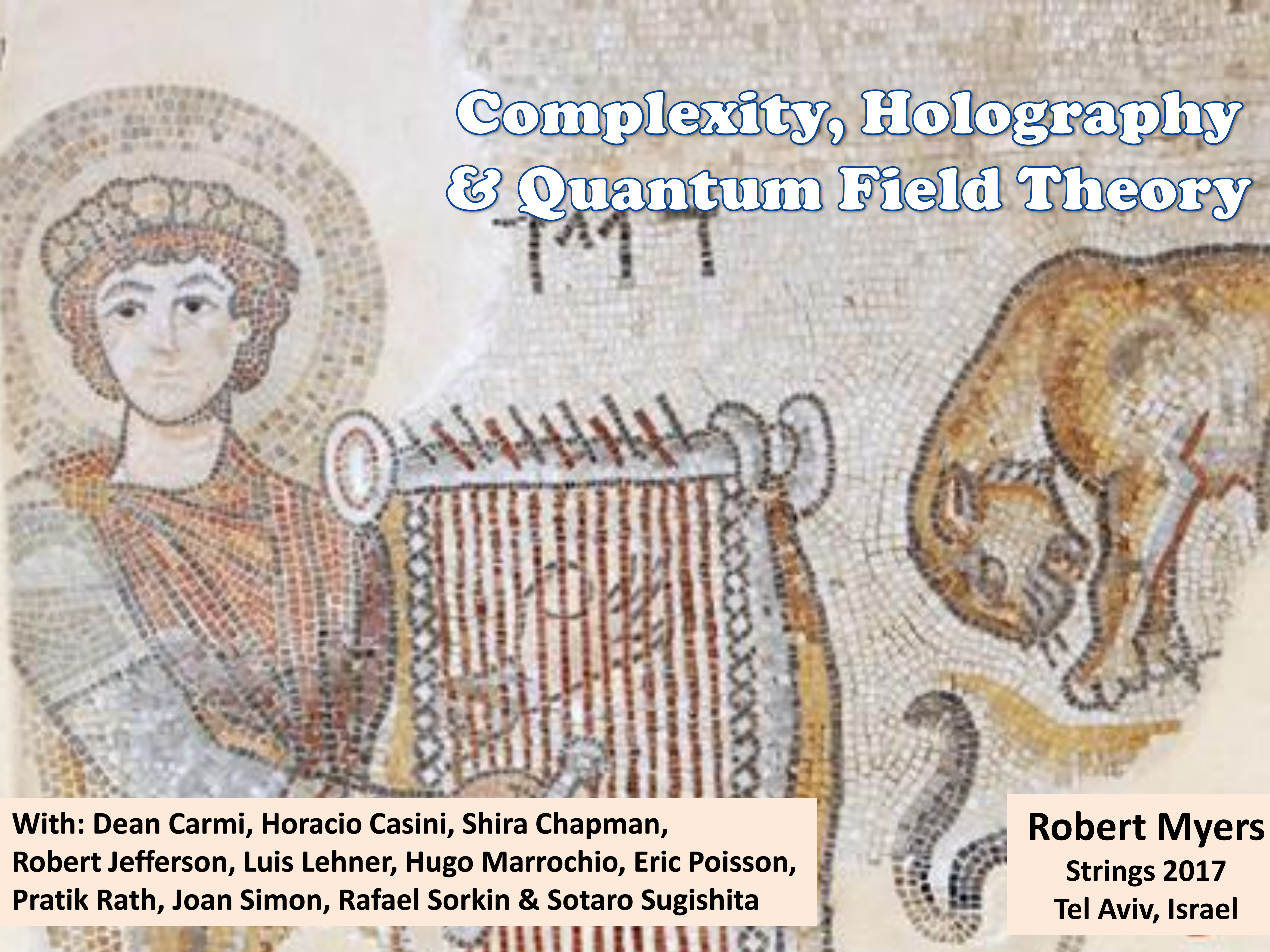
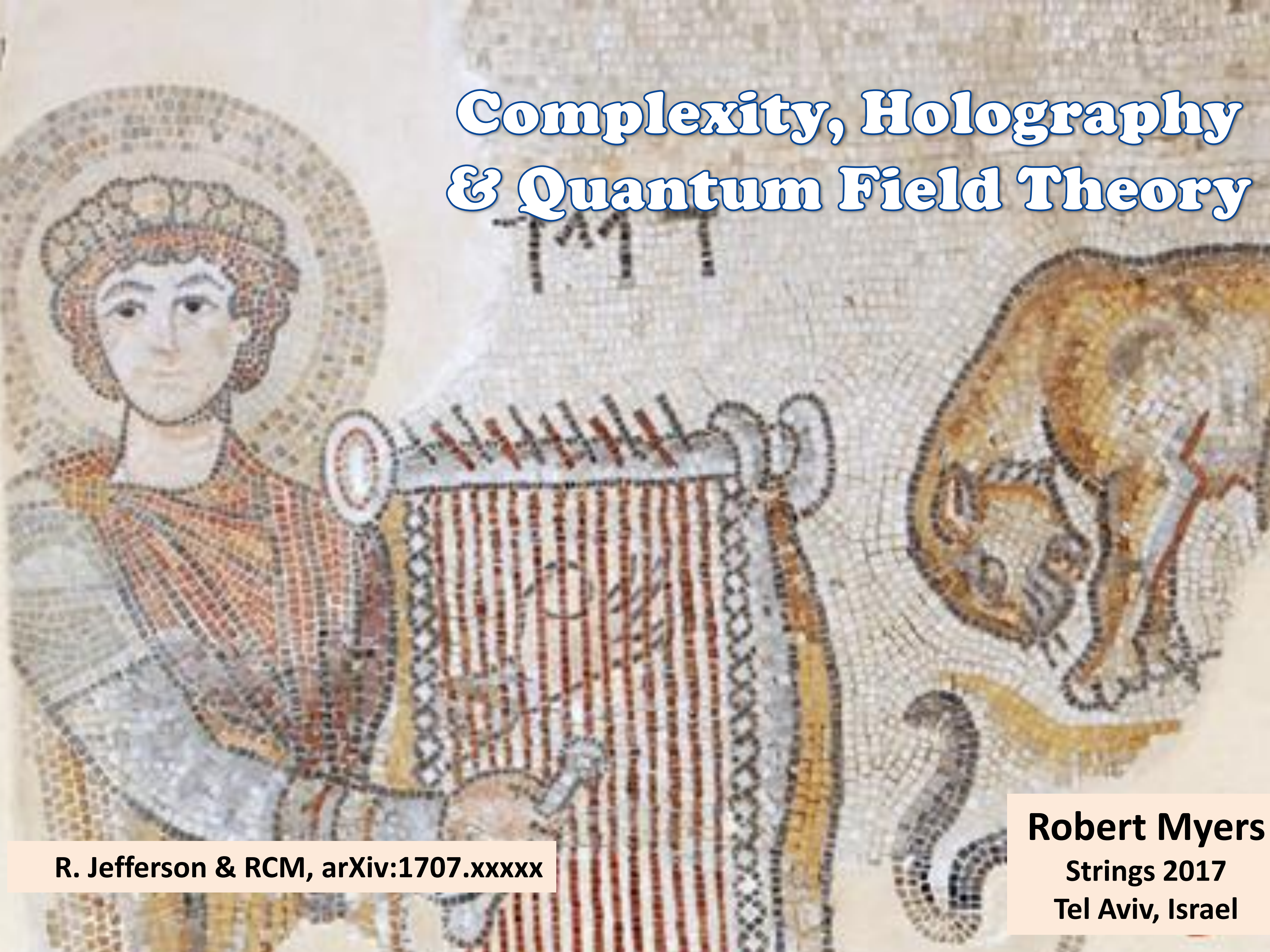


# Complexity, Holography & Quantum Field Theory

With: Dean Carmi, Horacio Casini, Shira Chapman,
Robert Jefferson, Luis Lehner, Hugo Marrochio, Eric Poisson,
Pratik Rath, Joan Simon, Rafael Sorkin & Sotaro Sugishita

**Robert Myers**
Strings 2017
Tel Aviv, Israel



# Complexity, Holography & Quantum Field Theory

R. Jefferson & RCM, arXiv:1707.xxxxx

**Robert Myers**

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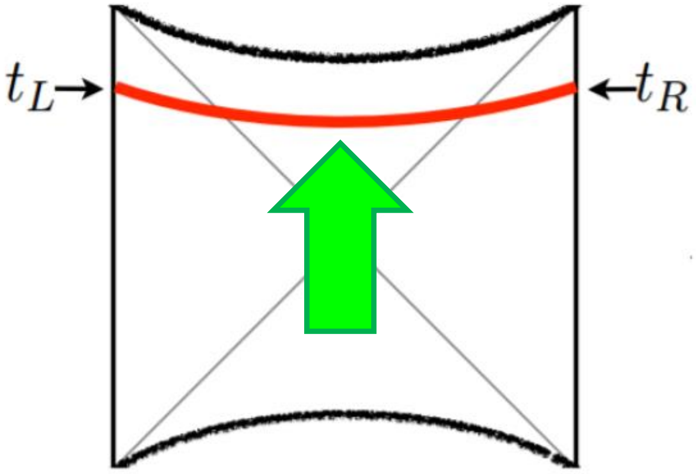
# **Holographic Complexity:** A Tale of Two Dualities

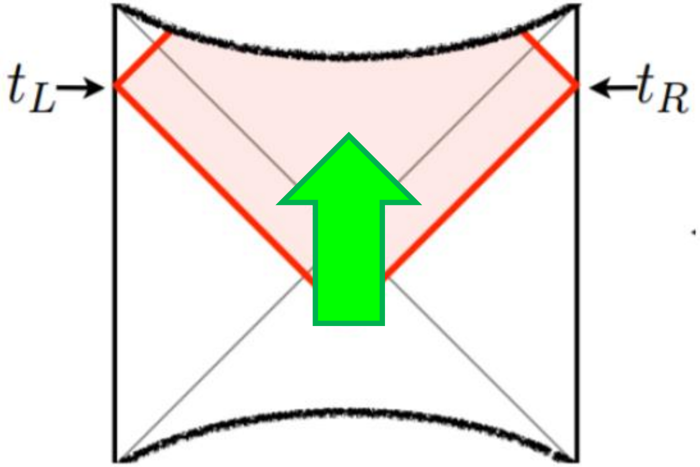
- complexity=volume: evaluate proper volume of extremal codim-one surface connecting Cauchy surfaces in boundary theory (cf holo EE)

(Stanford & Susskind)

Complexity = Volume



Complexity = Action



- complexity=action: evaluate gravitational action for Wheeler-DeWitt patch = domain of dependence of bulk time slice connecting boundary Cauchy slices in CFT (Brown, Roberts, Swingle, Susskind & Zhao)
- both of these gravitational "observables" probe the black hole interior (at arbitrarily late times on boundary)

# Questions?

- What is “holographic complexity”?
  - ➢ what is boundary dual of these gravitational observables?
  - ➢ QFT/path integral description of “complexity” in boundary CFT?

- is there a privileged role for (states on) null Cauchy surfaces?
  - ➢ provide distinguished reference states?

- is there a “renormalized holographic complexity”?
  - ➢ what’s it good for?; (EE vs mutual information versions of F)

- ambiguities? ambiguities? ambiguities?
  - ➢ connections between ambiguities in gravity and boundary?

- more boundary terms: higher codim. intersections; “complex” joint contributions; boundary “counterterms”

- why is complexity of formation positive?

- $\mathcal{C}_A$ contribution of spacetime singularity?
- subregion complexity?

# Questions?

- What is "holographic complexity"?
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**What does "complexity" mean in a quantum field theory?**

**(Tentative first steps w/ R. Jefferson)**

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- $\mathcal{C}_A$ contribution of spacetime singularity?
- subregion complexity?

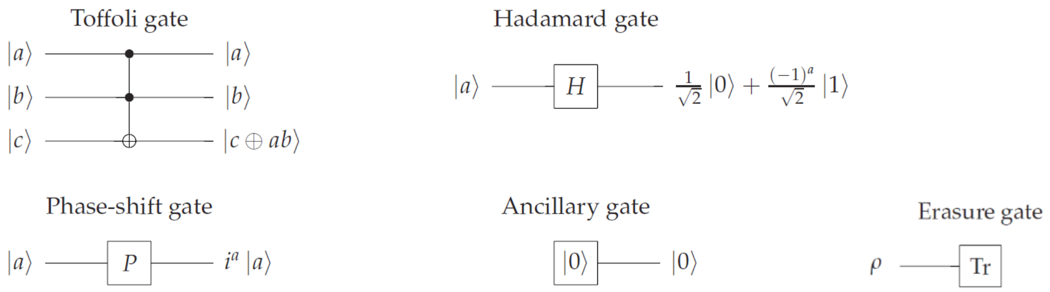
# Complexity in Quantum Field Theory??

- computational complexity: how difficult is it to implement a task? eg, how difficult is it to prepare a particular state?
- quantum circuit model:

$$|\psi\rangle = U\,|\psi_0\rangle$$

unitary operator built from set of simple gates

simple reference state eg, $|00000\cdots 0\rangle$



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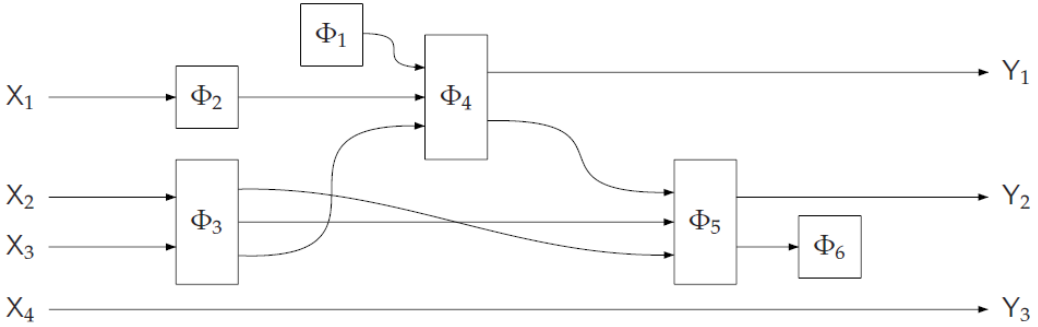

tolerance:

$$\big|\,|\psi\rangle - |\psi\rangle_{\rm Target}\,\big|^2 \leq \varepsilon$$

- **complexity** = minimum number of gates required to prepare the desired target state (ie, need to find optimal circuit)
- does the answer depend on the choices??

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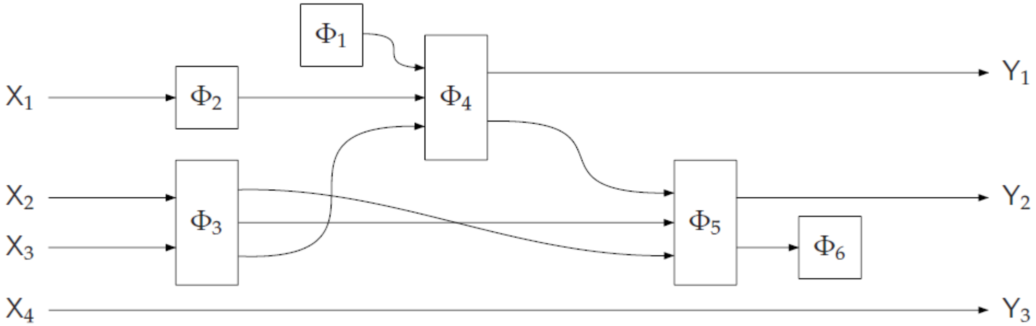

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- does the answer depend on the choices?? **YES!!**

## Quantum Field Theory:

- **free** scalar field theory (in d spacetime dimensions)

$$H = \frac{1}{2}\int d^{d-1}x\left[\pi(x)^2+\vec{\nabla}\phi(x)^2+m^2\phi(x)^2\right]$$

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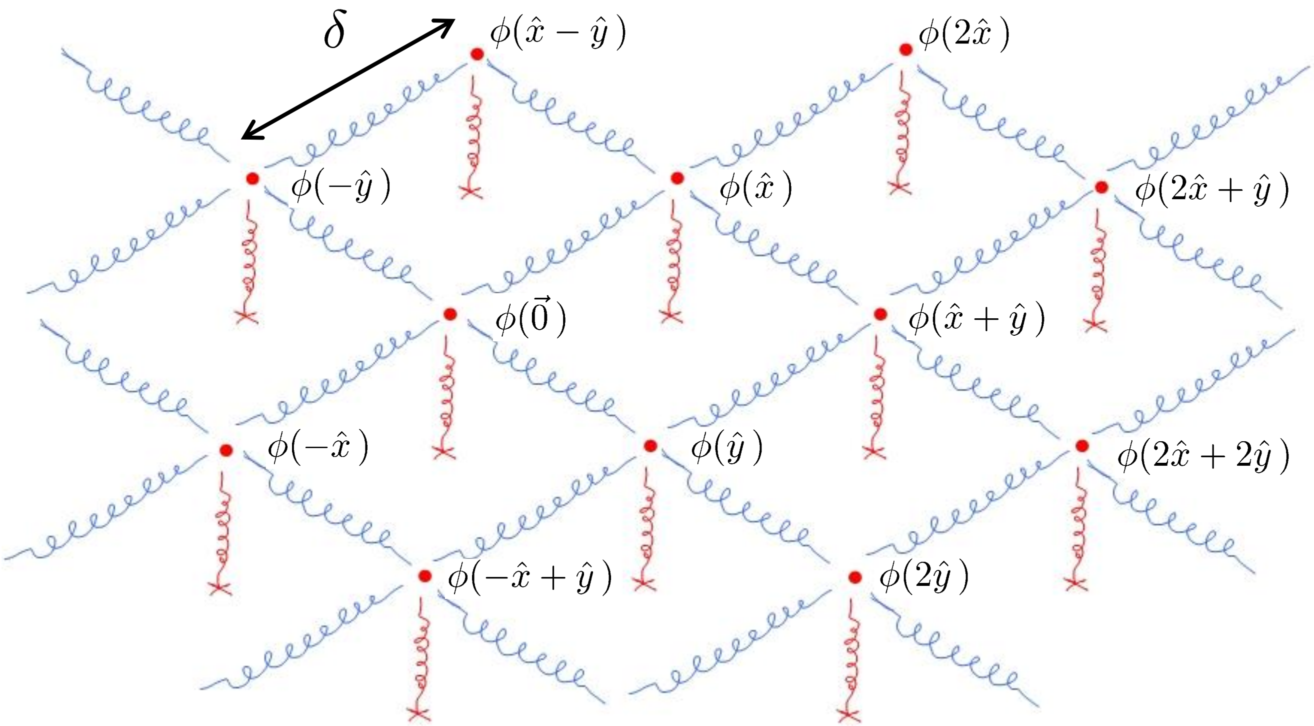
\begin{aligned}
H &= \frac{1}{2}\int d^{d-1}x\left[\pi(x)^2+\vec{\nabla}\phi(x)^2+m^2\phi(x)^2\right]\\
&= \frac{1}{2}\sum_{\vec{n}}\left[\frac{p(\vec{n})^2}{\delta^{d-1}}+\delta^{d-1}\left\{\frac{1}{\delta^2}\sum_i[\phi(\vec{n})-\phi(\vec{n}-\hat{x}_i)]^2+m^2\phi(\vec{n})^2\right\}\right]
\end{aligned}
$$



## Quantum Field Theory:

- an infinite family of coupled harmonic oscillators

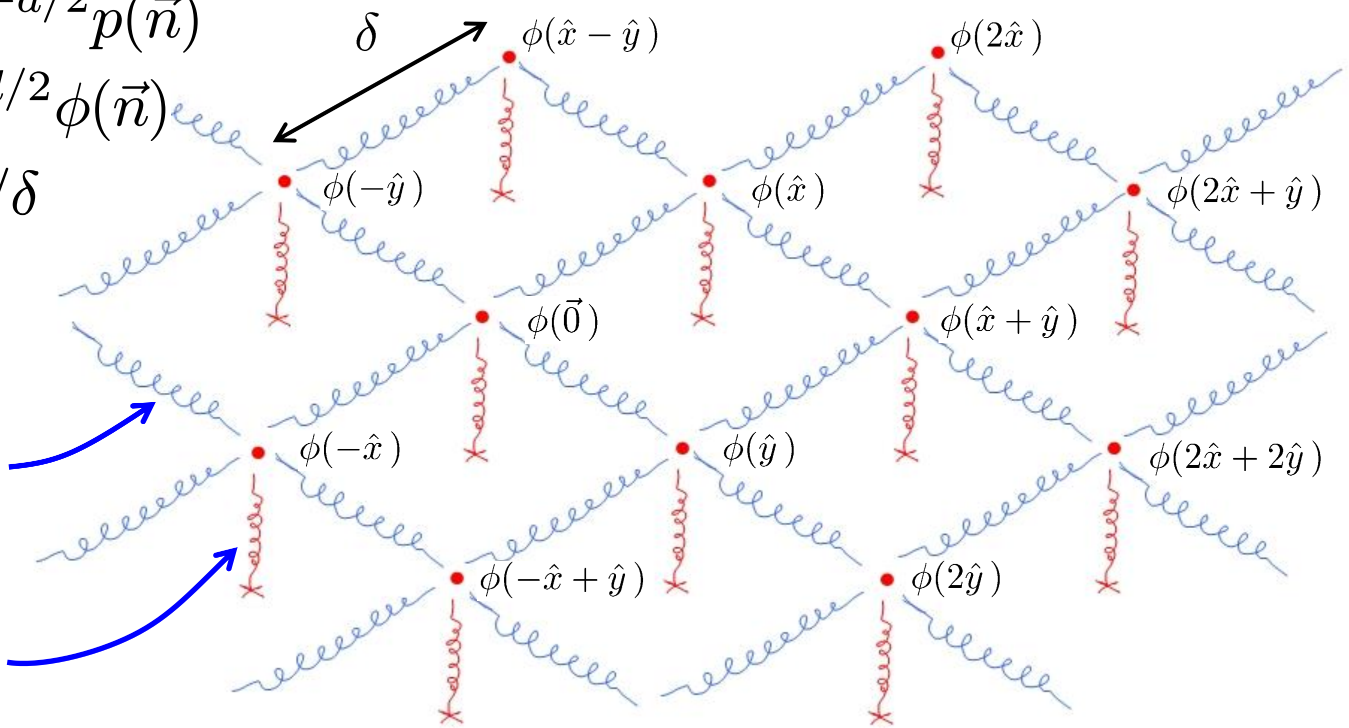
$$H = \frac{1}{2}\int d^{d-1}x\left[\pi(x)^2 + \vec{\nabla}\phi(x)^2 + m^2\phi(x)^2\right]$$

$$= \frac{1}{2}\sum_{\vec{n}}\left[\frac{P(\vec{n})^2}{M} + M\left\{\quad{}^2\sum_i[X(\vec{n}\,) - X(\vec{n}-\hat{x}_i\,)]^2 + \omega^2 X(\vec{n}\,)^2\right\}\right]$$

$$P(\vec{n}) = \delta^{-d/2}p(\vec{n})$$

$$X(\vec{n}) = \delta^{d/2}\phi(\vec{n})$$

$$M = 1/\delta$$

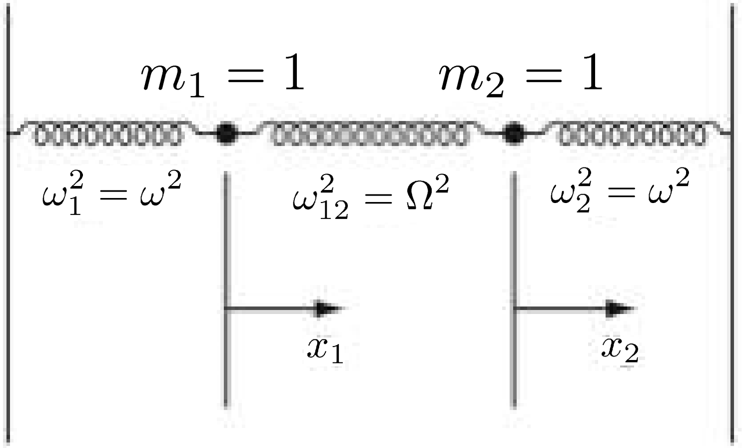


$$\Omega^2 = 1/\delta^2$$

$$\omega^2 = m^2$$

- **QFT** ⟶ **two coupled harmonic oscillators**

$$H = \frac{1}{2}\left[p_1^2 + p_2^2 + \omega^2\left(x_1^2 + x_2^2\right) + \Omega^2\left(x_1 - x_2\right)^2\right] \qquad (M = 1)$$



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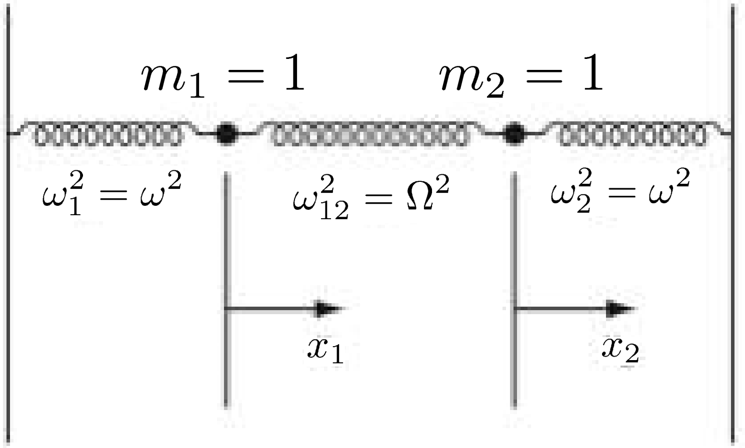
$$\omega_+^2 = \omega^2\,, \quad \omega_- = \omega^2 + 2\Omega^2$$

find normal modes; problem reduces to two independent SHO’s



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- ground-state wave-function:

$$\Psi_0(x_+, x_-) = \Psi_0(x_+)\,\Psi_0(x_-) = \frac{(\omega_+\omega_-)^{1/4}}{\sqrt{\pi}}\,\exp\left[-\frac{1}{2}\left(\omega_+ x_+^2 + \omega_- x_-^2\right)\right]$$

$$\Psi_0(x_1, x_2) = \frac{(\omega_+\omega_-)^{1/4}}{\sqrt{\pi}}\,\exp\left[-\frac{1}{4}\left(\omega_+\left(x_1 + x_2\right)^2 + \omega_-(x_1 - x_2)^2\right)\right]$$

$$= \frac{(\omega_1^2 - \beta^2)^{1/4}}{\sqrt{\pi}}\,\exp\left[-\frac{1}{2}\,\omega_1\,x_1^2 - \frac{1}{2}\,\omega_1\,x_2^2 - \beta\,x_1 x_2\right]$$

with $\omega_1 = \dfrac{\omega_+ + \omega_-}{2}$ and $\beta = \dfrac{\omega_+ - \omega_-}{2} < 0$

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**Target State!!**

$$= \frac{(\omega_1^2 - \beta^2)^{1/4}}{\sqrt{\pi}}\,\exp\left[-\frac{1}{2}\,\omega_1\,x_1^2 - \frac{1}{2}\,\omega_1\,x_2^2 - \beta\,x_1 x_2\right]$$

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**Target state:** $\psi_T(x_1, x_2) \simeq \exp\left[-\frac{1}{2}\,\omega_1\,x_1^2 - \frac{1}{2}\,\omega_1\,x_2^2 - \beta\,x_1 x_2\right]$

**Reference state:** ??????

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- factorized Gaussian: $(\omega_0\,x_i + i\,p_i)\,\psi_R(x_i) = 0$

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**Gates/Unitaries:** **??????**

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**Gates/Unitaries:**

- natural operators: $x_1\,,\quad x_2\,,\quad p_1\,,\quad p_2$ $\qquad [x_i,p_j] = i\,\delta_{ij}$

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→ $Q_{00} = \exp[i\epsilon\, x_0\, p_0]$ “add a small phase”

infinitesimal parameter $\epsilon \ll 1$

c-numbers

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(mulitply by small plane wave component)

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(mulitply by small plane wave component)

$Q_{ij} = \exp[i\epsilon\, x_i\, p_j] \quad (i \neq j)$ “shift $x_j$ by $\epsilon\, x_i$” (entangling)

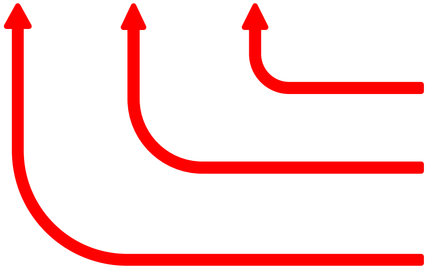
$$\begin{aligned} Q_{ii} &= \exp\left[i\frac{\epsilon}{2}(x_i\, p_i + p_i\, x_i)\right] \\ &= e^{\epsilon/2}\exp[i\epsilon\, x_i\, p_i] \end{aligned}$$

“rescale $x_i$ to $e^{\epsilon} x_i$” (scaling)

**Target state:** $\psi_T(x_1, x_2) \simeq \exp\left[-\frac{1}{2}\,\omega_1\,x_1^2 - \frac{1}{2}\,\omega_1\,x_2^2 - \beta\,x_1 x_2\right]$

**Reference state:** $\psi_R(x_1, x_2) \simeq \exp\left[-\frac{1}{2}\,\omega_0\,x_1^2 - \frac{1}{2}\,\omega_0\,x_2^2\right]$

- one path/circuit: $\psi_T(x_1, x_2) = Q_{22}^{\alpha_3}\ Q_{21}^{\alpha_2}\ Q_{11}^{\alpha_1}\ \psi_R(x_1, x_2)$



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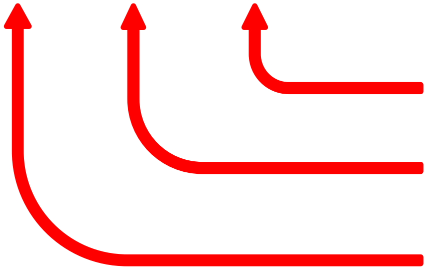

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"circuit depth": 
$$\begin{aligned} \mathcal{D}_1 &= |\alpha_1| + |\alpha_2| + |\alpha_3| \\ &= \frac{1}{2\epsilon}\log\left[\frac{\omega_1^2-\beta^2}{\omega_0^2}\right] + \frac{|\beta|}{\epsilon}\sqrt{\frac{\omega_0}{\omega_1}}\left(\omega_1^2-\beta^2\right)^{-1/2} \end{aligned}$$

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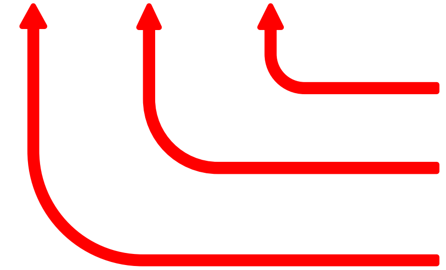

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**How do we find optimal circuit??**

- follow approach of **Mike Neilsen** (eg, Hamiltonian control theory)

Nielsen [arXiv:0502070]; Neilsen et al [arXiv:0603161]; Neilsen & Dowling [arXiv:0701004]

“What is the minimal size quantum circuit required to exactly implement
a specified n-qubit unitary operation, U, without the use of ancilla qubits?”

## Nielsen approach:

- work with smooth functions on a smooth space (rather than discrete)

$$\psi_T(x_1,x_2) = U\,\psi_R(x_1,x_2) \qquad \text{with} \qquad U = \mathcal{P}\exp\left[\int_0^1 ds\; Y^I(s)\,\mathcal{O}_I\right]$$

where
$$\mathcal{O}_{11} = \frac{i}{2}\,(x_1\,p_1 + p_1\,x_1)\,, \quad \mathcal{O}_{12} = i\,x_1\,p_2\,,$$
$$\mathcal{O}_{22} = \frac{i}{2}\,(x_2\,p_2 + p_2\,x_2)\,, \quad \mathcal{O}_{21} = i\,x_2\,p_1$$

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*s*: position label

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- consider trajectories:

$$U(s) = \mathcal{P}\exp\left[\int_0^s d\tilde{s}\; Y^I(\tilde{s})\,M_I\right] \qquad \text{where} \qquad U(s=0) = 1\,, \quad U(s=1) = U_{fin}$$

velocity: $Y^I(s) = \mathrm{Tr}\left[\partial_s U(s)\,U^{-1}(s)\,M_I\right]$

## “What is the minimal size quantum circuit required to exactly implement a specified n-qubit unitary operation, U, without the use of ancilla qubits?”

### Nielsen approach:

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velocity: $Y^I(s) = \mathrm{Tr}\left[\partial_s U(s)\, U^{-1}(s)\, M_I\right]$

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minimizing the cost function/action: $\mathcal{D} = \int_0^1 ds \sum_I |Y^I(s)|$

→ extremal path $U(s)$ is geodesic in a Finsler geometry

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# **Nielsen approach:** to find optimal circuit

$$\psi_T(x_1,x_2) = U(s=1)\,\psi_R(x_1,x_2) \quad \text{with} \quad U(s) = \mathcal{P}\exp\left[\int_0^s d\tilde{s}\; Y^I(\tilde{s})\,\mathcal{O}_I\right]$$

where
$$\mathcal{O}_{11} = \frac{i}{2}\,(x_1\,p_1 + p_1\,x_1)\,, \quad \mathcal{O}_{12} = i\,x_1\,p_2\,,$$
$$\mathcal{O}_{22} = \frac{i}{2}\,(x_2\,p_2 + p_2\,x_2)\,, \quad \mathcal{O}_{21} = i\,x_2\,p_1$$

- find the “geodesic” minimizing the “action”:

$$\mathcal{D} = \int_0^1 ds\Big[(Y^{11}(s))^2 + (Y^{22}(s))^2 + (Y^{21}(s))^2 + (Y^{12}(s))^2\Big]^{1/2}$$

- defining $Y^I(s) = \mathrm{Tr}\left[\partial_s U(s)\,U^{-1}(s)\,M_I\right]$ with $M_I \longrightarrow \mathcal{O}_I$ ?

(But see: Chapman, Heller, Marrochio & Pastawski)

## Alternative picture:

- at this stage thinking of Gaussian states, so consider space of states as space of (positive) quadratic forms

$$\psi \simeq \exp\left[-\frac{1}{2}x_i\, A_{ij}\, x_j\right] \longrightarrow \quad A_T = \begin{bmatrix} \omega_1 & \beta \\ \beta & \omega_1 \end{bmatrix} \qquad A_R = \begin{bmatrix} \omega_0 & 0 \\ 0 & \omega_0 \end{bmatrix}$$

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- unitary gates: $Q_{ij} = \exp[\epsilon\, \mathcal{O}_{ij}]$ with $\mathcal{O}_{ij} = i\, x_i\, p_j + \frac{1}{2}\delta_{ij}$

$$\longrightarrow \quad Q_{ij} = \exp[\epsilon\, M_{ij}] \qquad \text{with} \qquad [M_{ij}]_{ab} = \delta_{ia}\,\delta_{jb}$$

eg, $M_{11} = \begin{bmatrix}1 & 0 \\ 0 & 0\end{bmatrix}$

where $A' = Q_{ij}\, A\, Q_{ij}^T$

$\uparrow$

generators of GL(2,R)

# Alternative picture:

- what would the optimal circuit look like?

$$A_T = U(1)\, A_R\, U^T(1) \qquad \text{with} \quad U(s) = \mathcal{P}\exp\left[\int_0^s d\tilde{s}\; Y^I(\tilde{s})\, M_I\right]$$

$$A_R = \begin{bmatrix} \omega_0 & 0 \\ 0 & \omega_0 \end{bmatrix} \quad A_T = \begin{bmatrix} \omega_1 & \beta \\ \beta & \omega_2 \end{bmatrix} \qquad \text{where} \quad [M_{ij}]_{ab} = \delta_{ia}\,\delta_{jb}$$

generators of GL(2,R)

- we want to minimize: $\mathcal{D} = \int_0^1 ds \sum_{ij} \left[(Y^{ij}(s))^2\right]^{1/2}$

- defining $Y^I(s) = \mathrm{Tr}\left[\partial_s U(s)\, U^{-1}(s)\, M_I\right]$ is now straightforward

→ finding geodesics for some right invariant metric on GL(2,R)

$$ds^2 = \delta_{IJ}\, \mathrm{Tr}\left[dU(s)\, U^{-1}(s)\, M_I\right]\, \mathrm{Tr}\left[dU(s)\, U^{-1}(s)\, M_J\right]$$

- metric:

$$ds^2 = 2dy^2 + 2d\rho^2 + 2\cosh(2\rho)\cosh^2\rho\, d\tau^2 + 2\cosh(2\rho)\sinh^2\rho\, d\theta^2 - 8\sinh^2\rho\, \cosh^2\rho\, d\theta d\tau$$

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minimize in family of geodesics → $\theta_1-\tau_1$ **is not fixed!**

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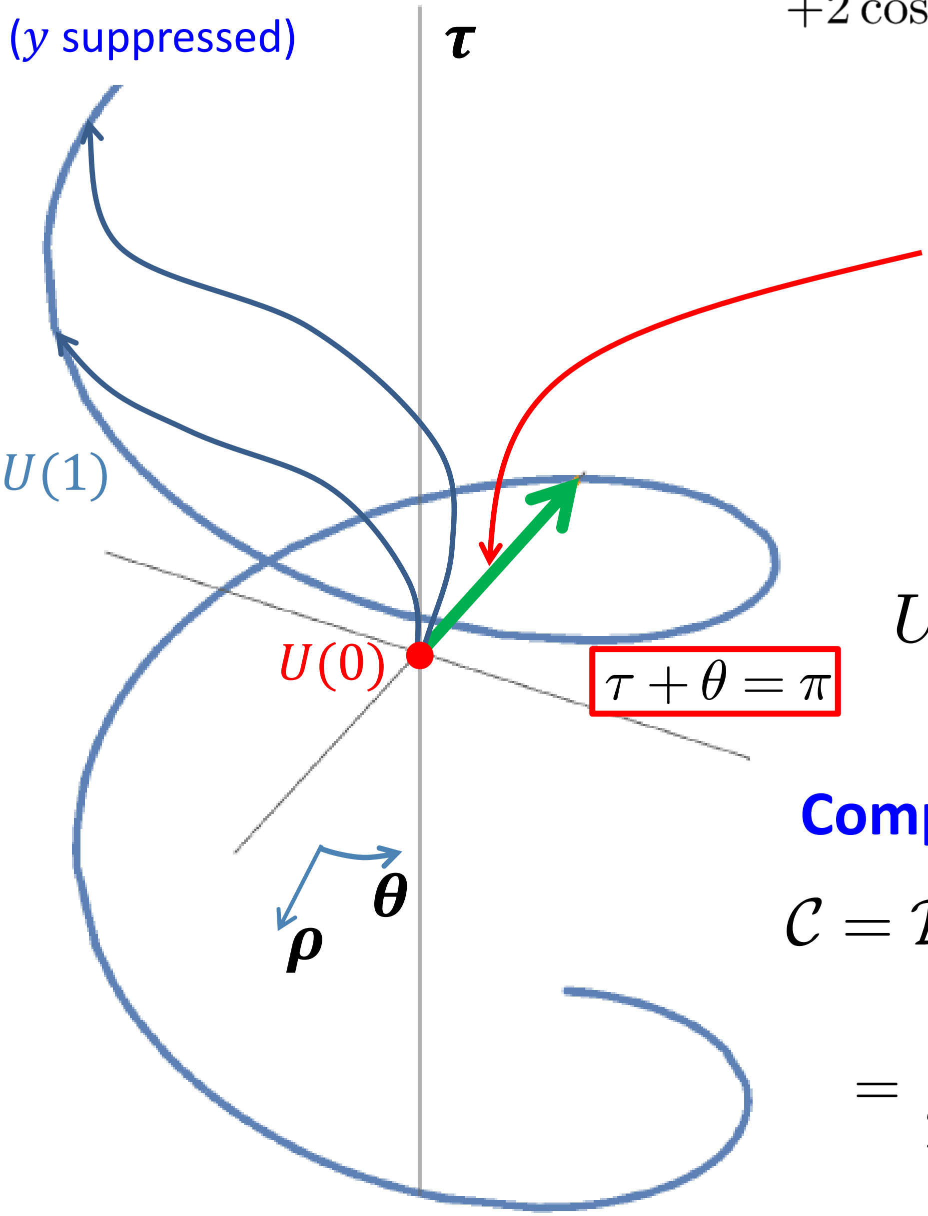
→

$$\begin{aligned} y_1 &= \frac{1}{2}\log\frac{\omega_1^2-\beta^2}{\omega_0^2} = \frac{1}{2}\log\frac{\omega_+\omega_-}{\omega_0^2}\\ \rho_1 &= \frac{1}{2}\log\frac{\omega_1+|\beta|}{\omega_1-|\beta|} = \frac{1}{2}\log\frac{\omega_-}{\omega_+}\\ \theta_1+\tau_1 &= \pi \end{aligned}$$

minimize in family of geodesics → $\theta_1 - \tau_1$ **is not fixed!**

- solve for geodesics:
$$\begin{aligned} ds^2 &= 2dy^2+2d\rho^2+2\cosh(2\rho)\cosh^2\rho\, d\tau^2 \\ &\quad +2\cosh(2\rho)\sinh^2\rho\, d\theta^2-8\sinh^2\rho\,\cosh^2\rho\, d\theta d\tau \end{aligned}$$

($y$ suppressed)



minimal geodesic:

$$\tau(s)=0,\ \ \theta(s)=\pi,$$
$$y(s)=y_1\, s,\ \ \rho(s)=\rho_1\, s$$

$$U(s)=\mathcal{P}\exp\left[\begin{bmatrix} y_1 & -\rho_1 \\ -\rho_1 & y_1 \end{bmatrix} s\right]$$

**Complexity:**

$$\mathcal{C}=\mathcal{D}_{min}=\sqrt{2\rho_1^2+2y_1^2}$$
$$=\frac{1}{2}\sqrt{\log^2\left(\frac{\omega_+}{2\omega_0}\right)+\log^2\left(\frac{\omega_-}{2\omega_0}\right)}$$

with $\omega_+^2 \;=\; \omega^2\,, \quad \omega_-^2=\omega^2+2\Omega^2$

## Normal modes:

- examine normal circuit in normal mode basis

$$\tilde{x} = Rx \quad \longleftrightarrow \quad \begin{bmatrix} x_- \\ x_+ \end{bmatrix} = R \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{with} \quad R = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\longrightarrow \tilde{A}_T = R\, A_T R^\dagger = \begin{bmatrix} \omega_- & 0 \\ 0 & \omega_+ \end{bmatrix}, \qquad \tilde{A}_R = \begin{bmatrix} \omega_0 & 0 \\ 0 & \omega_0 \end{bmatrix}$$

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$$\longrightarrow \tilde{U}(s) = R\, U(s) R^\dagger = \mathcal{P} \exp \left[ \begin{bmatrix} \frac{1}{2} \log \frac{\omega_-}{\omega_0} & 0 \\ 0 & \frac{1}{2} \log \frac{\omega_+}{\omega_0} \end{bmatrix} s \right]$$

- in normal mode basis, minimal circuit simply scales up diagonal entries

$$U(s) = \mathcal{P} \exp \left[ \frac{1}{2} M_{--} \log \frac{\omega_-}{\omega_0}\, s + \frac{1}{2} M_{++} \log \frac{\omega_+}{\omega_0}\, s \right]$$

with $M_{\pm\pm} = \frac{1}{2} \left( M_{11} + M_{22} \pm M_{12} \pm M_{21} \right)$

# And now for the main event . . . (almost)

- examine lattice of $N$ oscillators

$$\longrightarrow \quad A_T = m\,\mathbb{1}\,, \qquad A_R = \omega_0\,\mathbb{1}$$

"lower bound"

$$\longrightarrow \mathcal{C} = \mathcal{D}_{min} = \frac{1}{2}\sqrt{\sum \log^2(m/\omega_0)} = \frac{N^{1/2}}{2}\log(\omega_0/m)$$

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[ $F_1$ & $F_p$ ]

$$\mathcal{D} = \int_0^1 ds \sum (Y^I(s))^2 \qquad \mathcal{D} = \int_0^1 ds \sum p_I\, (Y^I(s))^2$$

[ $LF_2$ & $LF_{2p}$ ]

$$\mathcal{D} = \int_0^1 ds \sum |Y^I(s)|^3 \qquad \mathcal{D} = \int_0^1 ds \sum p_I\, |Y^I(s)|^3$$

[ $N_3$ & $N_{3p}$ ]

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$$\longrightarrow \mathcal{C} = \mathcal{D}_{min} = \frac{1}{2^\alpha} \sum |\log\big(\omega_{\vec{k}}/\omega_0\big)|^\alpha$$

where for $(d-1)$-dimensional (periodic square) lattice:

$$\omega_{\vec{k}}^2 = m^2 + \frac{4}{\delta^2} \sum_i \sin^2 \frac{\pi k_i}{N}\,, \qquad k_i = 0, 1, \cdots, N^{1/(d-1)} - 1$$

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1) $\omega_0 = 1/\ell_0$ (IR) with $\ell_0 \gg \delta$ ⟶ $\mathcal{C} \sim N \log^\alpha \left( \frac{\ell_0}{\delta} \right) + \cdots$

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2) $\omega_0 = \beta/\delta$ (UV) $\longrightarrow$ $\mathcal{C} \sim \# \frac{V}{\delta^{d-1}} + \cdots$ ⟵ similar to C=A and C=V ($\alpha$ unconstrained)

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$$\text{og} \left( \frac{\ell_0}{\delta} \right) + \cdots$$

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QFT: $\omega_0$

- reference state introduces scale
- $\omega_0 \sim {}^{1}/_{R}$: complexity is superextensive
- $\omega_0 = {}^{1}/_{\ell_0}$: complexity depends on new (unphysical) scale
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complexity = action: $\alpha = L/\ell$

- null normals introduce scale
- $\ell \sim V^{1/(d-1)}$: complexity is superextensive
- $\ell = \ell_0$: complexity depends on new (unphysical) scale
- $\ell \sim \delta$: IR contributions depend on UV cutoff, eg, $d\mathcal{C}_A/dt$

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# Conclusions/Questions:

- complexity model for free scalar shows surprising similarities to holographic proposals for complexity of boundary CFT states
- possible extensions of QFT model:
  - ➢ complexity for excited QFT states? in interacting QFT's?
  - ➢ appropriate gate set? appropriate cost functions?
- where is hyperbolic AdS geometry/cMERA in QFT discussion?

(See: Chapman, Heller, Marrochio & Pastawski)

- concrete connection to "holographic complexity"?
  - ➢ QFT/path integral description of "complexity" in boundary CFT?
  - ➢ what is boundary dual of these gravitational observables?
- need/want a better idea?

$\text{build } \rho_A \to S_{EE} = -\Sigma\, \lambda_n \log \lambda_n$ ⟶ Replica Trick

$\text{build optimal } U \to \mathcal{C} = \#\text{ gates}$ ⟶ **? ? ? ? ?**

⟶ preliminary suggestions:

Caputa, Kundu, Miyaji, Takayanagi & Watanabe (1703.00456; 1706.07056)

Czech (1706.00965)

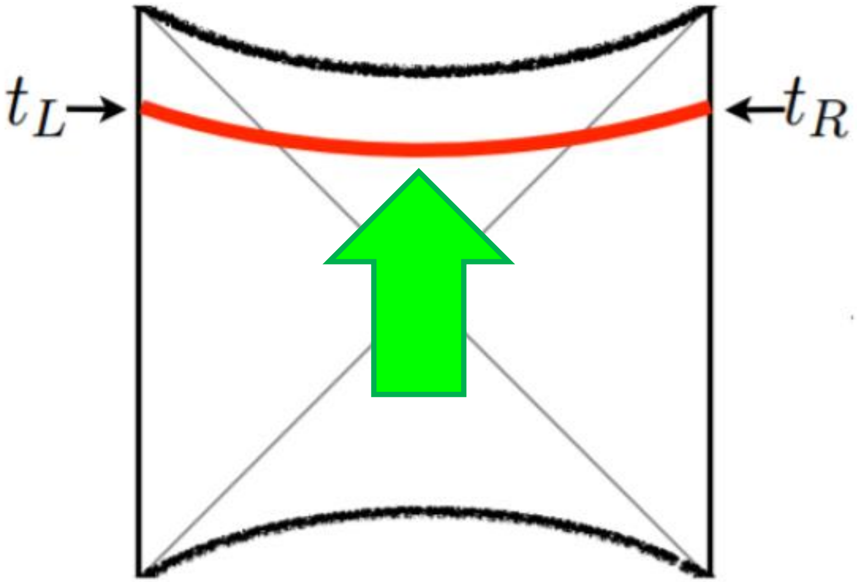
**See Takayanagi's talk!**

**[Talk ends here]**

Dean Carmi, RCM & Pratik Rath

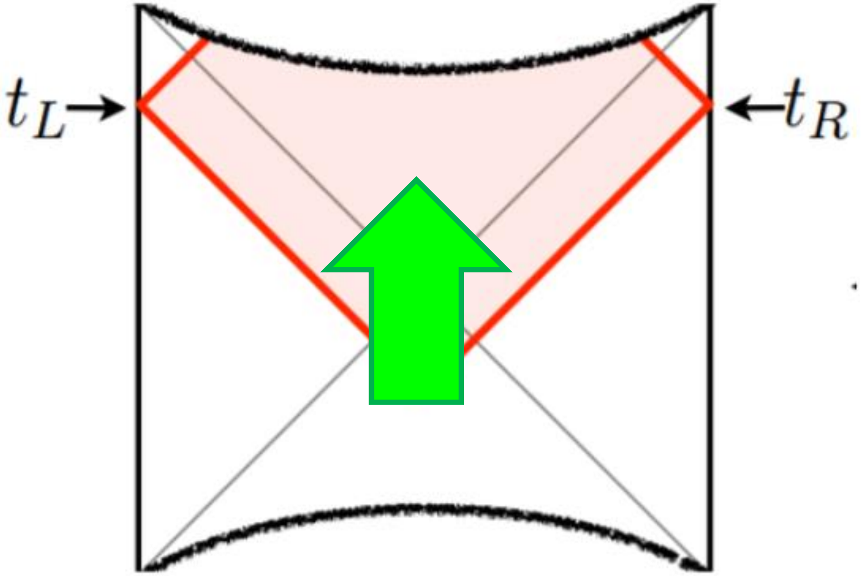
# Holographic Complexity:

Complexity = Volume



$$\mathcal{C}_{\rm V}(\Sigma) = \max_{\Sigma=\partial\mathcal{B}}\left[\frac{\mathcal{V}(\mathcal{B})}{G_N\,\ell}\right]$$

Complexity = Action



$$\mathcal{C}_{\rm A}(\Sigma) = \frac{I_{\rm WDW}}{\pi\,\hbar}$$

(Brown, Roberts, Swingle, Stanford, Susskind & Zhao)

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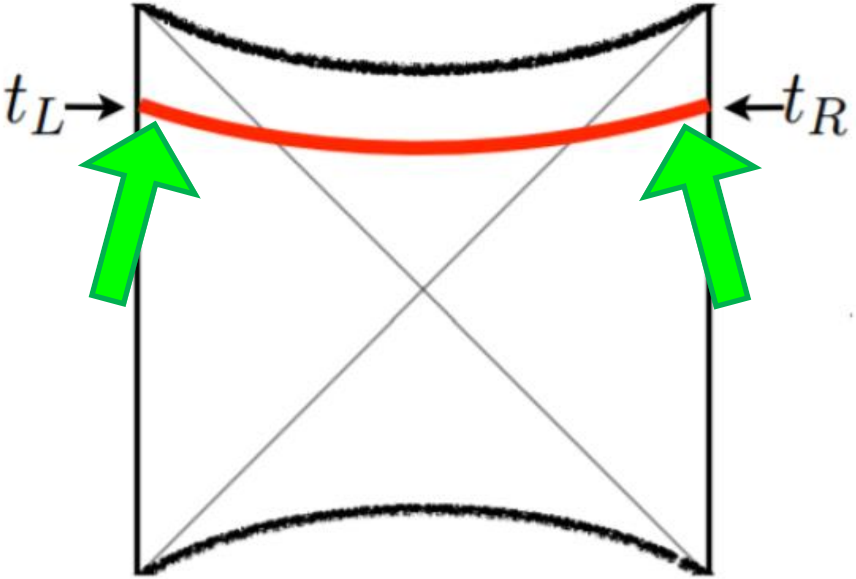

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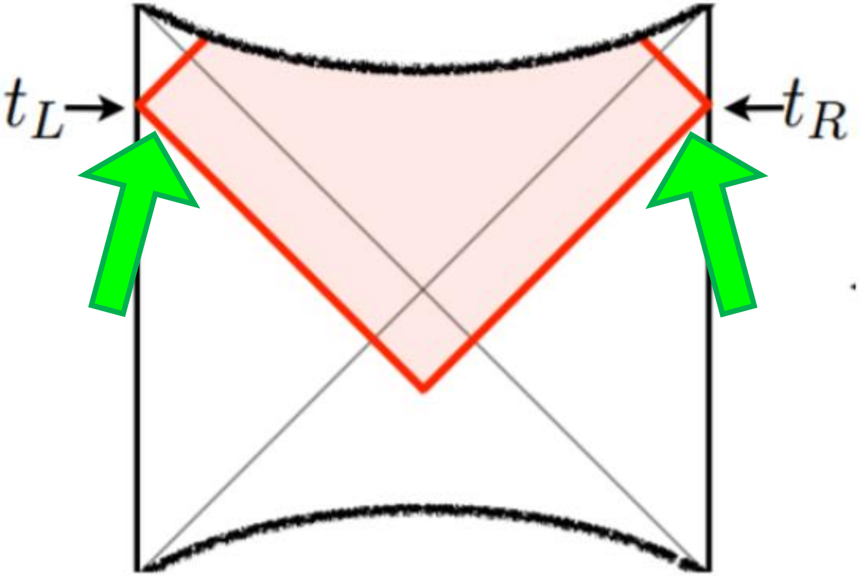

- UV divergences naturally associated with establishing correlations or entanglement down to arbitrarily small length scales

# Holographic Complexity:

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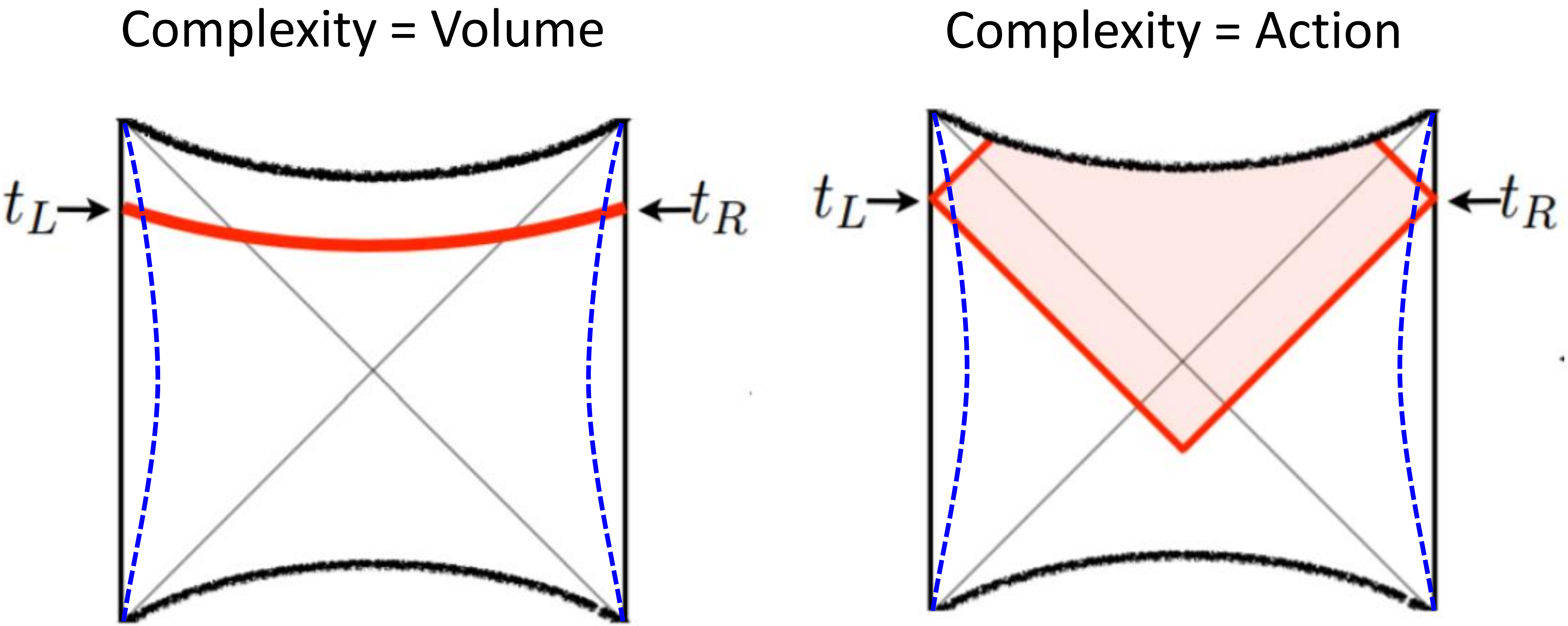

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- regulate volume/action with the introduction of UV regulator surface at large radius ($r_{max} = L^2/\delta$), as usual

# Holographic Complexity:

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- regulate volume/action with the introduction of UV regulator surface at large radius ($r_{max} = L^2/\delta$), as usual

$$\mathcal{C}_V(\Sigma) = \frac{8\pi^{\frac{d+2}{2}}\Gamma(d/2)}{\Gamma(d+2)}\,C_T\int_\Sigma d^{d-1}\sigma\sqrt{h}\,\Bigg[\frac{1}{\delta^{d-1}}$$

$$-\frac{(d-1)}{2(d-2)(d-3)\delta^{d-3}}\left(\mathcal{R}^a_a-\frac{1}{2}\mathcal{R}-\frac{(d-2)^2}{(d-1)^2}K^2\right)+\cdots\Bigg]$$

$$= \; f(d)\;C_T\,\frac{V_\Sigma}{\delta^{d-1}} \;+\; \text{“curvature corrections”}$$

- UV divergences appear as local integrals of geometric invariants (as with holographic entanglement entropy)

# Holographic Complexity:

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- regulate volume/action with the introduction of UV regulator surface at large radius ($r_{max} = L^2/\delta$), as usual

$$\mathcal{C}_V(\Sigma) = \frac{1}{\delta^{d-1}} \int_\Sigma d^{d-1}\sigma \sqrt{h}\; v_0(\mathcal{R}, K)$$

$$\mathcal{C}_A(\Sigma) = \frac{1}{\delta^{d-1}} \int_\Sigma d^{d-1}\sigma \sqrt{h}\; \left[v_1(\mathcal{R}, K) + \underline{\log\left(\frac{L}{\alpha\,\delta}\right)}\; v_2(\mathcal{R}, K)\right]$$

from boundary terms in action

with

$$v_k(\mathcal{R}, K) = \sum_{n=0}^{\lfloor \frac{d-1}{2} \rfloor} \sum_i c_{i,n}^{[k]}(d)\; \delta^{2n}\; [\mathcal{R}, K]_i^{2n}$$

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AdS scale

normalization

$\boldsymbol{k}\cdot\hat{\boldsymbol{t}} = \pm\alpha$

$$\mathcal{C}_A(\Sigma) = \frac{1}{\delta^{d-1}} \int_\Sigma d^{d-1}\sigma \sqrt{h}\; \left[ v_1(\mathcal{R}, K) + \underbrace{\log\left(\frac{L}{\alpha\,\delta}\right)}_{\text{from boundary terms in action}} v_2(\mathcal{R}, K) \right]$$

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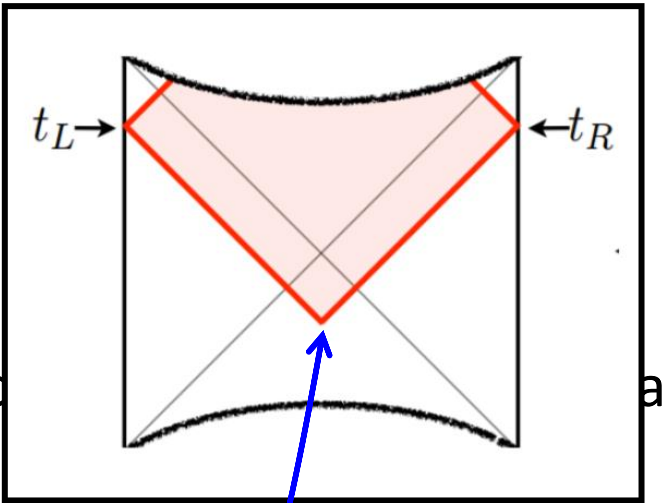
# Holographic Complexity:



- U … with c … s
- r … oduc … ace a … al

$$\alpha = L/\ell$$

- eliminates AdS scale ✓
- $\ell \sim V^{1/(d-1)}$: complexity is superextensive
- $\ell = \ell_0$: complexity depends on new (unphysical) scale
- $\ell \sim \delta$: IR contributions depend on UV cutoff, eg, $d\mathcal{C}_A/dt$

$\mathcal{C}_V$ … $K)$

normalization

$\boldsymbol{k}\cdot\hat{\boldsymbol{t}} = \pm\alpha$

AdS scale

$$\mathcal{C}_A \ldots , K) + \log\left(\frac{L}{\alpha\,\delta}\right)\ v_2(\mathcal{R},K)\Big]$$

from boundary terms in action

$$\sum_{n=0}\ \sum_i\ c_{i,n}^{[k]}(d)\ \delta^{2n}\ [\mathcal{R},K]_i^{2n}$$

- UV divergences appear as local integrals of geometric invariants (as with holographic entanglement entropy)

# And now for the main event . . .

- examine lattice of $N$ oscillators $\longrightarrow$ restore $\boldsymbol{\Omega = 1/\delta}$ **&** $\boldsymbol{\omega = m}$

$$\longrightarrow \mathcal{C} = \mathcal{D}_{min} = \frac{1}{2^\alpha} \sum |\log(\omega_{\vec{k}}/\omega_0)|^\alpha$$

where for $(d-1)$-dimensional (periodic square) lattice:

QFT: $\omega_0$

- reference state introduces scale
- $\omega_0 \sim {}^1/_R$: complexity is superextensive
- $\omega_0 = {}^1/_{\ell_0}$: complexity depends on new (unphysical) scale
- $\omega_0 \sim {}^1/_\delta$: IR contributions depend on UV cutoff

complexity = action: $\alpha = L/\ell$

- null normals introduce scale
- $\ell \sim V^{1/(d-1)}$: complexity is superextensive
- $\ell = \ell_0$: complexity depends on new (unphysical) scale
- $\ell \sim \delta$: IR contributions depend on UV cutoff, eg, $d\mathcal{C}_A/dt$

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