Meromorphic functions and giant topology

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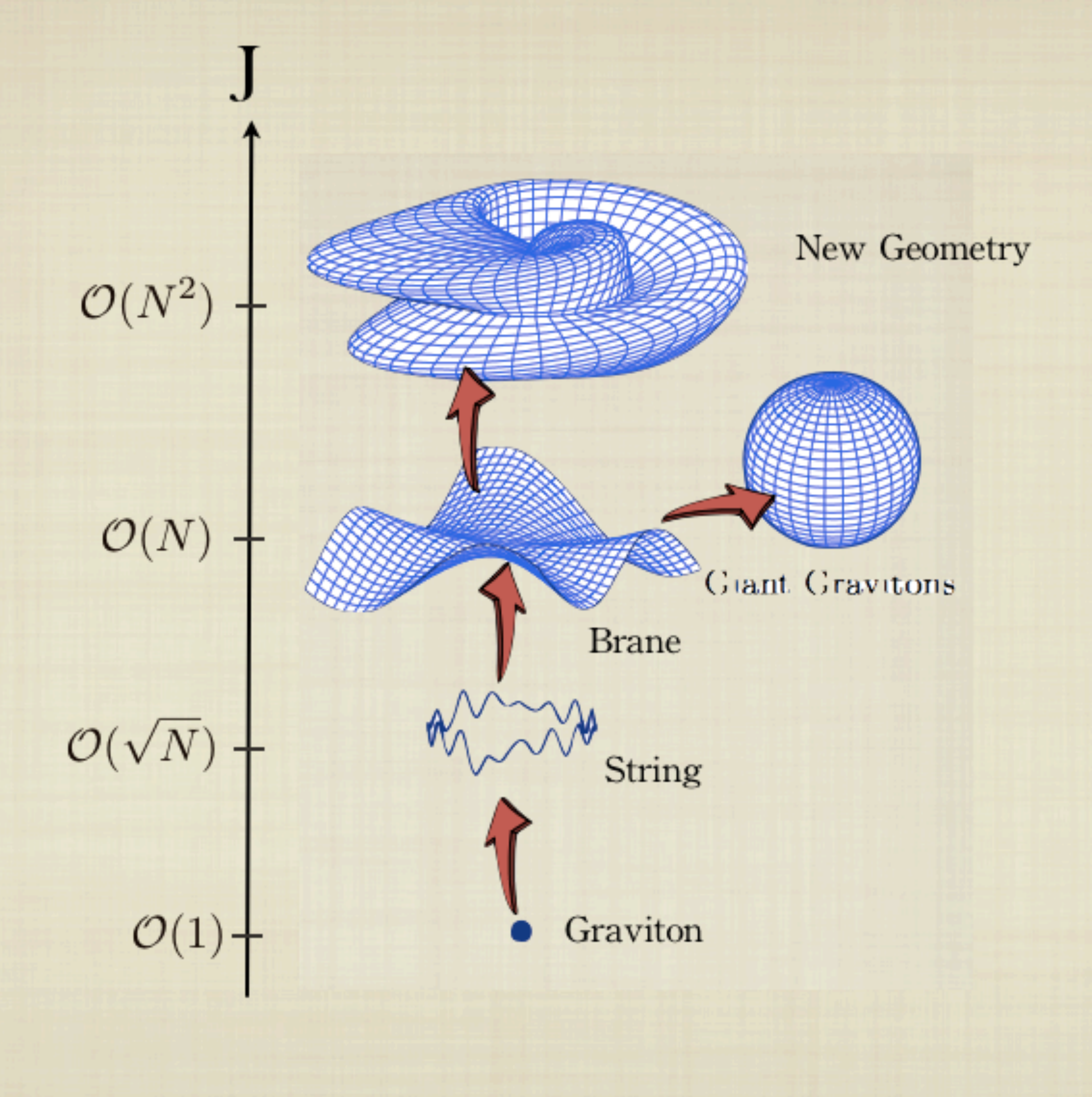
Motivation

How is spacetime emergent?

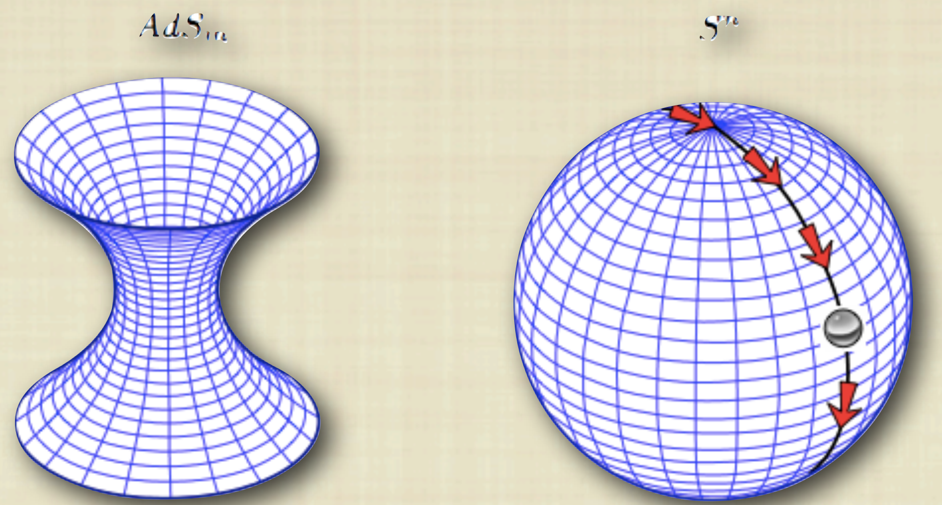
Gauge/Gravity Duality



Giant Gravitons in Gravity

- In type IIB string theory giant gravitons are D3-branes wrapping a 3-sphere in $AdS_5 \times S^5$

[McGreevy et.al. 00]



- Some properties:
  - They are classically stabilized through coupling to an RR 5-form flux in the supergravity background

    [Grisaru et.al. 00]

  - admit a microscopic description as a large number of coincident gravitons that couple to the background RR C_{p+1} potential and polarize into a macroscopic Dp-brane.

    [Myers 99, Lozano et.al. 02]

  - They come in two forms, depending on which 3-sphere the D3-brane wraps.

Giant Gravitons in Gauge Theory

- In the dual $\mathcal{N}=4$ SYM, giant gravitons are identified with Schur polynomial operators: [Corely-Jevicki-Ramgoolam '01]

$$\chi_R(Z) = \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma) Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_n}_{i_{\sigma(n)}}$$

- Some properties:
  - The Schur label R is a Young diagram with n boxes.
  - In the completely antisymmetric representation $\chi_R(Z)$ collapses to a subdeterminant operator. [Balasubramanian et al '01]
  - Schurs satisfy a nice product rule that follows from the Schur-Weyl duality

$$\chi_R(Z)\chi_S(Z) = \sum_T f_{RS;T}\chi_T(Z)$$

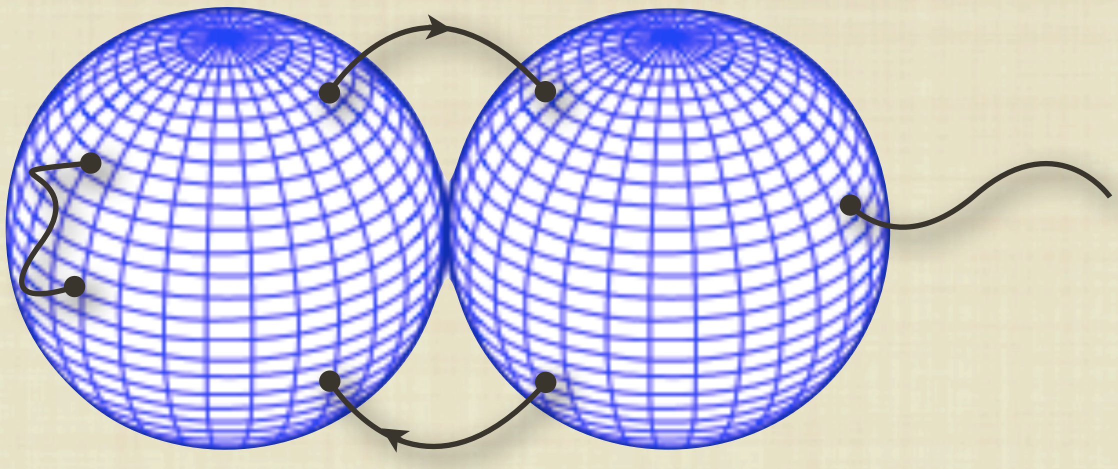
Emergent topology

How is the topology of the brane encoded in the Schur operators?

Emergent topology

- The spherical D3-brane giant graviton must satisfy **Gauss' Law:** every source must match with a corresponding sink:

[Balasubramanian et al '02]



- Giants with strings attached are dual to **restricted Schur** operators:

[de Mello Koch et al '07-14]

$$\chi^{(k)}_{R,R_1} = \frac{1}{(n-k)!}\sum_{\sigma\in S_n}\mathrm{Tr}_{R_1}\left(\Gamma_R(\sigma)\right)\mathrm{Tr}\left(\sigma Z^{\otimes n-k}\left(W^{(1)}\right)^{i_{n-k+1}}_{i_{\sigma(n-k+1)}}\cdots\left(W^{(k)}\right)^{i_n}_{i_{\sigma(n)}}\right)$$

- The number of operators that can be constructed for a given representation matches precisely the number of allowed states in the string theory.

- How do we see **non-trivial topology**?

[Nishioka-Takayanagi '09 Berenstein-Park '09]

Giants & Holomorphic Surfaces

- A holomophic function $f : \mathbb{C}^3 \to \mathbb{C}$ defines a supersymmetric D3-brane in $\mathbb{R} \times S^5 \in AdS_5 \times S^5$ as a surface [Mikhailov '00]

$$f(e^{-it}Z_1,\, e^{-it}Z_2,\, e^{-it}Z_3) = 0, \qquad \sum_i |Z_i|^2 = 1$$

- The amount of SUSY preserved depends on the number of arguments:

$$f(Z_1) = 0 \Rightarrow \frac{1}{2}\,\mathrm{BPS}; \quad f(Z_1, Z_2) = 0 \Rightarrow \frac{1}{4}\,\mathrm{BPS}; \quad f(Z_1, Z_2, Z_3) = 0 \Rightarrow \frac{1}{8}\,\mathrm{BPS}$$

- Example **(1,0,0)**: The usual sphere giant corresponds to the linear polynomial $f(Z_1) = Z_1 - \alpha = 0$
  - The spatial part of the brane worldvolume Σ is parameterized by Z_2 and Z_3 with

$$|Z_2|^2 + |Z_3|^2 = 1 - \alpha^2$$

  - The brane rigidly rotates in the Z_1-plane.
  - Σ is an S^3 with radius $\sqrt{1-\alpha^2}$ so that the maximal giant corresponds to $\alpha = 0$
  - A holomorphic function with multiple zeros leads to multiple concentric giants - case **(m,0,0)**

1/4-BPS giants (i): meromorphic functions

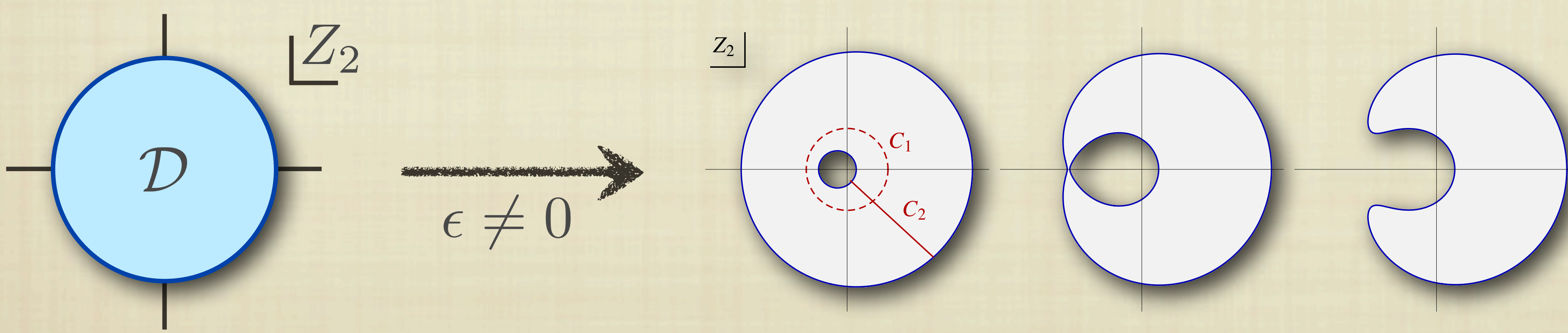
- Now let's add to $f(Z_1)$ some meromorphic function of Z_2 and see if we can read off the topology.
- Example **(1,1,0):** $f(Z_1, Z_2) = Z_1 - \alpha + \dfrac{\epsilon}{Z_2}$

[Abbott,JM,Prinsloo & Rughoonauth '14]

- The spatial part of the D3 worldvolume is parameterized by ϕ_3 and some portion of the Z_2-plane

$$\mathcal{D} = \left\{ Z_2 \middle| \left|\alpha - \frac{\epsilon}{Z_2}\right|^2 + |Z_2|^2 \right\} \leq 1$$

- The topology of the 3-manifold Σ can be read off from $\mathcal{D}$
- When $\epsilon = 0$ the base space is a disc and the giant is a round 3-sphere.



- For a simple pole:

$$\Sigma \simeq S^2 \times S^1 \text{ when } 0 < \epsilon < \epsilon_{crit}$$

$$\Sigma \simeq S^3 \text{ when } \epsilon > \epsilon_{crit}$$

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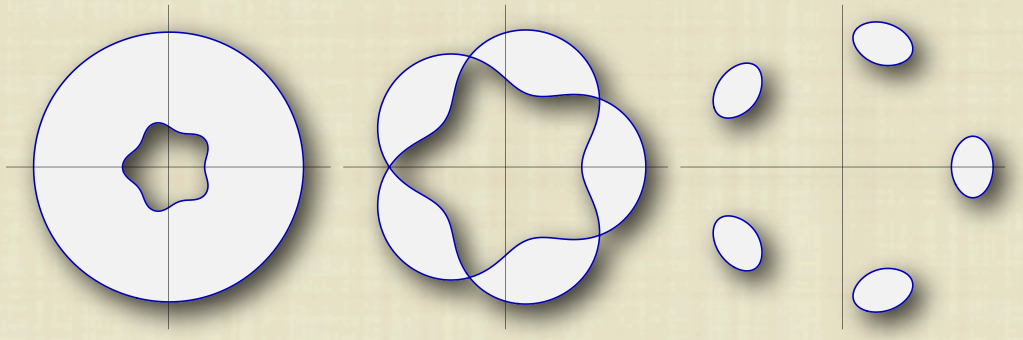
1/4-BPS giants (iii): higher order and multiple poles

- **(1,1,0) with higher order poles:** $f(Z_1, Z_2) = Z_1 - \alpha + \frac{\epsilon}{(Z_2)^N}$

  - The geometry exhibits and additional an N-fold symmetry under $Z_2 \to e^{i2\pi/N} Z_2$

  - For example, when N=5:



  - For a class (1,1,0) meromorphic function with order N pole:

$$\Sigma \simeq S^1 \times S^2 \quad \text{when} \quad 0 < \epsilon < \epsilon_{crit}$$

$$\Sigma \simeq \bigsqcup_{j=1}^{N} S^3 \quad \text{when} \quad \epsilon > \epsilon_{crit}$$

- **(1,n,0) with multiple simple poles:** $f(Z_1, Z_2) = Z_1 - \alpha + \sum_{j=1}^{n} \frac{\epsilon_j}{Z_2 - \beta_j}$

  - The topology is a **connected sum:** $\Sigma \simeq \sharp^n (S^2 \times S^1)$

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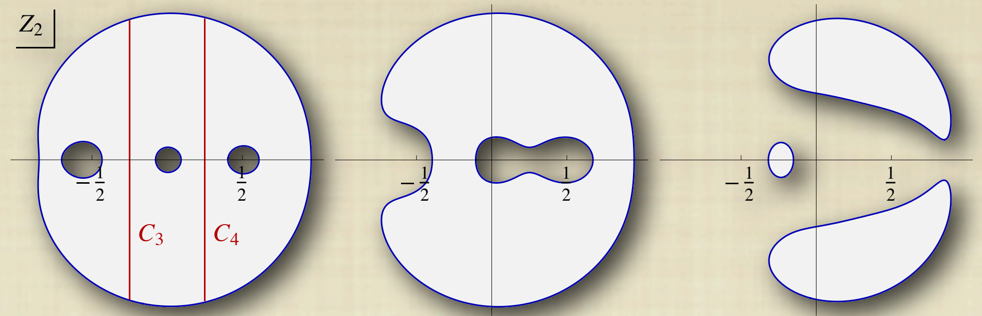
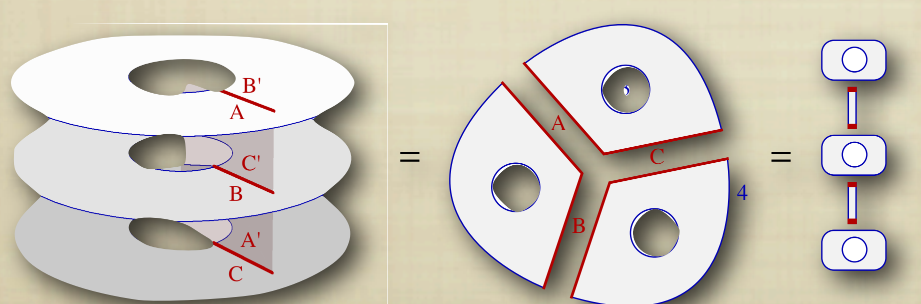
1/4-BPS giants (iii): higher order and multiple poles

- **(1,3,0) with simple poles:**



  - To see the topology, cut $\mathcal{D}$ along the red lines to isolate the poles.
  - Each line is an $[0,1]\times S^1$ with the S^1 shrinking to zero size on $\partial\mathcal{D}$
  - Gluing everything back together $\Sigma \simeq (S^2\times S^1)\sharp(S^2\times S^1)\sharp(S^2\times S^1)$
  - Increasing the residue is more complicated with multiple poles.
  - The $\epsilon\longrightarrow 0$ limit gives a clean interpretation as 4 intersecting D-branes with $f(Z_1,Z_2)\rightarrow\left(Z_1-\frac{1}{2}\right)Z_2\left(Z_2^2-\frac{1}{4}\right)$

- A useful check **(1,3,0) = (3,1,0):** $f(Z_1,Z_2)=(Z_1^3-\alpha^3)Z_2+\epsilon$
  - Here the area of the Z_2-plane occupied by the solution $\mathcal{D}$ is a 3-sheeted Riemann surface.
  - Each sheet is a disc with one hole and a branch cut from $Z_2=0\longrightarrow\epsilon/\alpha^3$

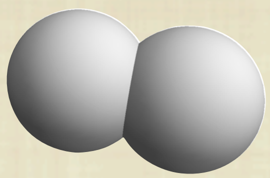
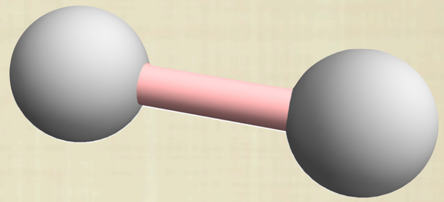


$$\Rightarrow \Sigma\simeq\sharp^3(S^2\times S^1)$$

1/4-BPS giants (iv): Jouet d'enfant

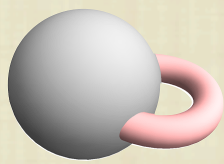
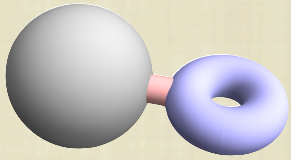
- The **connected sum** of two 3-manifolds $\mathcal{M}$ and $\mathcal{N}$, is formed by joining a point on $\mathcal{M}$ to one on $\mathcal{N}$ with a tube $S^2 \times [0,1] \simeq S^3$ with two punctures:

$$\mathcal{M}\sharp\mathcal{N} = \mathcal{M}\sharp S^3 \sharp\mathcal{N} \quad \longleftrightarrow$$

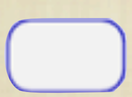
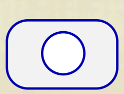
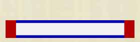
 = 

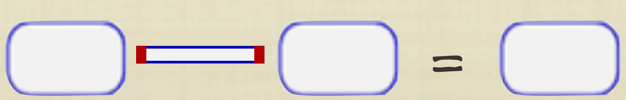
- Connecting the tube between two points on the **same manifold** adds an $S^2 \times S^1$:

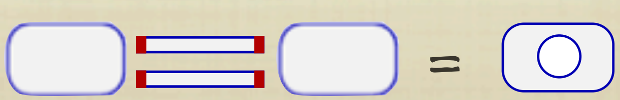
$$\mathcal{M} + (S^2 \times [0,1]) = \mathcal{M}\sharp(S^2 \times S^1) \quad \longleftrightarrow$$

 = 

- For more complicated topologies, it will be useful to introduce some **new notation**:

 = S^3;  = $S^2 \times S^1$;  = $S^2 \times [0,1]$

- Examples (trivial):  (3-sphere is the identity for prime factorisation)

 (adding a handle connects an $S^2 \times S^1$)

1/4-BPS giants (v): Jouer avec les jouets

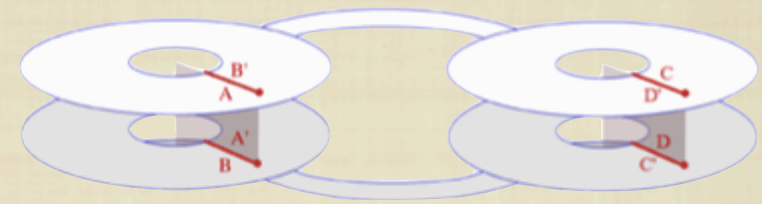
- Example (less trivial): The **(2,2,0)** polynomial $f(Z_1, Z_2) = (Z_1^2 - \alpha^2)(Z_2^2 - \beta^2) + \epsilon$

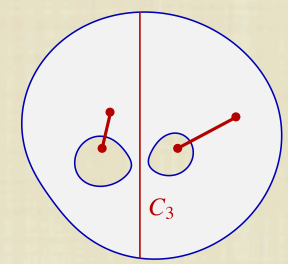
- Old way: - Solve $f(Z_1, Z_2) = 0$ for $Z_1(Z_2)$

  - Determine the branch points and cut lines

  - Draw the Riemann surfaces

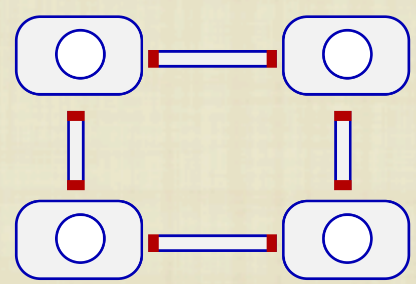
  - Cut and glue

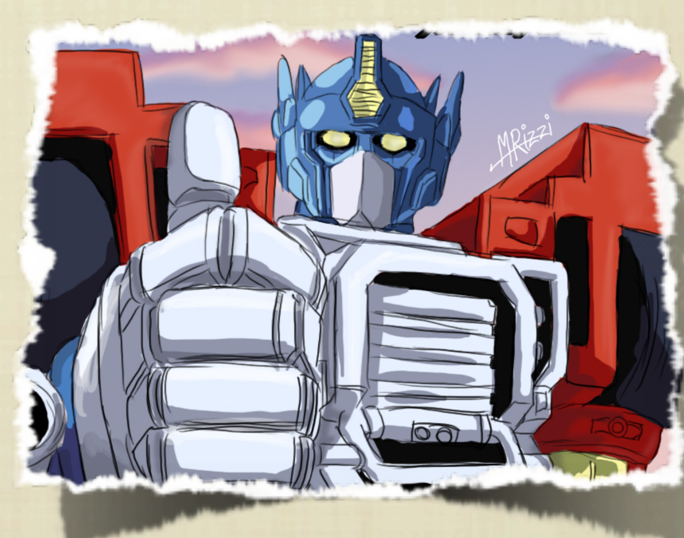




- New way: - Use prime factorisation

  - Count connections:





- Either way: $\Sigma = \sharp^5(S^2 \times S^1)$

1/4-BPS giants (vi): A general formula

- **(m,n,0) with all simple poles:**
  - Draw an m x n grid of 's
  - Connect all sites on the lattice with or

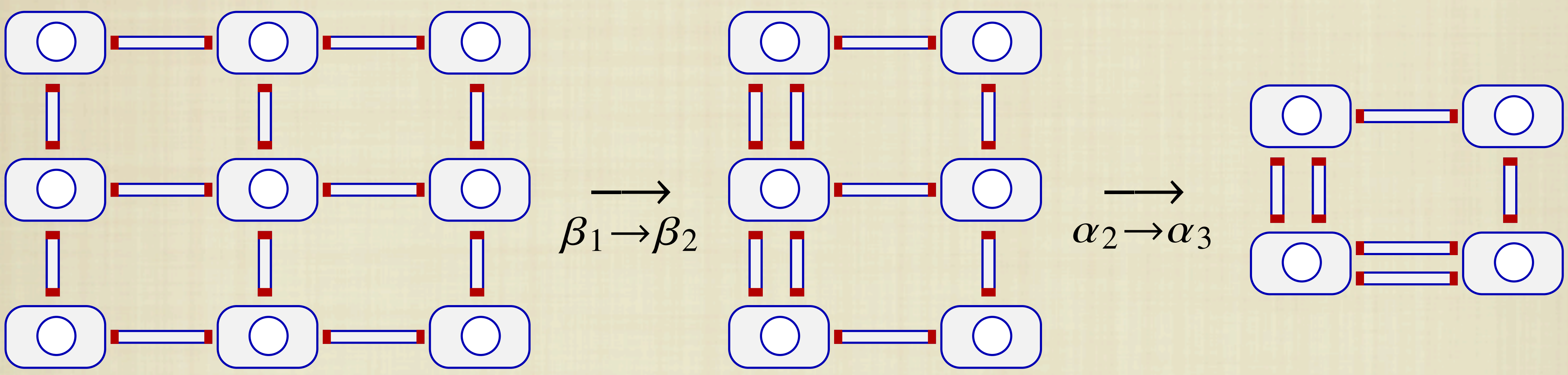
$$\Sigma = \sharp^K (S^2 \times S^1), \qquad K = mn + (m-1)(n-1)$$

- **(m,n,0) with higher order poles:**
  - Let n (m) be the number of poles in $Z_2 (Z_1)$ with total order N (M)
  - Draw an m x n grid of 's
  - Connect all sites on the lattice with M 's and N 's

$$\Sigma = \begin{cases} \sharp^K (S^2 \times S^1), \quad K = 1 + M(n-1) + N(m-1), & \epsilon < \epsilon_{crit}, \\ \bigsqcup_{i=1} \left[\sharp^{K_i} (S^2 \times S^1)\right] \bigsqcup_j S^3, & \epsilon > \epsilon_{crit} \end{cases}$$

1/4-BPS giants (vi): A general formula

- A worked example with **(3,3,0)**:



$M=m=N=n=3,\ K=13$ $\qquad$ $N=3,\ n=2,\ K=10$ $\qquad$ $M=N=3,\ m=n=2,\ K=7$

1/8-BPS giants

- A generic **(m,n,l)** holomorphic polynomial $f(Z_1, Z_2, Z_3)$ lacks the isometries required for our analysis, making a topological **classification of 1/8-BPS giants difficult**.

- However, our methods still hold for the class of functions **(m,n,m):**

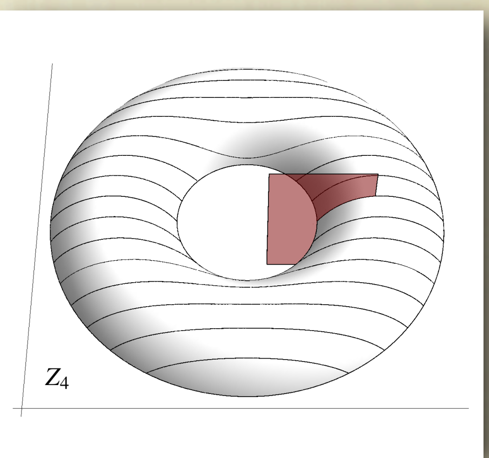
$$f(Z_1, Z_2, Z_3) = 1 + \sum_{k=1}^{m} \frac{\epsilon'_k}{Z_1 Z_3 - \gamma_k} + \sum_{j=1}^{n} \frac{\epsilon_j}{Z_2 - \beta_j}$$

- For this case, the topology of the 3-manifold (for small ϵ)

$$\Sigma = S^1 \times \natural^K T^2, \quad K = mn + (m-1)(n-1)$$

- A check with **(3,1,3):** $f(Z_1, Z_2, Z_3) = \left((Z_1 Z_2)^3 - \alpha^3\right) Z_2 + \epsilon$

  - This case is simple enough to study directly.
  - $\mathcal{D}$ is a 3-sheeted Riemann surface and $\Sigma = S^1 \times \natural^3 T^2$



Conclusions

- If $g(Z_1, Z_2)$ is a **meromorphic** function with number of poles in Z_1 and Z_2 counted by (m,M) and (n,N) respectively, the 1/4-BPS giant in $AdS_5 \times S^5$ given by $f(Z_1, Z_2) = 1 + \epsilon\, g(Z_1, Z_2)$ has topology given by the **prime decomposition**

$$\mathcal{M} = \#^K (S^2 \times S^1), \qquad K = 1 + M(n-1) + N(m-1)$$

- As ϵ is increased, K generically decreases and **the brane may break up** into several disjoint pieces, either 3-spheres or connected sums of $S^2 \times S^1$
- The case 1/8-BPS giants is generally much **more difficult** but we have some limited results
- We are left with **more questions** than answers:

  - Is there a corresponding systematic treatment of **1/8-BPS giants**?
  - What are the **operators dual** to these giants and how is the topology of the giant coded in the operators?
  - Is there a similar classification for giants in the **AdS4/CFT3** correspondence?

감사합니다 спасибі!

நன்றி Ndiyabulela! Ke a leboha!

σας ευχαριστώ! Ngeyabonga! Baie Dankie!

Ukhani! **Thank You!** Merci!

Obrigado!

Inkomu! Siyabonga! Danke!

¡Gracias धन्यवाद Grazie! ありがとう

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