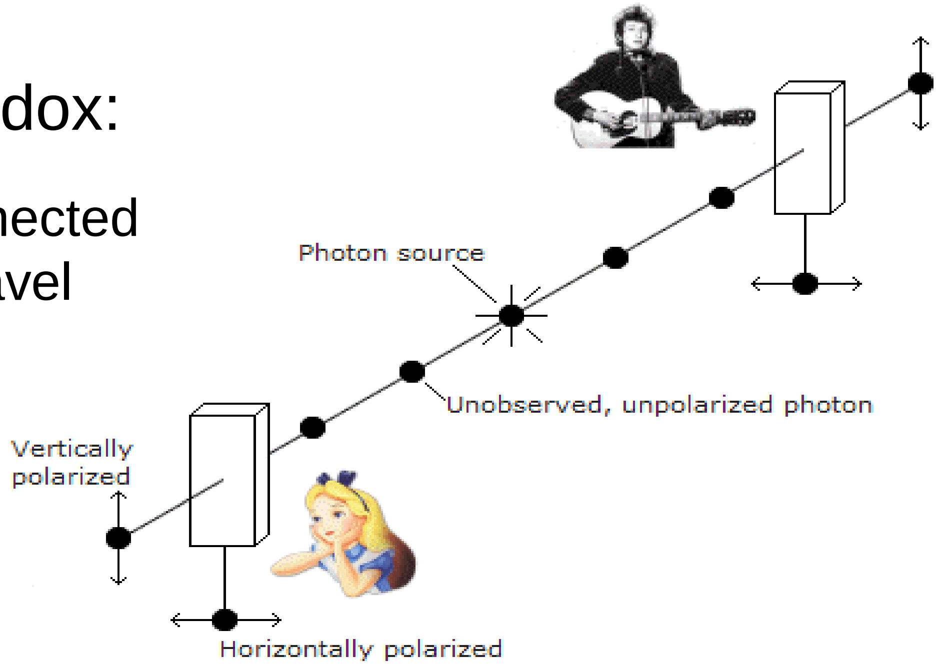


$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\Big)$$

compare: $$|\psi'\rangle = \frac{1}{2}\Big(|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle + |\downarrow\downarrow\rangle\Big)$$

$$= \frac{1}{2}\Big(|\uparrow\rangle + |\downarrow\rangle\Big) \otimes \Big(|\uparrow\rangle + |\downarrow\rangle\Big)$$ ⟶ **No Entanglement!!**

$$|\psi''\rangle = \frac{1}{2}\Big(|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\Big)$$ ⟶ **Entangled!!**

Entanglement Entropy:

Ågeneral diagnostic: divide quantum system into two parts and use entropy as measure of correlations between subsystems

Åprocedure:

- Ådivide system into two subsystems, eg, A and B
- Åtrace over degrees of freedom in subsystem B
- Åremaining dof in A are described by a density matrix ρ_A
- Åcalculate von Neumann entropy: $S_{EE} = -Tr\left[\rho_A\,\log\rho_A\right]$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\Big) \longrightarrow \rho = \mathrm{Tr}_2\left(|\psi\rangle\langle\psi|\right) = \frac{1}{2}\left(|\downarrow\rangle\langle\downarrow| + |\uparrow\rangle\langle\uparrow|\right)$$

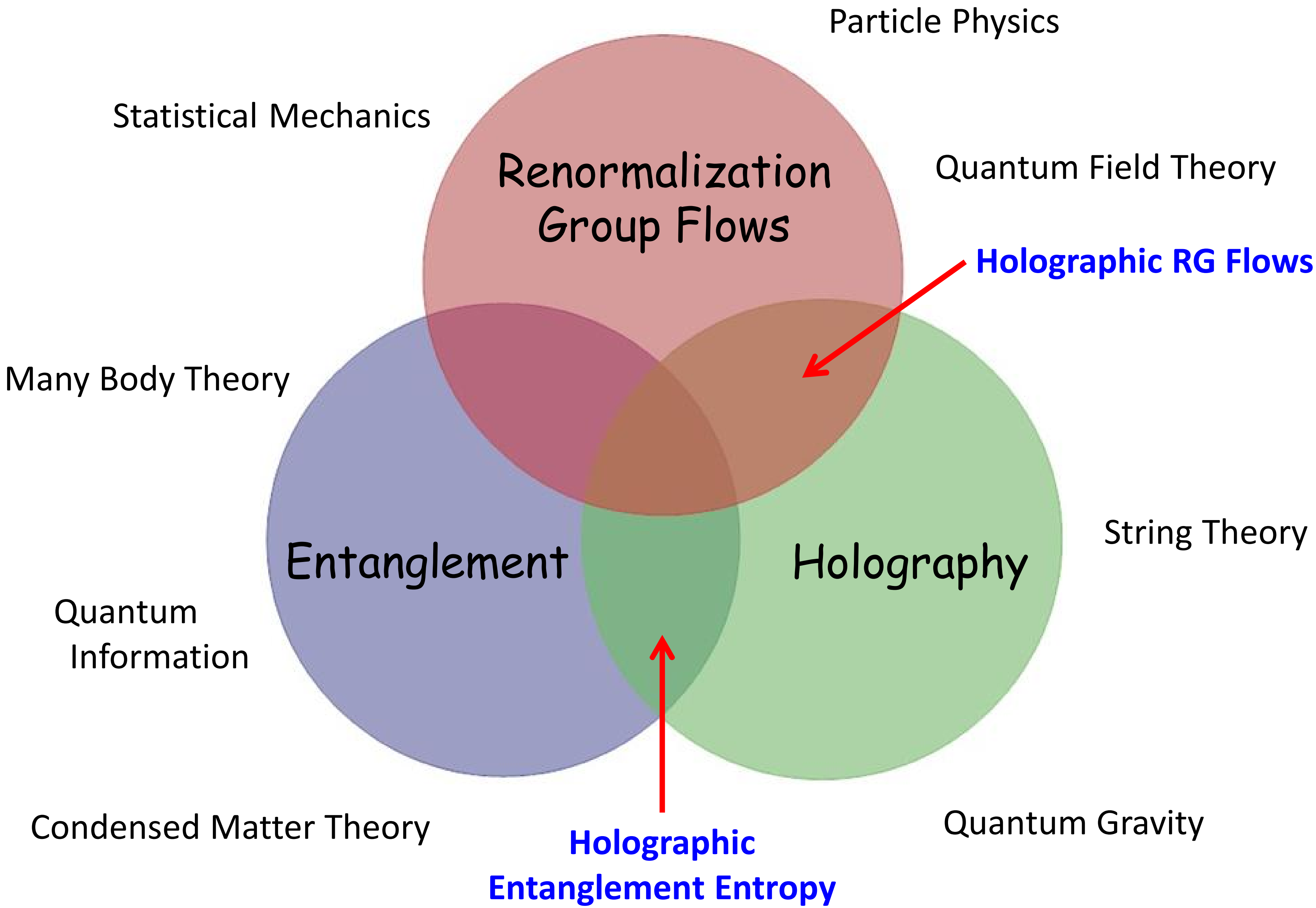
$$\longrightarrow S_{EE} = \log 2$$

compare: $$|\psi'\rangle = \frac{1}{2}\Big(|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle + |\downarrow\downarrow\rangle\Big)$$

$$= \frac{1}{2}\Big(|\uparrow\rangle + |\downarrow\rangle\Big)\otimes\Big(|\uparrow\rangle + |\downarrow\rangle\Big) \Longrightarrow S_{EE} = 0$$ No Ent!!

$$|\psi''\rangle = \frac{1}{2}\Big(|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\Big) \Longrightarrow S_{EE} = \log 2$$ Ent

New Dialogues in Theoretical Physics:



Entanglement Entropy 2:

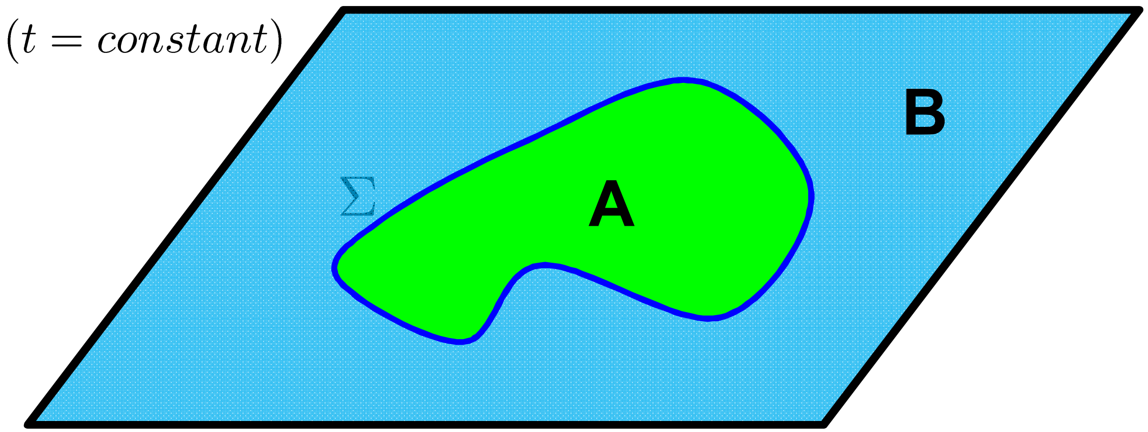
Åin the context of holographic entanglement entropy, S_{EE} is applied in the context of quantum field theory

Åin QFT, typically introduce a (smooth) boundary or entangling surface Σ which divides the space into two separate regions

Åintegrate out degrees of freedom in ñoutsideòregion

Åremaining dof are described by a density matrix ρ_A

→ calculate von Neumann entropy: $S_{EE} = -Tr\left[\rho_A \log \rho_A\right]$



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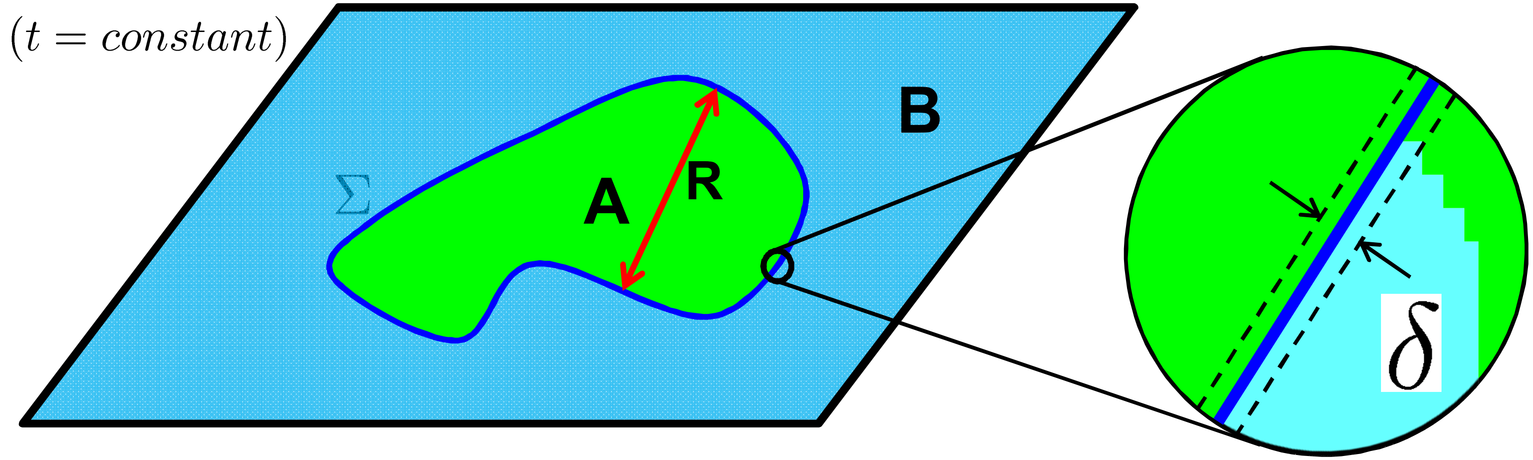

Åresult is UV divergent!

Åmust regulate calculation: δ = short-distance cut-off

$$S = c_0 \frac{R^{d-2}}{\delta^{d-2}} + c_2 \frac{R^{d-4}}{\delta^{d-4}} + \cdots$$

d = spacetime dimension

Åcareful analysis reveals geometric structure, eg, $S = \tilde{c}_0 \frac{\mathcal{A}_\Sigma}{\delta^{d-2}} + \cdots$

Entanglement Entropy 2:

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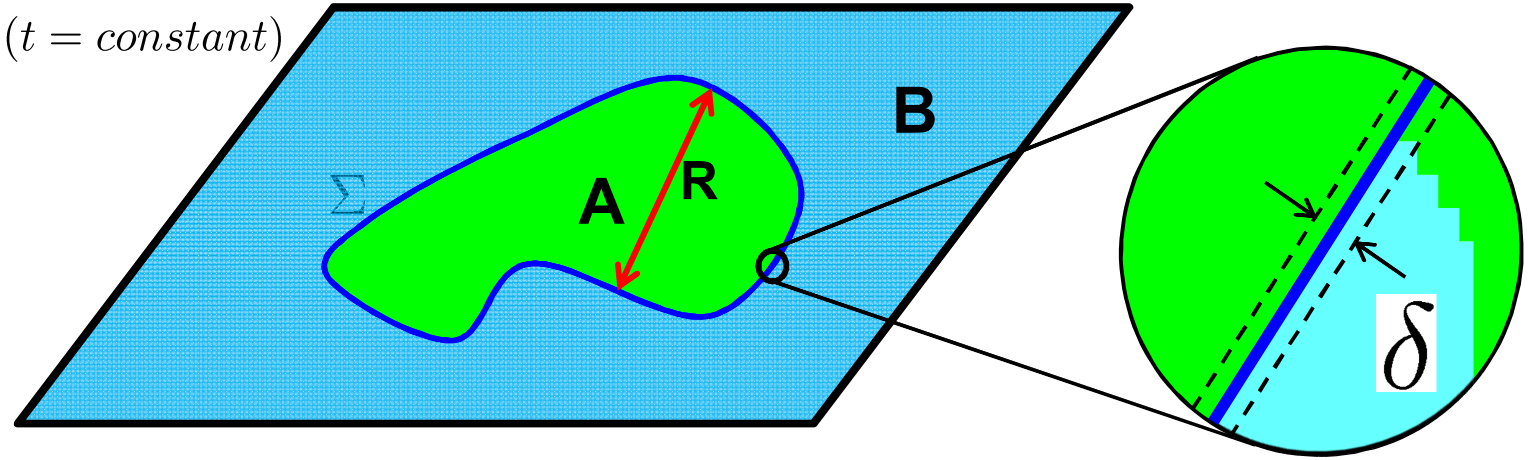
→ calculate von Neumann entropy: $S_{EE} = -Tr\left[\rho_A \log \rho_A\right]$



Åmust regulate calculation: δ = short-distance cut-off

$$S = c_0 \frac{R^{d-2}}{\delta^{d-2}} + c_2 \frac{R^{d-4}}{\delta^{d-4}} + \cdots$$

d = spacetime dimension

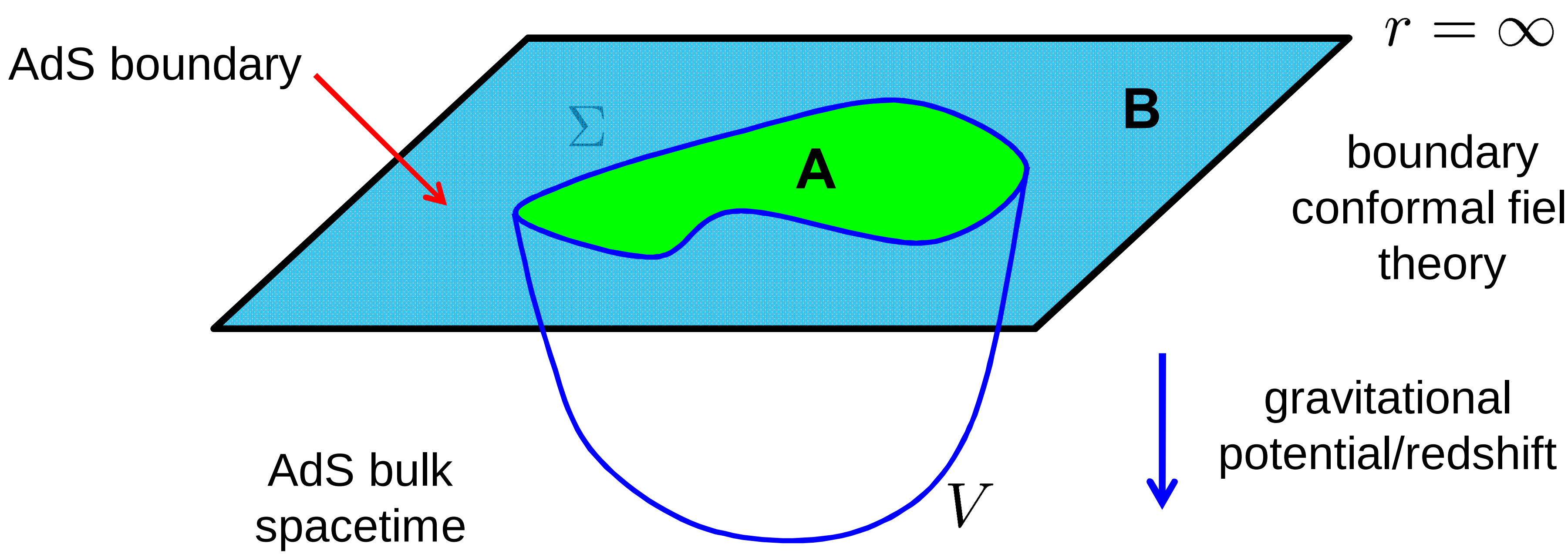
Åleading coefficients sensitive to details of regulator, eg, $\delta \to 2\delta$

Åfind universal information characterizing underlying QFT in subleading terms, eg,

$$S = \cdots + c_d \log\left(R/\delta\right) + \cdots$$

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:



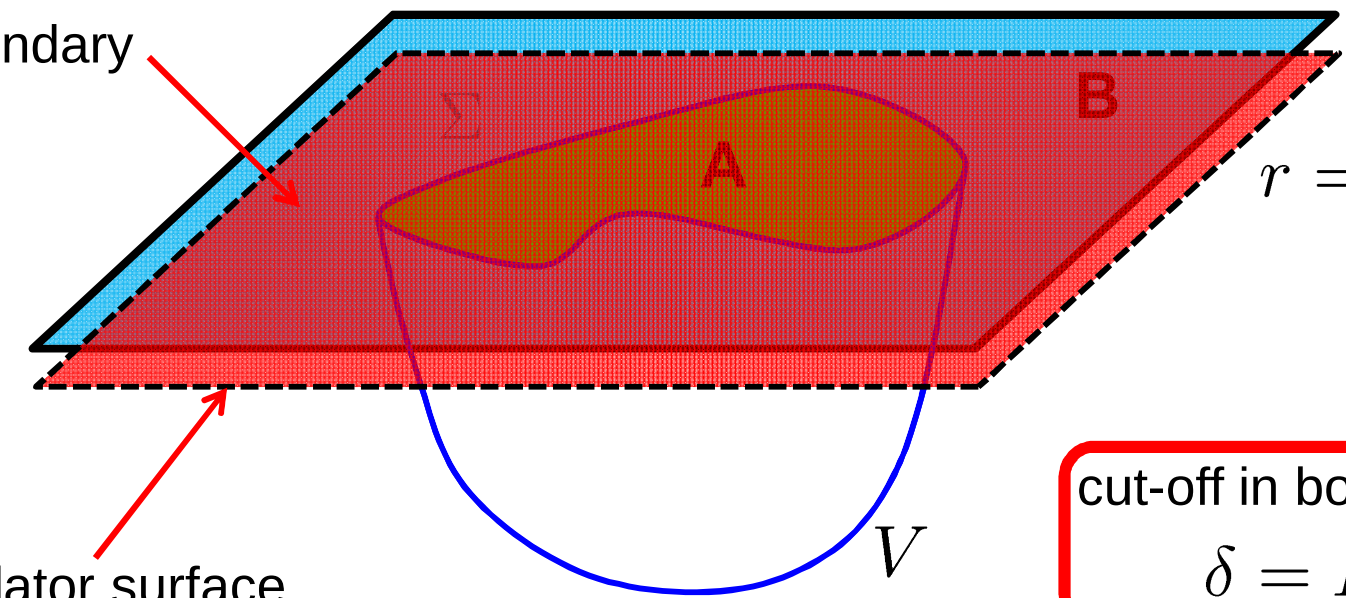
$$S(A) = \text{??‹t}_{\partial V = \Sigma} \frac{A_V}{4G_N} = \infty!!$$

ÅñUV divergenceòbecause area integral extends to $r = \infty$

looks like
BH entropy!

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:



cut-off in boundary CFT:
$\delta = L^2/R_0$

$$S(A) = \operatorname*{ext}_{\partial V = \Sigma} \frac{A_V}{4G_N} \simeq \frac{L^{d-1}}{G_N}\, \frac{\mathcal{A}_\Sigma}{\delta^{d-2}} + \cdots$$

- "UV divergence" because area integral extends to $r = \infty$
- as usual, introduce "regulator surface" at large radius: $r = R_0$

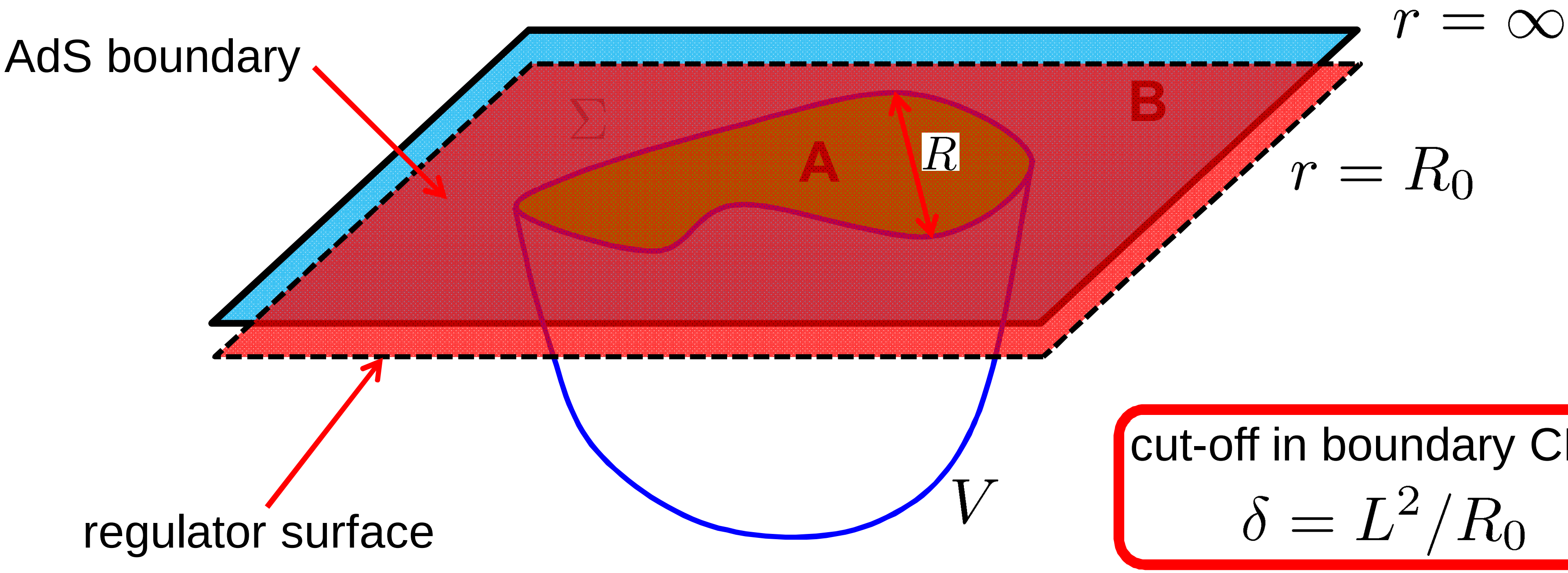
→ short-distance cut-off in boundary theory: $\delta = L^2/R_0$

central charge $(L/\ell_{Planck})^{d-1}$ (counts dof)

"Area Law"

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:



cut-off in boundary CFT:
$\delta = L^2/R_0$

general expression (as desired):

$$S(A) \simeq c_0(R/\delta)^{d-2} + c_2(R/\delta)^{d-4} + \cdots$$

$$\begin{cases} +c_{d-2}\log(R/\delta) + \cdots & \text{(d even)} \\ +\ c_{d-2}\ + \cdots & \text{(d odd)} \end{cases}$$

universal contributions

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:

$$S(A) = \underset{\partial V = \Sigma}{\text{ext}} \frac{A_V}{4G_N}$$

conjecture

Extensive consistency tests:

1) leading contribution yields ñarea lawò $S \simeq \frac{L^{d-1}}{G_N} \frac{\mathcal{A}_\Sigma}{\delta^{d-2}} + \cdots$

2) recover known results for d=2 CFT:

(Holzhey, Larsen & Wilczek)
(Calabrese & Cardy)

$$S = \frac{c}{3} \log\left(\frac{C}{\pi\,\delta} \sin \frac{\pi \ell}{C} \right)$$

3) $S(A) = S(\bar{A})$ in a pure state

⟶ A and $\bar{A}$ both yield same bulk surface V

4) for thermal bath: $S(A) \supset S_{therm} = \alpha\, T^{d-1} \times volume$

$R \gg 1/T$

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:

$$S(A) = \underset{\partial V = \Sigma}{\text{ext}} \frac{A_V}{4G_N}$$

conjecture

Extensive consistency tests:

5) strong sub-additivity: $S(A \cup B) + S(A \cap B) \leq S(A) + S(B)$

(Headrick & Takayanagi)

[further monogamy relations: Hayden, Headrick & Maloney]

6) for even d, connection of universal/logarithmic contribution in S_{EE} to central charges of boundary CFT, eg, in d=4

$$S_{uni} = \log(R/\delta)\, \frac{1}{2\pi} \int_\Sigma d^2x \sqrt{h} \left[\mathbf{c} \left(C^{ijkl}\, \tilde{g}^{\perp}_{ik}\, \tilde{g}^{\perp}_{jl} - K^{i\,b}_{a} K^{ia}_{b} + \frac{1}{2} K^{i\,a}_{a} K^{i\,b}_{b} \right) - \mathbf{a}\, \mathcal{R} \right]$$

(Hung, RM & Smolkin)

7) derivation of holographic EE for spherical entangling surfaces

(Casini, Huerta & RM, RM & Sinha)

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:

$$S(A) = \underset{\partial V = \Sigma}{\text{ext}} \frac{A_V}{4G_N}$$

~~conjecture~~

Extensive consistency tests: → **new proof!!!**

(Lewkowycz & Maldacena)

• generalization of Euclidean path integral calc's for S_{BH}, extended to "periodic" bulk solutions without Killing vector

• for AdS/CFT, translates replica trick for boundary CFT to bulk

$$\Delta\tau = 2\pi \to 2\pi n \longrightarrow \log Z(n) = \log \mathrm{Tr}\left[\rho^n\right] = -I_{grav}(n)$$

$$\longrightarrow S = -n\partial_n \left[\log Z(n) - n \log Z(1)\right]\Big|_{n=1}$$

• at $n \sim 1$, linearized gravity eom demand: $K^\alpha = h^{ij}\, K^\alpha_{ij} = 0$

→ τ shrinks to zero on an extremal surface in bulk

• evaluating Einstein action yields $S = A/4G_N$ for extremal surface

Topics currently trending in **Holographic S_{EE}**:

(Ryu & Takayanagi `06 → 111 cites in past year of total of 317)

- **thermodynamic properties of S_{EE} for excited states**
thermodynamic properties of S_{EE} for excited states

  (Bhattacharya, Nozaki, Takayanagi & Ugajin; . . .)
(Bhattacharya, Nozaki, Takayanagi & Ugajin)

- “entanglement tsunami” probe of holo-quantum quenches

  (Liu & Suh)

- probe of large-N phase transitions at finite volume (Johnson)

- phase transitions in holographic Renyi entropy

  (Belin, Maloney & Matsuura)

- holographic S_{EE} in higher spin gravity

  (Ammon, Castro & Iqbal; de Boer & Jottar)

- holographic S_{EE} beyond classical gravity

  (Barrella, Dong, Hartnoll & Martin)

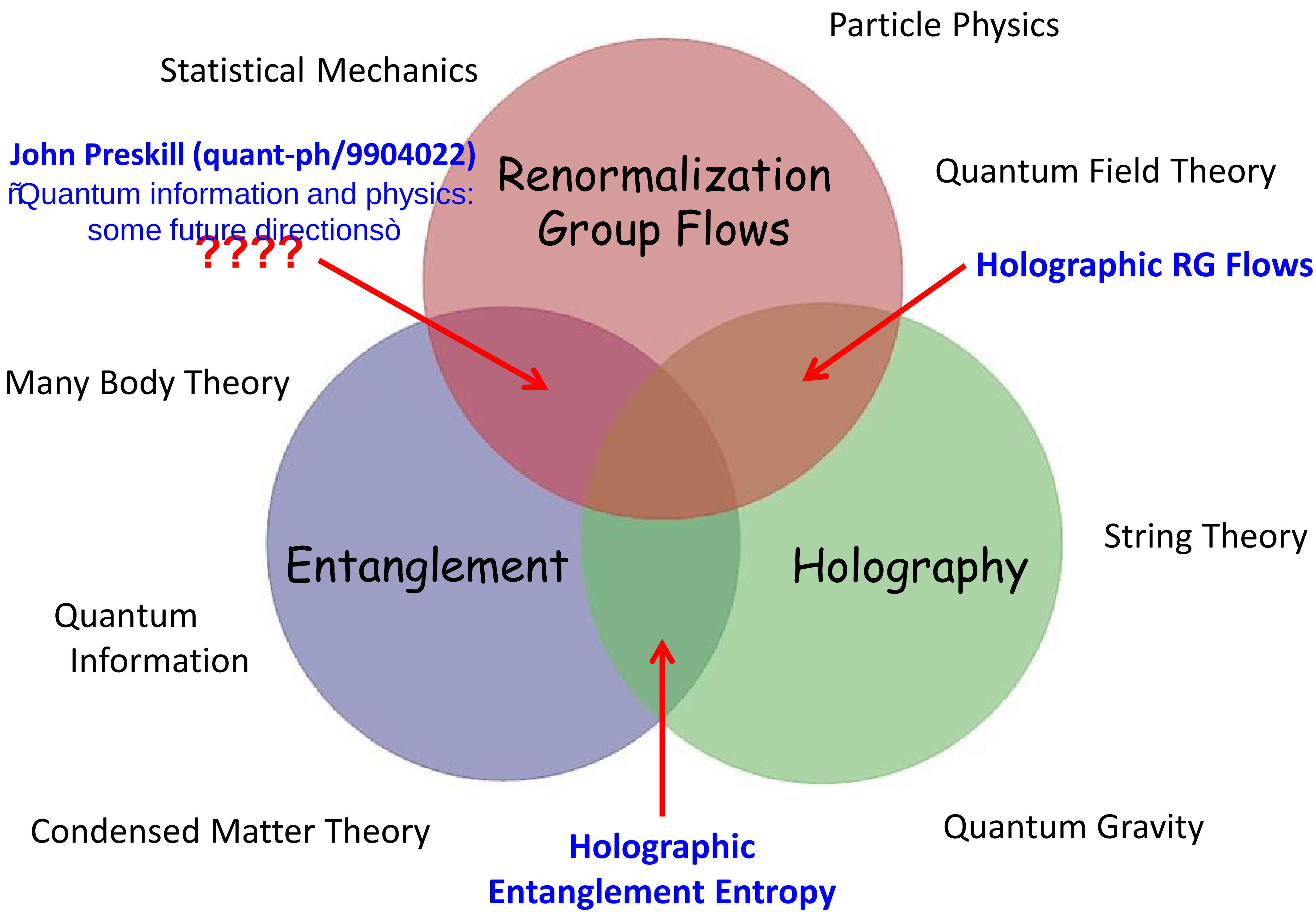
- probing causal structure in the bulk

  (Hubeny, Maxfield, Rangamani & Tonni)

- holographic Renyi entropy for disjoint intervals (Faulkner; Hartman)

. . . . . .

New Dialogues in Theoretical Physics:



Zamolodchikov c-theorem (1986):

Årenormalization-group (RG) flows can seen as one-parameter motion

$$\frac{d}{dt} \equiv -\beta^i(g)\,\frac{\partial}{\partial g^i}$$

in the space of (renormalized) coupling constants $\{g^i,\ i = 1, 2, 3, \cdots\}$ with beta-functions as ñvelocitiesò

Åfor unitary, renormalizable QFTôs in two dimensions, there exists a positive-definite real function of the coupling constants $c(g)$:

1. monotonically decreasing along flows: $\frac{d}{dt} c(g) \leq 0$

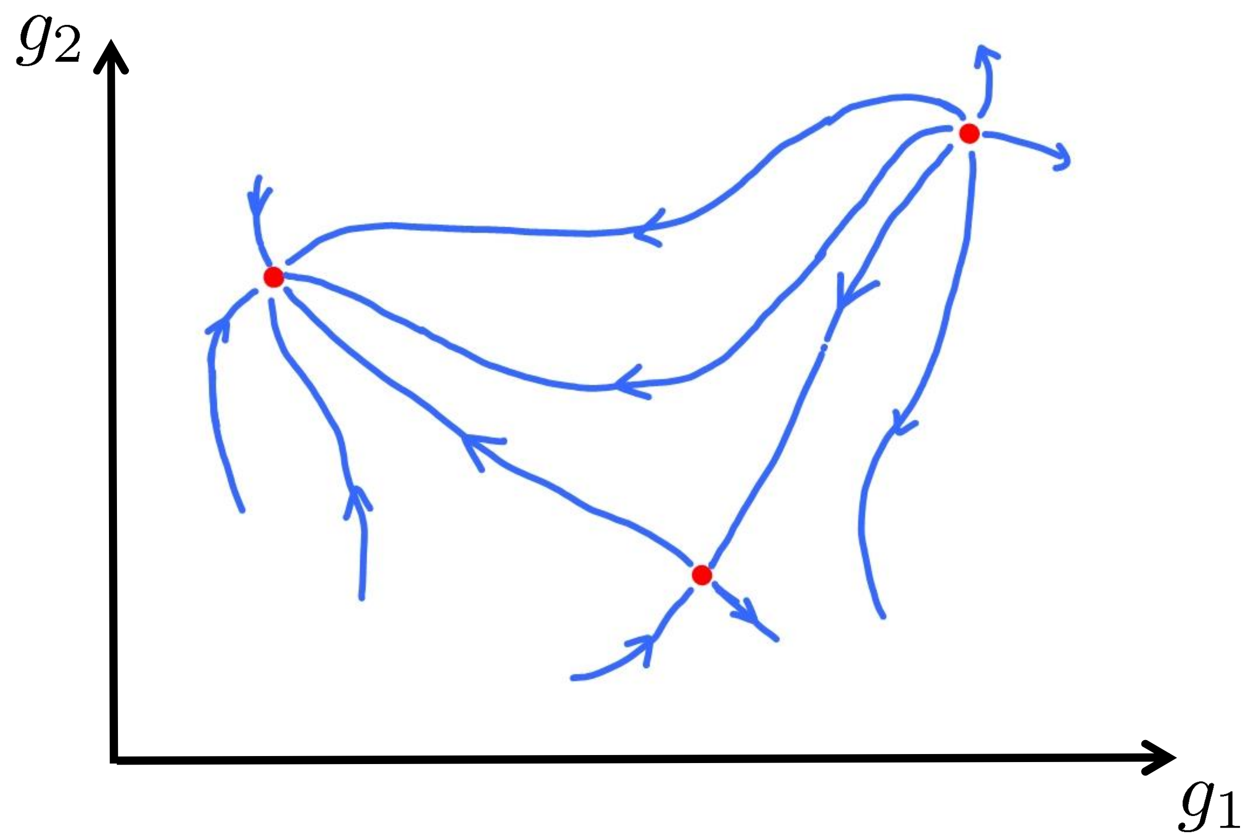
2. ñstationaryò at fixed points $g^i = (g^*)^i$:

$$\beta^i(g^*) = 0 \longleftrightarrow \frac{\partial}{\partial g^i} c(g) = 0$$

3. at fixed points, it equals central charge of corresponding CFT

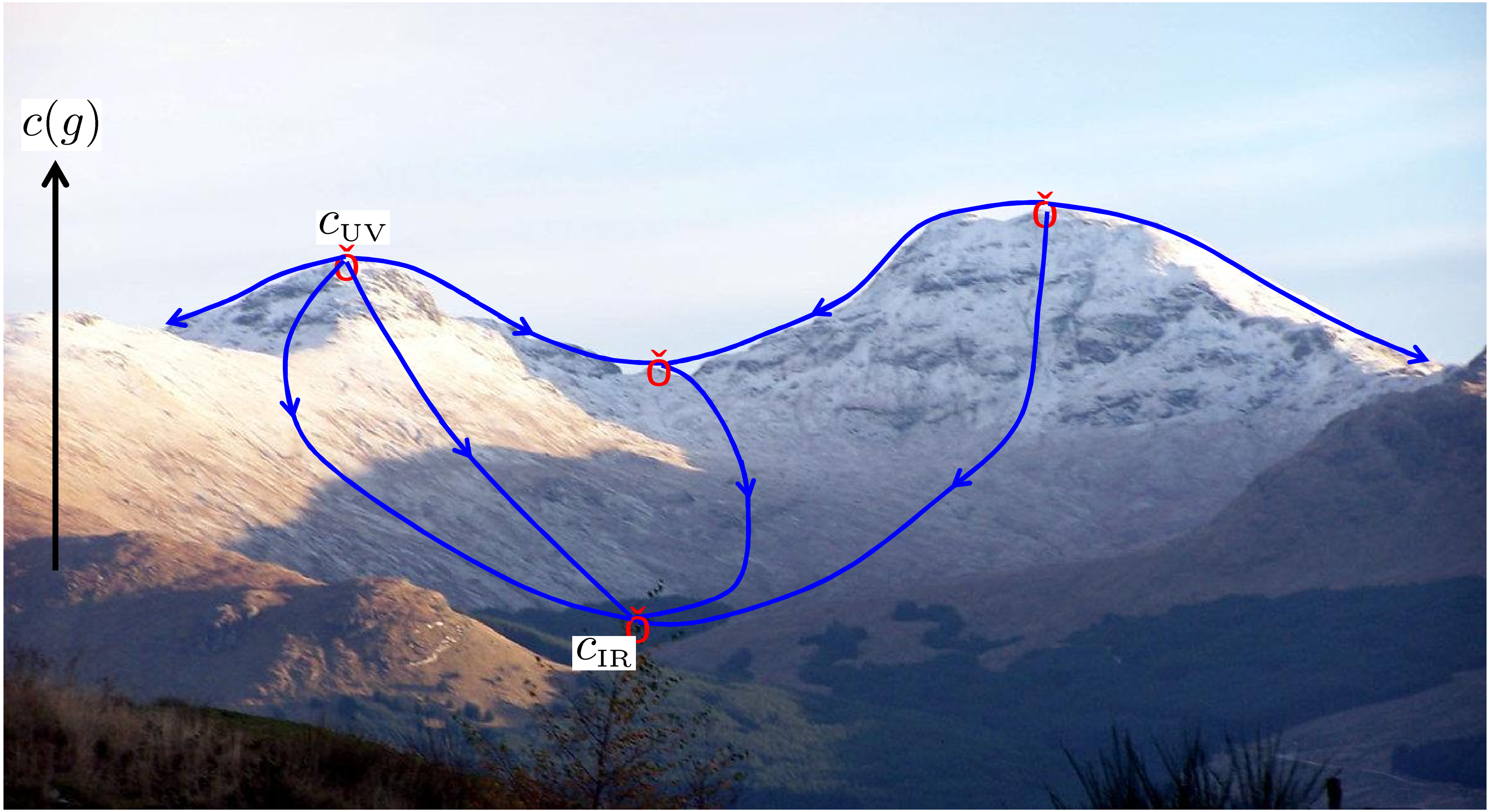
$$c(g^*) = c$$

with Zamolodchikov's framework:



BECOMES

with Zamolodchikov's framework:



Consequence for any RG flow in d=2: $c_{\mathrm{UV}} > c_{\mathrm{IR}}$

C-theorems in higher even dimensions??

d=2: $$\langle T_\mu{}^\mu \rangle = -\frac{c}{12} R$$

d=4: $$\langle T_\mu{}^\mu \rangle = \frac{c}{16\pi^2} I_4 - \frac{a}{16\pi^2} E_4 - \frac{a'}{16\pi^2} \nabla^2 R$$

$I_4 = C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$ and $E_4 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2$

Åin 4 dimensions, have three central charges: $c,\ a,\ a'$

Ådo any of these obey a similar ñc-theoremò under RG flows?

✗ a'-theorem: a' is scheme dependent (not globally defined)

✗ c-theorem: there are numerous counter-examples

Cardyôs conjecture (1988):

a-theorem: for any RG flow in d=4, $a_{\rm UV} > a_{\rm IR}$

Ånumerous nontrivial examples, eg, perturbative fixed points (Jack & Osborn), SUSY gauge theories (Anselmi et al; Intriligator & Wecht)

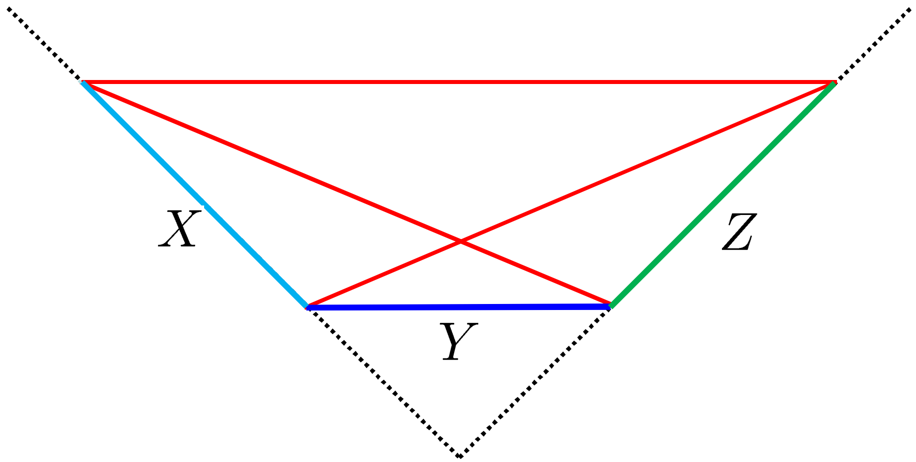
ÅJP: perhaps QI can provide insight into c-theorems for **odd** dimôs

(Casini & Huerta '04)

Entanglement proof of c-theorem:

Åc-theorem for d=2 RG flows can be established using unitarity, Lorentz invariance and **strong subaddivity inequality**:

$$S(X \cup Y \cup Z) - S(X \cup Y) - S(Y \cup Z) + S(Y) \leq 0$$



Ådefine: $C(\ell) = 3\,\ell\;\partial_\ell S(\ell)$

$$\longrightarrow \quad \partial_\ell C(\ell) \leq 0$$

Åfor d=2 CFT: $S(\ell) = \frac{c}{3}\log(\ell/\delta) + a_0$

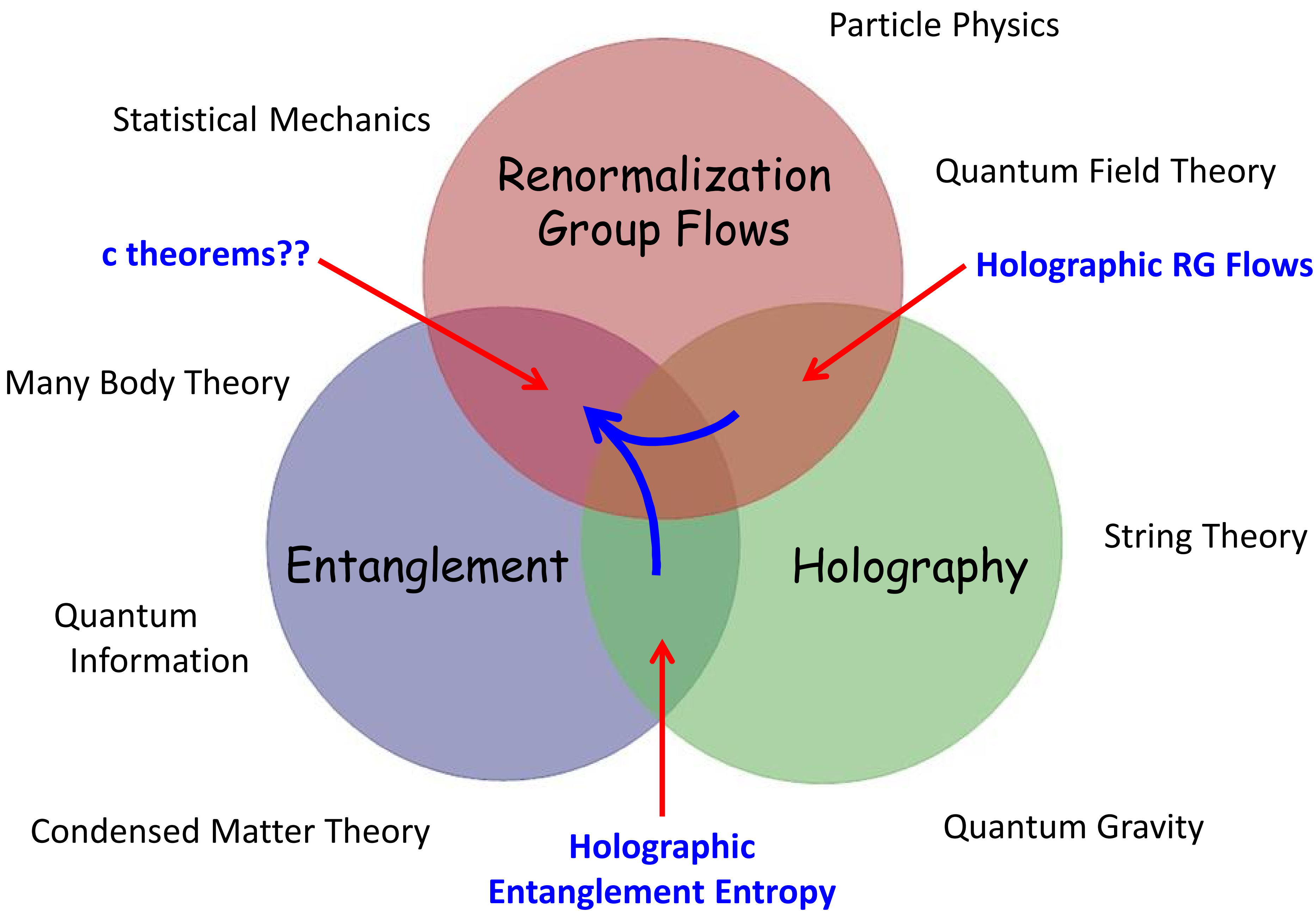
(Holzhey, Larsen & Wilczek)

(Calabrese & Cardy)

$$\longrightarrow \quad C_{\rm CFT}(\ell) = c$$

Åhence it follows that: $\boxed{c_{\rm UV} > c_{\rm IR}}$

New Dialogues in Theoretical Physics:



(Girardello, Petrini, Porrati and Zaffaroni, hep-th/9810126)
(Freedman, Gubser, Pilch & Warner, hep-th/9904017)

Holographic RG flows:

$$I = \frac{1}{2\ell_P^3}\int d^5x\,\sqrt{-g}\left[R - \frac{1}{2}(\partial\phi)^2 - V(\phi)\right]$$

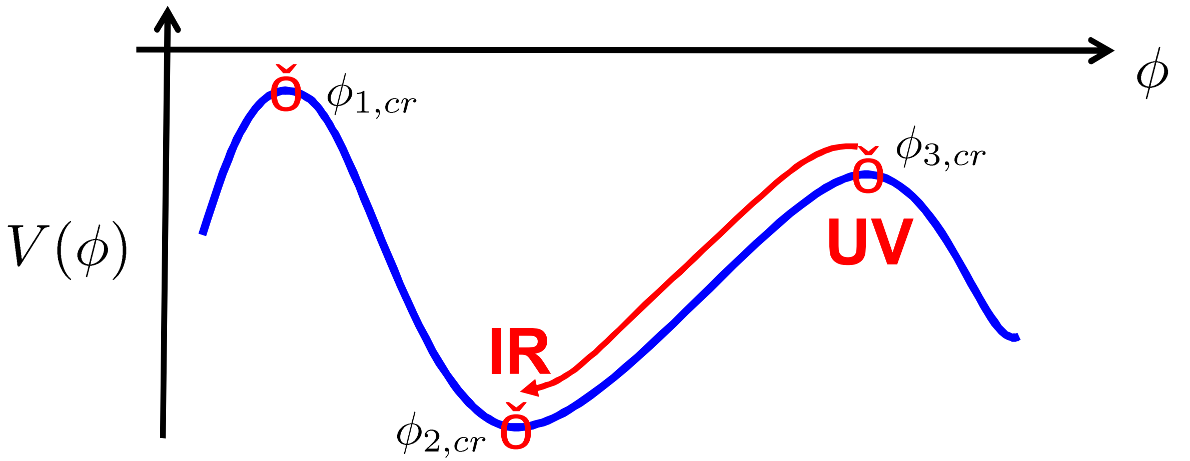
Åimagine potential has stationary points giving negative ş

$$\longrightarrow \quad V(\phi_{i,cr}) = -\frac{12}{L^2}\alpha_i^2$$

Åconsider metric: $ds^2 = e^{2A(r)}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + dr^2$

Åat stationary points, AdS_5 vacuum: $A(r) = r/\tilde{L}$ with $\tilde{L} = L/\alpha_i$

ÅHRG flow: solution starts at one stationary point at large radius and ends at another at small radius ï connects CFT_{UV} to CFT_{IR}



(Girardello, Petrini, Porrati and Zaffaroni, hep-th/9810126)

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Holographic RG flows:

Åfor general flow solutions, define: $a(r) \equiv \frac{\pi^2}{\ell_P^3 A'(r)^3}$

$$a'(r) = -\frac{3\pi^2}{\ell_P^3 A'(r)^4} A''(r) = -\frac{\pi^2}{\ell_P^3 A'(r)^4}\left(T^t{}_t - T^r{}_r\right) \geq 0$$

Einstein equations ⤴ **null energy condition** ⤴

$(\, T_{\mu\nu}\, \ell^\mu\, \ell^\nu \geq 0\,)$

Åat stationary points, $a(r) \to a^* = \pi^2\, \tilde{L}^3/\ell_P^3$ and hence

$$\boxed{a^*_{UV} \geq a^*_{IR}}$$

Åusing holographic trace anomaly: $a^* = a$

(e.g., Henningson & Skenderis)

⟶ supports Cardyôs conjecture

Åfor Einstein gravity, central charges equal$(a = c)$: $c_{UV} \geq c_{IR}$

(Freedman, Gubser, Pilch & Warner, hep-th/9904017)

Holographic RG flows:

$$I = \frac{1}{2\ell_P^{d-1}} \int d^{d+1}x \sqrt{-g} \left[R - \frac{1}{2}(\partial\phi)^2 - V(\phi) \right]$$

Åsame story is readily extended to (d+1) dimensions

Ådefining: $a_d(r) \equiv \dfrac{\pi^{d/2}}{\Gamma\left(d/2\right)\left(\ell_P A'(r)\right)^{d-1}}$

$$a'(r) \;=\; -\frac{(d-1)\pi^{d/2}}{\Gamma\left(d/2\right)\ell_P^{d-1} A'(r)^d} A''(r) = -\frac{\pi^{d/2}}{\Gamma\left(d/2\right)\ell_P^{d-1} A'(r)^d}\left(T^t{}_t - T^r{}_r\right) \geq 0$$

Einstein equations **null energy condition**

Åat stationary points, $a(r) \to a^* = \pi^{d/2}/\Gamma(d/2)\,(\tilde{L}/\ell_P)^{d-1}$ and so

$$\boxed{a^*_{UV} \geq a^*_{IR}}$$

Åusing holographic trace anomaly: $a^* \propto$ central charges

for even d! what about odd d? (e.g., Henningson & Skenderis)

Improved Holographic RG Flows:

Åadd higher curvature interactions to bulk gravity action

$\longrightarrow$ provides holographic field theories with, eg, $a \neq c$
so that we can clearly distinguish evidence of a-theorem

(Nojiri & Odintsov; Blau, Narain & Gava)

Åconstruct ñtoy modelsòwith fixed set of higher curvature terms
(where we can maintain control of calculations)

What about the swampland?

Åconstrain gravitational couplings with consistency tests
(positive fluxes; causality; unitarity) and use best judgement

Åultimately one needs to fully develop string theory for
interesting holographic backgrounds!

Å*“if certain general characteristics are true for all CFT’s, then
holographic CFT’s will exhibit the same features”*

Toy model:

(RM & Robinsion; RM, Paulos & Sinha)

$$I = \frac{1}{2\ell_P^3}\int d^5x\,\sqrt{-g}\left[\frac{12}{L^2}\alpha^2 + R + L^2\frac{\lambda}{2}\chi_4 + L^4\frac{7\mu}{4}\mathcal{Z}_5\right]$$

with $\chi_4 = R^{abcd}R_{abcd} - 4R_{ab}R^{ab} + R^2$

$$\mathcal{Z}_5 = R_a{}^c{}_b{}^d R_d{}^e{}_c{}^f R_e{}^a{}_f{}^b + \frac{1}{56}\left(21R_{abcd}R^{abcd}R - 72R_{abcd}R^{abc}{}_eR^{de} + 120R_{abcd}R^{ac}R^{bd} + 144R_a{}^bR_b{}^cR_c{}^a - 132R_a{}^bR_b{}^aR + 15R^3\right)$$

Å three dimensionless couplings: $L/\ell_P\,,\ \lambda\,,\ \mu$

Å again, gravitational eom and null energy conditon yield:

(RM & Sinha)

$$\boxed{a_{UV} \geq a_{IR}}$$

where $a = \frac{\pi^2}{f_\infty^{3/2}}\frac{L^3}{\ell_P^3}\left(1 - 6\lambda f_\infty + 9\mu f_\infty^2\right)$ ← central charge a of boundary CFT

with $\alpha^2 - f_\infty + \lambda f_\infty^2 + \mu f_\infty^3 = 0$

Å toy model supports for Cardyô s conjecture in four dimensions

(RM & Sinha)

Åfor holographic RG flows with general d, find:

$$(a_d^*)_{UV} \geq (a_d^*)_{IR}$$

where $a_d^* = \frac{\pi^{d/2} L^{d-1}}{\Gamma(d/2) f_\infty^{\frac{d-1}{2}} \ell_P^{d-1}} \left(1 - \frac{2(d-1)}{d-3}\lambda f_\infty - \frac{3(d-1)}{d-5}\mu f_\infty^2\right)$

with $\alpha^2 - f_\infty + \lambda f_\infty^2 + \mu f_\infty^3 = 0$

Åtrace anomaly for CFTȇs with even d: (Deser & Schwimmer)

$$\langle T_\mu{}^\mu \rangle = \sum B_i (\text{Weyl invariant})_i - 2(-)^{d/2} A \, (\text{Euler density})_d$$

Åverify that we have precisely reproduced central charge

$$a_d^* = A$$

(Henningson & Skenderis; Nojiri & Odintsov; Blau, Narain & Gava; Imbimbo, Schwimmer, Theisen & Yankielowicz)

⟶ **agrees with Cardy's conjecture**

What about odd *d*??

(Casini, Huerta & RM; RM & Sinha)

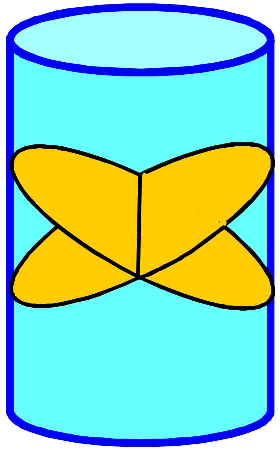
Holographic Entanglement Entropy:

ÅS_{EE} for CFT in d-dim. flat space and choose S^{d-2} with radius R

Åconformal mapping relate to thermal entropy on $\mathcal{H} = R \times H^{d-1}$ with $\mathcal{R} \sim 1/R^2$ and T=1/2´ R

Åholographic dictionary: thermal bath in CFT = black hole in AdS

$$S_{EE} = S_{thermal} = S_{horizon}$$



Ådesired ñblack holeòis a hyperbolic foliation of AdS

Åbulk coordinate transformation implements desired conformal transformation on boundary

Åapply Waldôs formula (for any gravity theory) for horizon entropy:

universal contributions:

$$S = \cdots + (-)^{\frac{d}{2}-1} 4\, a_d^* \log(2R/\delta) + \cdots$$ for even d

$$\cdots + (-)^{\frac{d-1}{2}} 2\pi\, a_d^* + \cdots$$ for odd d

(RM & Sinha)

C-theorem conjecture:

Åidentify central charge with universal contribution in entanglement entropy of ground state of CFT across sphere S^{d-2} of radius R:

$$S_{univ} = \begin{cases} (-)^{\frac{d}{2}-1}\, 4\, a_d^* \log(2R/\delta) & \text{for even } d \\ (-)^{\frac{d-1}{2}}\; 2\pi\, a_d^* & \text{for odd } d \end{cases}$$

(any gravitational action)

Åfor RG flows connecting two fixed points

$$(a_d^*)_{UV} \geq (a_d^*)_{IR}$$

(ñunitaryòmodels)

⟶ unified framework to consider c-theorem for **odd** or even *d*

⟶ connect to Cardyõs conjecture: $a_d^* = A$ for any CFT in even *d*

F-theorem:

(Jafferis, Klebanov, Pufu & Safdi)

Åexamine partition function for broad classes of 3-dimensional quantum field theories on three-sphere (SUSY gauge theories, perturbed CFTôs & O(N) models)

Åin all examples, F= ï log Z(S^3)>0 and decreases along RG flows

$\longrightarrow$ **conjecture:** $F_{UV} > F_{IR}$

Åalso naturally generalizes to higher odd *d*

Åcoincides with entropic c-theorem (Casini, Huerta & RM)

Åfocusing on renormalized or universal contributions, eg,

$$F_3 = -\log Z|_{finite} = -S_{univ} = 2\pi\, a_3^*\,.$$

Ågeneralizes to general odd d:

$$F_d = -\log Z|_{finite} = -S_{univ} = (-)^{\frac{d+1}{2}}2\pi\, a_d^*\,.$$

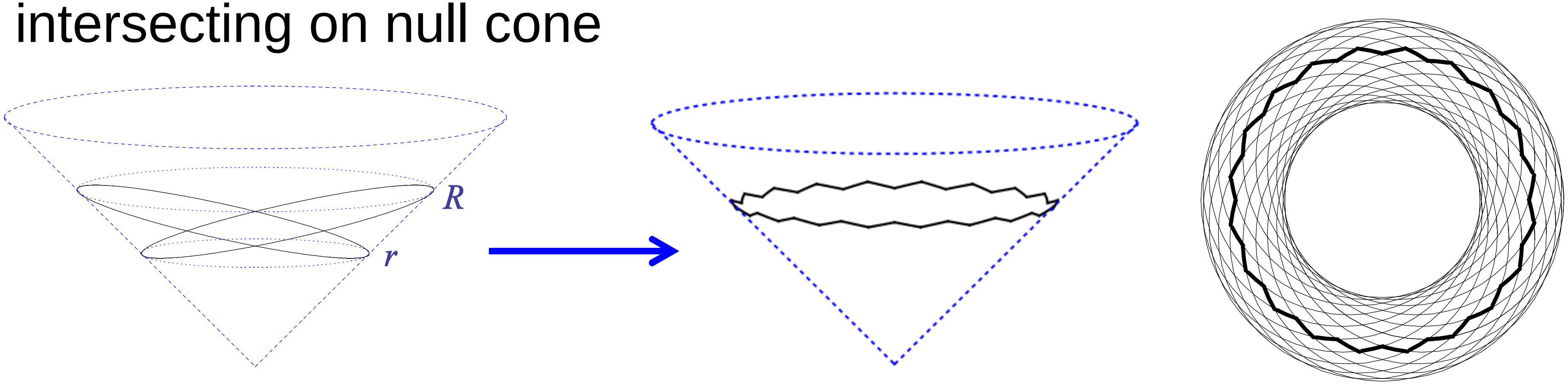
(Casini & Huerta ‘12)

Entanglement proof of F-theorem:

ÅF-theorem for d=3 RG flows established using unitarity, Lorentz invariance and **strong subaddivity**

$$\sum_i S(X_i) \geq S(\cup_i X_i) + S(\cup_{\{ij\}}(X_i \cap X_j)) + S(\cup_{\{ijk\}}(X_i \cap X_j \cap X_k)) + \ldots + S(\cap_i X_i)$$

Ågeometry more complex than d=2: consider many circles intersecting on null cone



(no corner contribution from intersection in null plane)

Ådefine: $C(R) = RS'(R) - S(R)$ $\longrightarrow$ $\partial_R C(R) \leq 0$

Åfor d=3 CFT: $S(R) = c_0\, R - 2\pi a_3$ $\longrightarrow$ $C_{\rm CFT}(R) = 2\pi a_3$

Åhence it follows that: $[a_3]_{\rm UV} > [a_3]_{\rm IR}$

(Liu & Mezei)

“Renormalized” Entanglement Entropy:

ÅS_{EE} is UV divergent, so must take care in defining universal term

Ådivergences determined by local geometry of entangling surface with covariant regulator,

$$S \;=\; c_0(\mu_i\delta)\,\frac{R^{d-2}}{\delta^{d-2}} \;+\; c_2(\mu_i\delta)\,\frac{R^{d-4}}{\delta^{d-4}} \;+\; \cdots + (-)^{\frac{d-1}{2}}\; 2\pi\, a_d(\mu_i\delta) + O(\delta/R)$$

Åcan isolate finite term with appropriate manipulations, eg,

d=3: $\mathcal{S}_3(R) = RS'(R) - S(R)$ ← c-function of Casini & Huerta

d=4: $\mathcal{S}_4(R) = R^2 S''(R) - RS'(R)$

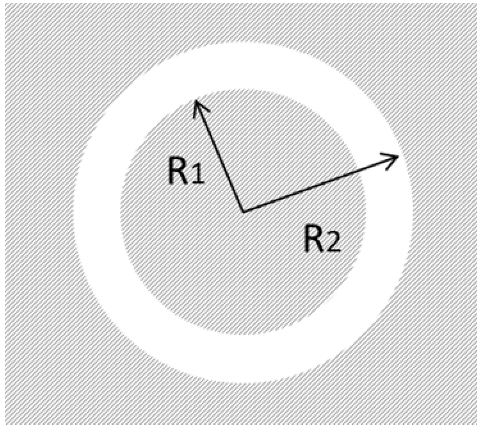
Åunfortunately, holographic experiments indicate $\mathcal{S}_d(R)$ are **not** good c-functions for *d*>3

(Casini, Huerta, RM & Yale)

"Renormalized" Entanglement Entropy 2:

ÅS_{EE} is UV divergent, so must take care in defining universal term

Åmutual information is intrinsically finite and so offers alternative approach to regulate S_{EE}



$$I(A,B) = S(A) + S(B) - S(A \cup B)$$

Åwith $R_{1,2} = R \pm \frac{\varepsilon}{2}$ and $R \gg \varepsilon \gg \delta$,

$$I(A,B) = 2\left(\frac{\tilde{a}}{\varepsilon} + b\right) R - 4\pi a_3 + O(\varepsilon)$$

Åchoice ensures that a_3 is not polluted by UV fixed point

Ånaturally extends to defining a_d in higher odd dimensions

Åfor d=3, entropic proof of F-theorem can be written in terms of mutual information

(Komargodski & Schwimmer; see also: Luty, Polchinski & Rattazzi)

a-theorem and Dilaton Effective Action

Åanalyze RG flow as ñbroken conformal symmetryò (Schwimmer & Theisen)

Åcouple theory to ñdilatonò(conformal compensator) and organize effective action in terms of $\hat{g}_{\mu\nu} = e^{-2\tau} g_{\mu\nu}$

diffeo X Weyl invariant: $g_{\mu\nu} \to e^{2\sigma} g_{\mu\nu} \quad \tau \to \tau + \sigma$

Åfollow effective dilaton action to IR fixed point, eg,

$$S_{anomaly} = -\delta a \int d^4x \sqrt{-g}\left(\tau E_4 + 4(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R)\partial_\mu\tau\partial_\nu\tau - 4(\partial\tau)^2\Box\,\tau + 2(\partial\tau)^4\right)$$

$\delta a = a_{UV} - a_{IR}$: ensures UV & IR anomalies match

Åwith $g \to \eta$, only contribution to 4pt amplitude with null dilatons:

$$S_{anomaly} = 2\,\delta a \int d^4x\,(\partial\tau)^4$$

Ådispersion relation plus optical theorem demand: $\delta a > 0$

(Solodukhin)

a-theorem, Dilaton and Entanglement Entropy

Åfind anomaly contribution for S_{EE}

$$S_{anomaly} = -\delta a \int d^4x \sqrt{-g} \left(\tau E_4 + 4(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R)\partial_\mu \tau \partial_\nu \tau - 4(\partial\tau)^2 \Box\, \tau + 2(\partial\tau)^4 \right)$$

$$[S_{EE}]_{anom} = \frac{\pi}{8}\, \delta a \int_\Sigma d^2\sigma \sqrt{\gamma}\, [\tau \mathcal{R}_\Sigma(\gamma) + (\partial_\Sigma \tau)^2]$$

Åfor conformally flat background and flat entangling surface,

$$\longrightarrow \quad [S_{EE}]_{anom} = \frac{\pi}{8}\, \delta a \int_\Sigma d^2\sigma \sqrt{\gamma}\, (\partial_\Sigma \tau)^2$$

Åcan express coefficient in terms of spectral density for $\langle T(x) T(y) \rangle$

$$\delta a = \frac{1}{90\pi^2} \int_0^\infty \frac{d\mu}{\mu^2} C^{(0)}(\mu) \;>\; 0$$

Åanalogous to effective-dilaton-action analysis for d=2

(Komargodski)

Questions:

- how much of Zamalodchikov's structure for d=2 RG flows extends higher dimensions?

→ d=3 entropic c-function not always stationary at fixed points

(Klebanov, Nishioka, Pufu & Safdi)

- can c-theorems be proved for higher dimensions? eg, d=5 or 6

→ dilaton-effective-action would require subtle refinement for d=6

(Elvang, Freedman, Hung, Kiermaier, RM & Theisen; Elvang & Olson)

- does scale invariance imply conformal invariance beyond d=2?

(Nakayama)

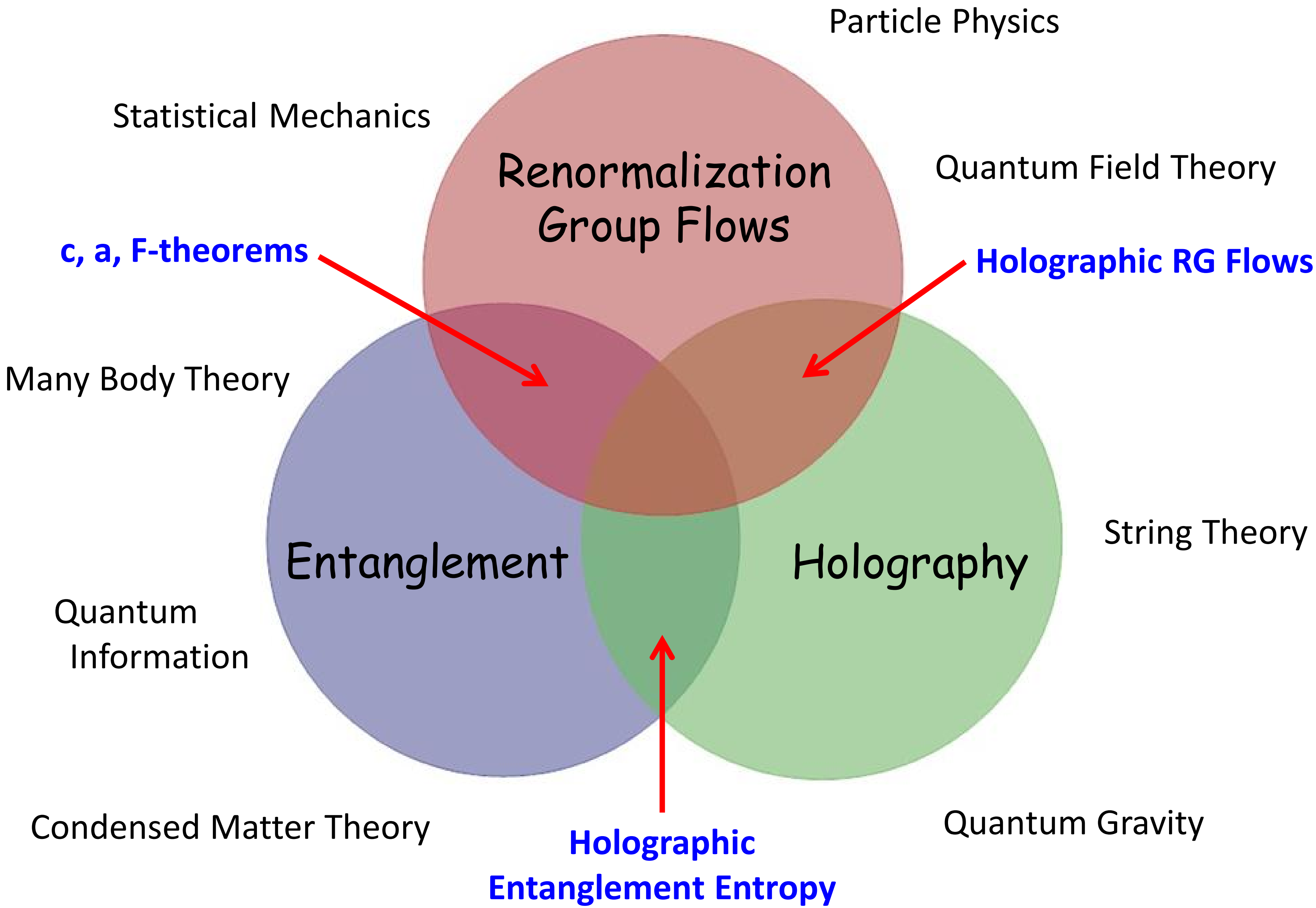
→ at least, perturbatively in d=4 (Luty, Polchinski & Rattazzi)

- further lessons for RG flows and entanglement from holography?

→ translation of "null energy condition" to boundary theory?

- what can entanglement entropy/quantum information really say about renormalization group and holography?

New Dialogues in Theoretical Physics:



(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:

$$S(A) \;=\; \underset{\partial V = \Sigma}{\mathrm{ext}} \;\frac{A_V}{4G_N}$$

<u>conjecture</u>

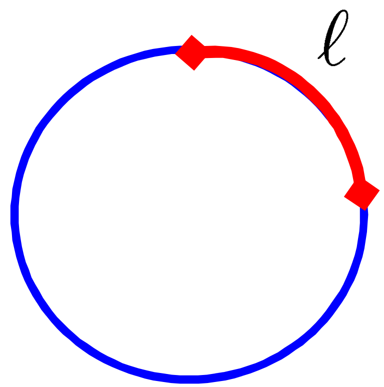
Extensive consistency tests:

1) leading contribution yields ñarea lawò $\quad S \simeq \dfrac{L^{d-1}}{G_N}\,\dfrac{\mathcal{A}_\Sigma}{\delta^{d-2}} + \cdots$

2) recover known results of Calabrese & Cardy
for d=2 CFT

$$S \;=\; \frac{c}{3}\log\left(\frac{C}{\pi\,\delta}\sin\frac{\pi\ell}{C}\right)$$

(also result for thermal ensemble)



ℓ

$C = \text{circumference}$

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:

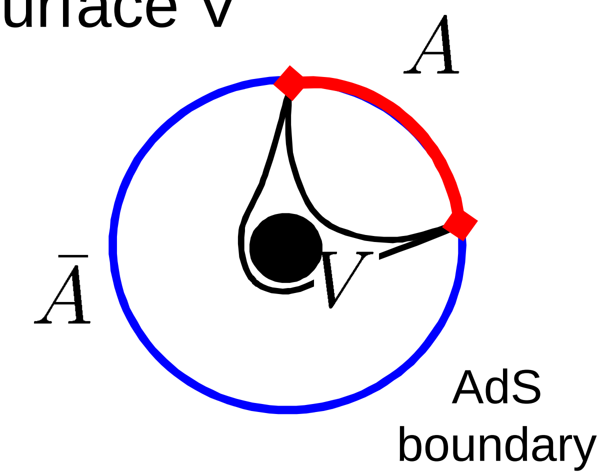
$$S(A) \;=\; \underset{\partial V\,=\,\Sigma}{\mathrm{ext}}\; \frac{A_V}{4G_N}$$

<u>conjecture</u>

Extensive consistency tests:

3) $S(A) = S(\bar{A})$ in a pure state

→ A and $\bar{A}$ both yield same bulk surface V



cf: thermal ensemble Í pure state

horizon in bulk → $S(A) \neq S(\bar{A})$

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:

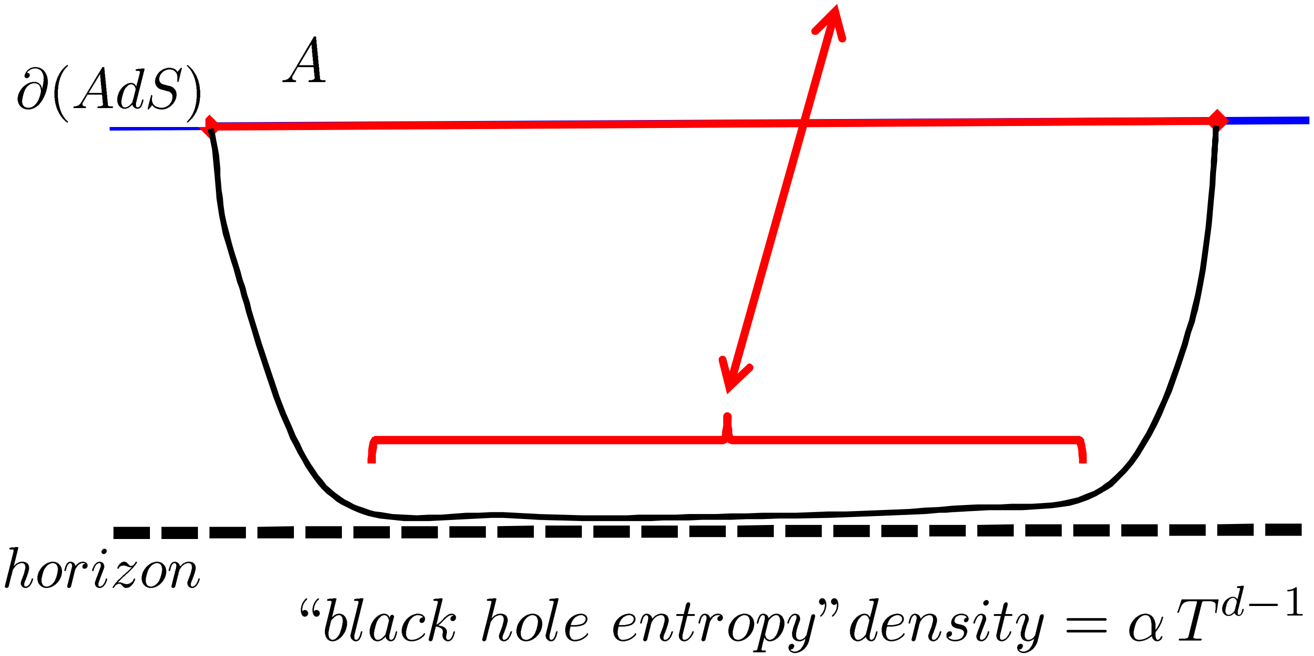
$$S(A) = \underset{\partial V = \Sigma}{\mathrm{ext}} \frac{A_V}{4G_N}$$

conjecture

Extensive consistency tests:

4) for thermal bath: $S(A) \supset S_{therm} = \alpha\, T^{d-1} \times volume$

$R \gg 1/T$



(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:

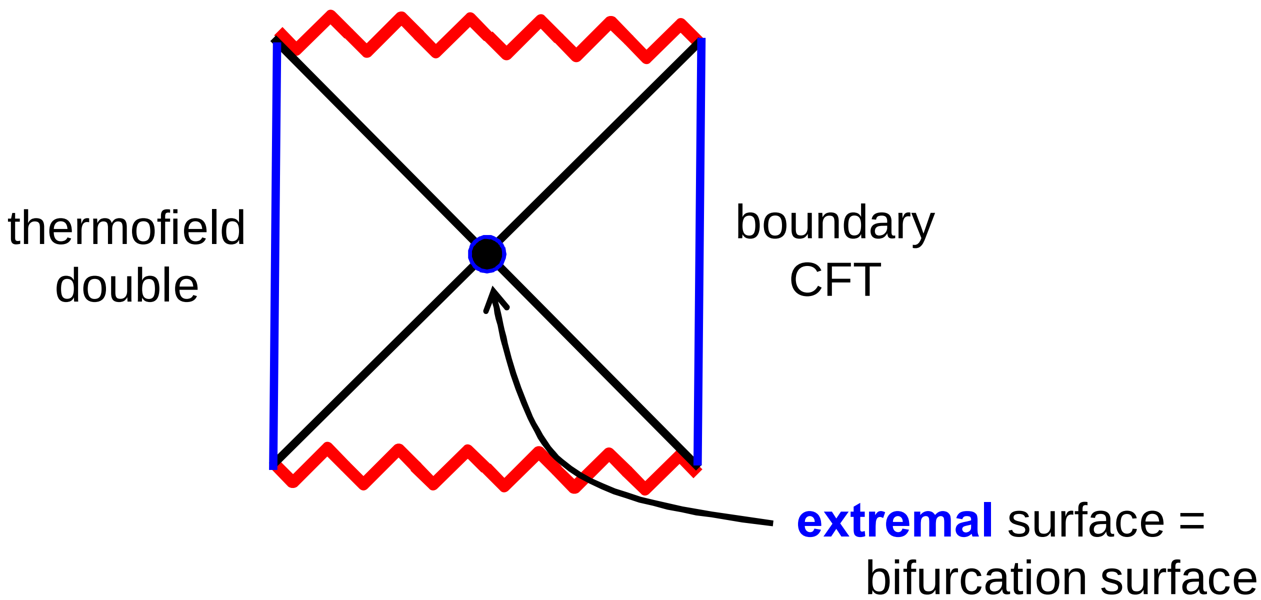
$$S(A) = \underset{\partial V = \Sigma}{\mathrm{ext}} \frac{A_V}{4G_N}$$

<u>conjecture</u>

Extensive consistency tests:

4) Entropy of eternal black hole =
entanglement entropy of boundary CFT & thermofield double

(Maldacena; Headrick)



(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:

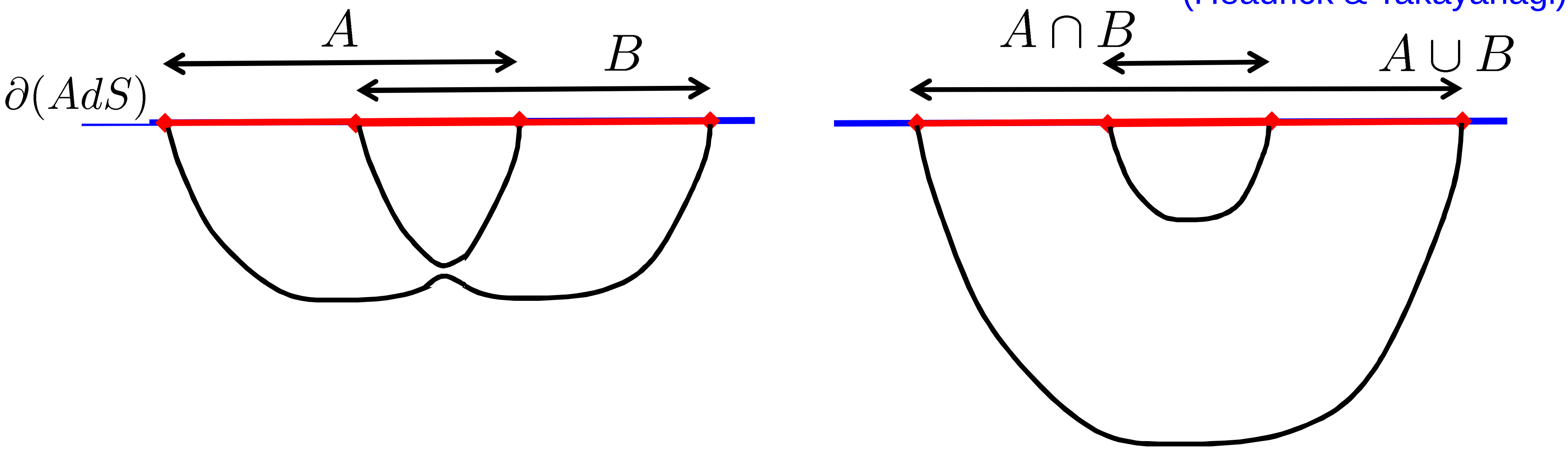
$$S(A) = \underset{\partial V = \Sigma}{\text{ext}} \frac{A_V}{4G_N}$$

conjecture

Extensive consistency tests:

5) strong sub-additivity: $S(A \cup B) + S(A \cap B) \leq S(A) + S(B)$

(Headrick & Takayanagi)



[further monogamy relations: Hayden, Headrick & Maloney]

Holographic Entanglement Entropy beyond Einstein:

$$S(A) = \underset{\partial V = \Sigma}{\text{ext}} \frac{A_V}{4G_N}$$

Recall consistency tests:

4) Entropy of eternal black hole = entanglement entropy of boundary CFT & thermofield double

5) strong sub-additivity: $S(A \cup B) + S(A \cap B) \leq S(A) + S(B)$

(Headrick & Takayanagi)

➡ for more general holographic framework, expect

$$S(A) = \underset{\partial V = \Sigma}{\text{ext}} \left[S_{horizon} \right]$$

includes α' corrections

g_s corrections

for more general holographic framework, expect

$$S(A) = \underset{\partial V = \Sigma}{\mathrm{ext}} \left[S_{horizon} \right] + \cdots$$

(deBoer, Kulaxizi & Parnachev)

(Hung, Myers & Smolkin)

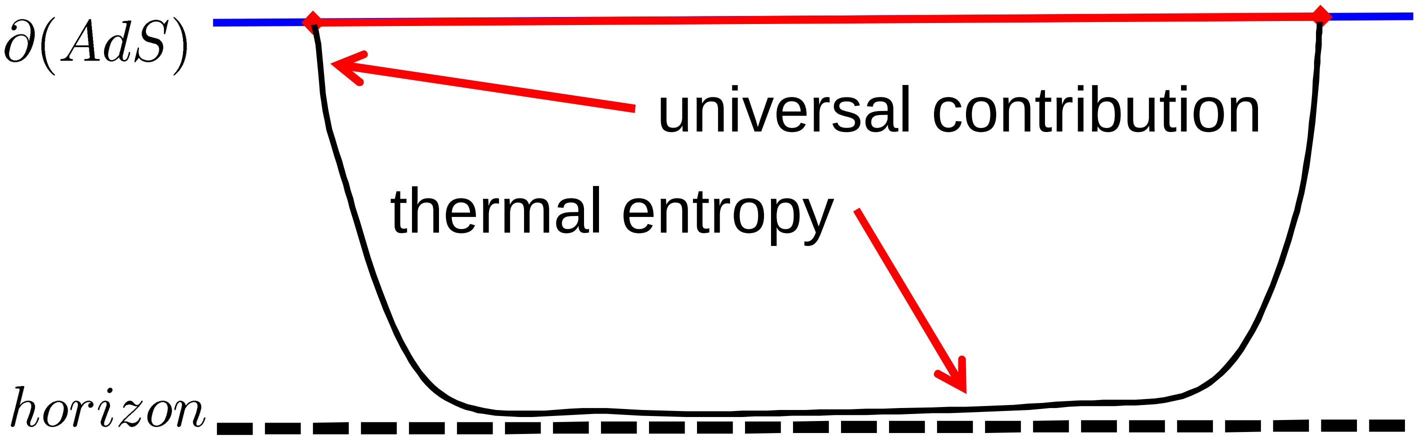
some progress with classical higher curvature gravity:

Ånote $S_{horizon}$ is **not** unique! and S_{Wald} is **wrong** choice!

Åcorrect choice understood for ñLovelock theoriesò

Åtest with universal term for d=4 CFT: (Solodukhin)

$$S_{uni} = \log(R/\delta)\, \frac{1}{2\pi} \int_\Sigma d^2x \sqrt{h} \left[\mathbf{c} \left(C^{ijkl}\, \tilde{g}^{\perp}_{ik}\, \tilde{g}^{\perp}_{jl} - K_a^{\ i\,b} K_b^{ia} + \frac{1}{2} K_a^{\ i\,a} K_b^{\ i\,b} \right) - \mathbf{a}\, \mathcal{R} \right]$$



for more general holographic framework, expect

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Åseems consistent with Lewkowycz-Maldacena proof

(Bhattacharyya, Kaviraj & Sinha; Fursaev, Patrushev & Solodukhin)

(Chen & Zhang??)

Lessons from Holographic EE:

Åcompare two theories:

$$I_0 = \frac{1}{2\ell_p^3}\int d^5x\,\sqrt{-g}\left[\frac{12}{\tilde{L}^2}+R\right] \quad \Longrightarrow \quad S_{BH} = 2\pi\,\mathcal{A}/\ell_p^3$$

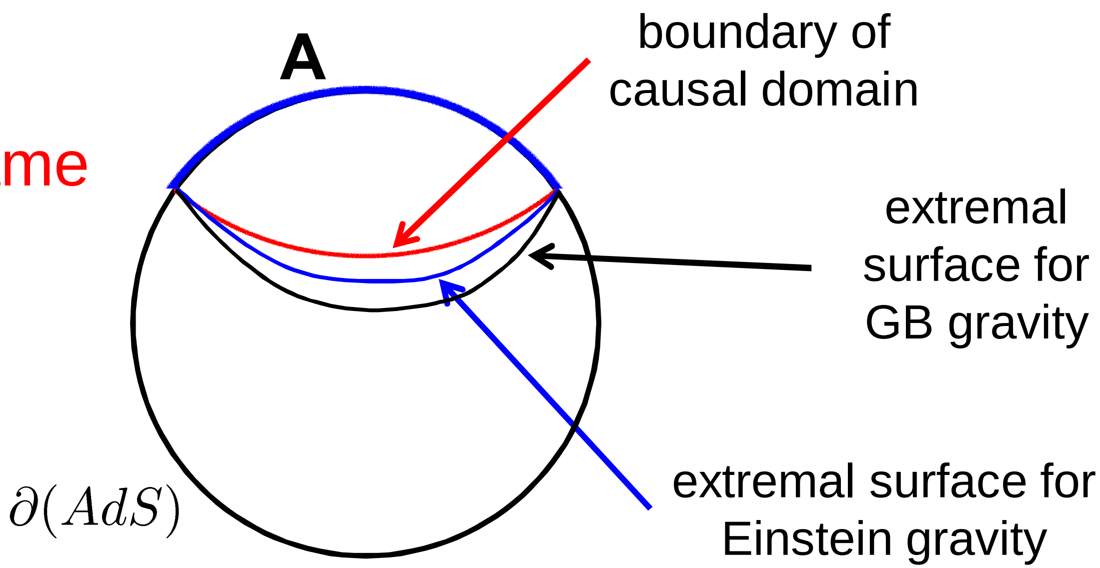
$$I_1 = \frac{1}{2\ell_p^3}\int d^5x\,\sqrt{-g}\left[\frac{12}{L^2}+R+L^2\frac{\lambda}{2}\,\left(R^{abcd}R_{abcd}-4R_{ab}R^{ab}+R^2\right)\right]$$

$$\Longrightarrow \quad S_{JM} = \frac{2\pi}{\ell_p^3}\int d^3x\sqrt{h}\,\left[1+\lambda\,L^2\,\mathcal{R}\right]$$

Åtune: $\tilde{L}^2 = \frac{L^2}{2}\left(1+\sqrt{1+4\lambda}\right)$

⟶ both theories have same AdS vacuum

Åclearly S_{EE} is not tied to causal structure or even geometry alone



F-theorem:

(Jafferis, Klebanov, Pufu & Safdi)

Åexamine partition function for broad classes of 3-dimensional quantum field theories (SUSY and non-SUSY) on three-sphere

Åin all examples, F= ï log Z >0 and decreases along RG flows

Åcoincides with our conjectured c-theorem! (Casini, Huerta & RM)

Åconsider S_{EE} of d-dimensional CFT for sphere $S^{\text{dï 2}}$ of radius R

Åconformal mapping: causal domain $\mathcal{D} \rightarrow$ (static patch of) dS_d

curvature ~ 1/R and thermal state: $\rho = \exp[-2\pi R\, H_\tau]/Z$

$$\longrightarrow \quad S_{EE} = S_{thermal}$$

Åstress-energy fixed by trace anomaly ï vanishes for odd d!

Åupon passing to Euclidean time with period $2\pi R$:

$$S_{EE} = \log Z|_{S^d}$$ for any odd d

F-theorem:

Åmust focus on renormalized or universal contributions, eg,

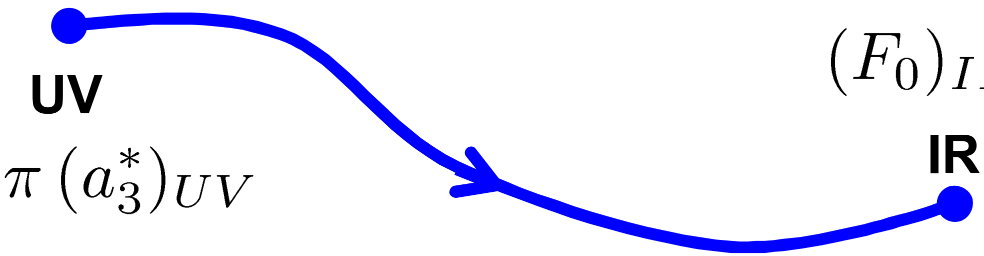
$$F_0 = -\,\log Z|_{finite} = -S_{univ} = 2\pi\, a_3^*\,.$$

Ågeneralizes to general odd d:

$$(-)^{\frac{d-1}{2}}\,\log Z|_{finite} = (-)^{\frac{d-1}{2}} S_{univ} = 2\pi\, a_d^*\,.$$

Åequivalence shown only for fixed points but good enough:

$$(F_0)_{IR} = 2\pi\,(a_3^*)_{IR}$$



$$(F_0)_{UV} = 2\pi\,(a_3^*)_{UV}$$

Åevidence for F-theorem (SUSY, perturbed CFTôs & O(N) models) supports present conjecture and our holographic analysis provides additional support for F-theorem