Gromov–Witten invariants without mirror symmetry

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Based on

- arXiv:1208.6244 with H. Jockers, V. Kumar, J. Lapan, and M. Romo,
- arXiv:1305.3278 with J. Halverson and V. Kumar,
- work in progress with J. Halverson, H. Jockers, and J. Lapan
- parallel work by S. Hosono

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There were many subsequent extensions: inclusion of D-branes ("homological mirror symmetry"), mirrors beyond Calabi–Yau manifolds, and so on.

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Traditional approaches:

- **Indirect approach:** Mirror Symmetry
  Powerful and applicable for Calabi-Yau manifolds with known mirror manifolds
- **Direct approach:**
  Localization techniques on the moduli space of stable maps (Kontsevich; Givental; Lian, Liu, Yau)

Basic mirror correspondence

The 2D nonlinear $\mathcal{N} = (2,2)$ sigma model with Calabi–Yau target manifold X of dimension d has two kinds of marginal operators, with geometrical identifications: those in the Hodge group $H^{d-1,1}(X)$ are identified with tangent vectors to $\mathcal{M}_{\mathsf{cx}}$ (the "complex structure moduli"), while those in $H^{1,1}(X,\mathbb{C})$ are tangent vectors to the "(complexified) Kähler moduli" $\mathcal{M}_{\mathsf{Käh}}$ of X (which includes the B field as well as the class of the Kähler metric). Non-renomormalization theorems imply that the complex structure moduli receive no quantum corrections, while the Kähler moduli are corrected by instantons of the 2D theory.

Basic mirror correspondence

A mirror manifold of X is a Calabi–Yau manifold Y whose sigma model is identified with that of X after a change of $\mathcal{N} = (2,2)$ action. The change is such that $H^{d-1,1}(Y) \cong H^{1,1}(X,\mathbb{C})$ and in fact $\mathcal{M}_{\text{cx}}(Y) = \mathcal{M}_{\text{Käh}}(X)$. This provides a tool to study quantum corrections to $\mathcal{M}_{\text{Käh}}(X)$: identify the quantum corrected space with $\mathcal{M}_{\text{cx}}(Y)$ and study that space semiclassically.

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This provides a tool to study quantum corrections to $\mathcal{M}_{\mathrm{Käh}}(X)$: identify the quantum corrected space with $\mathcal{M}_{\mathrm{cx}}(Y)$ and study that space semiclassically.
This strategy led to the spectacular result of Candelas, de la Ossa, Green, and Parkes predicting the number of holomorphic maps $\mathbb{CP}^1 \to X$ of fixed degree by calculating with the mirror manifold.

Basic mirror correspondence

The sigma model is a good approximation at large volume, which is a certain "large radius limit" point on the boundary of $\overline{\mathcal{M}_{\text{Käh}}}$. Integral shifts of the B-field leave the moduli space invariant and affect the choice of coordinates near the large radius limit: if $D_1, \ldots, D_k$ are a basis of $H^{1,1}(X)$ with general element $\sum t_j D_j$, then the natural coordinates are $q_j = \exp(2\pi i t_j)$ (using the standard convention that the B-field is real and the Kähler class is imaginary).

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The classical terms in physical quantities such as correlation functions and the metric on $\mathcal{M}_{\text{Käh}}$ are computed directly in terms of $H^{1,1}(X)$, and involve quantities such as the d-fold intersection pairing $\int_X D_{j_1} \cup \cdots \cup D_{j_d}$.

Basic mirror correspondence

On the moduli space of the mirror manifold Y, we must locate the corresponding "large complex structure" point in $\overline{\mathcal{M}_{\text{cx}}(Y)}$ with corresponding coordinates $z_1, \ldots, z_k$. The basic mirror correspondence will then involve logarithms $\frac{1}{2\pi i} \log z_j$, reflecting the integer shifts in the B-field of X. This kind of logarithmic structure near the boundary of moduli space was already studying in mathematics as the appropriate limiting behavior of Hodge structures when the complex structure acquired singularities.

Metrics on moduli spaces

According to Tian and Todorov, the Kähler potential $\mathcal{K}$ for the so-called Weil–Petersson metric on the (complex structure) moduli space can be written as

$$\mathcal{K} = i^{\dim X} \int_X \Omega_z \wedge \overline{\Omega_z}.$$

With the help of the periods, this Kähler potential can be reexpressed in the form

$$i^{\dim X} \sum \left(\int_{A_j} \overline{\Omega_z} \int_{B^j} \Omega_z - \int_{A_j} \Omega_z \int_{B^j} \overline{\Omega_z} \right.$$

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The logarithmic behavior of those period integrals near the large complex structure limit point allows one to extract the individual periods by making an asymptotic expansion of the Kähler potential.

Metrics on moduli spaces

If we pick a basis $D_1, \ldots, D_k$ of $H^{1,1}$ with corresponding coordinates $t_1, \ldots, t_k$, then the general form of the function $\exp(-K)(t_1, \ldots, t_k)$ is

$$\left(\sum_{j=0}^{d} \int_X \alpha_j \cup \sum_{\ell_1,\ldots,i_{\ell-j}} (D_{\ell_1} \cup \cdots \cup D_{\ell_{d-j}}) t_{\ell_1} \cdots t_{\ell_{d-j}} \right) + O(e^{-t}),$$

for some cohomology classes $\alpha_j \in H^{2j}(X, \mathbb{Q})$ which specify the perturbative corrections, where $O(e^{-t})$ represents instanton corrections. We normalize things so that $\alpha_0 = 1 \in H^0(X)$ corresponds to the classical term.
On general grounds, the α_j take a universal form obtained by integrating polynomials in the curvature of the Calabi–Yau metric. Thus, the α_j can be expressed in terms of the Chern classes of (the tangent bundle of) X.

Note that, as usual, the Kähler potential is only well-defined up to adding the norm of a holomorphic function. In particular, we can choose Ω corresponding to the flat coordinates by making sure that the coefficient of $(\ln z)^3$ is 1. The coefficient of $(\ln z)^2$ then allows one to read off the Gromov–Witten invariants N_d. In the case of the quintic hypersurface, this is equivalent to the original computation of Candelas, de la Ossa, Green, and Parkes.

The JKLMR proposal

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- Then

$$Z_{S^2} = e^{-K(q,\bar{q})},$$

where $K(q,\bar{q})$ is the Kähler potential on $\mathcal{M}_{\text{Käh}}$, the (Kähler) moduli space of the IR conformal field theory.

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- This partition function has been calculated by Benini & Cremonesi and by Doroud, Gomis, Le Floch, & Lee (analogous to Nekrasov, Pestun, et al.). One remarkable feature is that it only depends on $\mathcal{M}_{\text{Käh}}$ and not on $\mathcal{M}_{\text{cx}}$.

Kähler moduli

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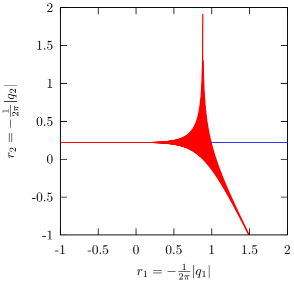
We also found the explicit form of quantum corrected $\mathcal{M}_{\text{Käh}}$. For example, for the resolution of a hypersurface in $\mathbb{P}^{(1,1,2,2,2)}$, the corrected moduli space is



Kähler moduli

Subsequent mathematical work of Givental and Lian–Liu–Yau – not unrelated to our instanton sums – gave a mathematical proof for the Candelas et al. formula which can be regarded as a statement purely on the Kähler side, using the mirror only for motivation.

Hypersurface $X_{d+2} \subset \mathbb{CP}^{d+1}$ and its Mirror

- (quintic case is $d = 3$)
- projective hypersurface
  $X_{d+2} = \{F_{d+2}(x_1, \ldots, x_{d+2}) = 0\} \subset \mathbb{CP}^{d+1}$

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- hypersurface–mirror: $Y_{d+2} \to \overline{Y_{d+2}}$, where $\overline{Y_{d+2}} = \{\sum x_j^{d+2} - (d+2)\psi \prod x_j = 0\}/(\mathbb{Z}_{d+2})^d \subset \mathbb{CP}^{d+1}/(\mathbb{Z}_{d+2})^d$

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- ordinary ("complex structure") moduli space, with parameter $z = (-(d+2)\psi)^{-d-2}$
- the identification between the two is made with the help of periods $\Phi(z) = \int_\Gamma \Omega_z$, for $\Gamma \in H_d(Y, \mathbb{Z})$ and Ω_z a holomorphic d-form on Y_z

Periods

There are two approaches to periods:

- Identify a particular cycle Γ for which $\Phi(z) := \int_\Gamma \Omega_z$ is single-valued as z approaches the large complex structure limit $(z \to 0)$. The cycle is closely related to $(S^1)^{d+1}$, a real torus inside $(\mathbb{C}^*)^{d+1} \subset \mathbb{CP}^{d+1}$, as will be explained.

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- Describe the differential operators annihilating $\int_\Gamma \Omega_z$ in terms of the intersection ring of the mirror, and build solutions by power series methods.
- To implement the first one, we write

$$\int_\Gamma \Omega_z = \operatorname{Res} \int_{(S^1)^{d+1}} \frac{\prod dx_j}{F_{d+2}}$$

and then explicitly integrate, obtaining an expression involving Gamma functions.

Periods

We illustrate the second approach in the case of the quintic, where $\Phi(z)$ satisfies an algebraic differential equation $\mathcal{D}\Phi = 0$, where, for an appropriate choice of Ω_z,

$$\mathcal{D} = \left(z\frac{d}{dz}\right)^4 - 5z\left(5z\frac{d}{dz}+1\right)\left(5z\frac{d}{dz}+2\right)\left(5z\frac{d}{dz}+3\right)\left(5z\frac{d}{dz}+4\right)$$

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It is easy to find a single power series solution near $z = 0$:

$$\Phi_0(z) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5}\, z^n$$

but the other three solutions are elusive.

The recursion relations implied by the equation lead one to a formal power series of the form

$$\Phi(z,\alpha) = \sum_{n=0}^{\infty} \frac{(5\alpha+1)(5\alpha+2)\cdots(5\alpha+5n)}{[(\alpha+1)(\alpha+2)\cdots(\alpha+n)]^5}\, z^{\alpha+n}\,;$$

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one finds that $\mathcal{D}(\Phi(z,\alpha)) = \alpha^4 z^\alpha$ and so we must have $\alpha^4 = 0$ in order to obtain a solution. In fact, the formal solution can be interpreted with α taken from the ring $\mathbb{C}[\alpha]/(\alpha^4)$ as follows: each coefficient

$$\frac{(5\alpha+1)(5\alpha+2)\cdots(5\alpha+5n)}{[(\alpha+1)(\alpha+2)\cdots(\alpha+n)]^5}$$

can be evaluated in that ring, and written as a polynomial in α of degree 3; moreover, z^α can be expanded as $1 + \alpha \ln z + \frac{1}{2}\alpha^2 (\ln z)^2 + \frac{1}{6}\alpha^3 (\ln z)^3$.

GLSM

A GLSM requires specification of a gauge group G, some chiral superfields Φ transforming under a representation of G, and a gauge-invariant superpotential W (a polynomial in Φ), with a Lagrangian

$$\mathcal{L}_{\text{GLSM}} = \int d^4\theta \bar{\Phi} e^{2Q(\Phi)V} \Phi + \\ + \left(\int d^2\theta \, W + \int d^2\tilde{\theta} \widetilde{W} + \text{c.c.} \right)$$

where $\widetilde{W} = (r + i\theta)\Sigma$ and Σ is the (twisted chiral) field strength of a vector superfield V.

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The GLSM for the Calabi–Yau hypersurface above is well-known: we use $G = U(1)$, $d + 2$ fields Φ of charge 1 and a field P of charge $(-d - 2)$, with superpotential

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If $r \gg 0$, the D-terms produce a bundle over $\mathbb{CP}^{d+1}$ with fiber coordinate P, and the F-terms restrict us to the zero section of that bundle $P = 0$ and to lie on the locus $F_{d+2}(x) = 0$, where x is the leading scalar in Φ.

Hypersurface Z_{S^2}

The perturbative part of Z_{S^2} for the Calabi–Yau hypersurface X of degree $d+2$ in $\mathbb{CP}^{d+1}$ can be written as:

$$Z_{\mathrm{pert}} = \mathrm{Res}_{\epsilon=0} \left[\frac{\pi^{d+1} \sin((d+2)\pi\epsilon)}{\sin(\pi\epsilon)^{d+2}} e^{2\pi i \epsilon (t-\bar{t})} \frac{\Gamma(1-(d+2)\epsilon)^2}{\Gamma(1-\epsilon)^{2d+4}} \right]$$

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We can rewrite this formula to the more suggestive form:

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With the identification $2\pi i\epsilon = H$, we evaluate the residue as

$$Z_{\rm pert} = \frac{1}{(2\pi i)^d}\int_X e^{H(t-\bar{t})}\frac{\Gamma(1-\frac{(d+2)H}{2\pi i})}{\Gamma(1-\frac{H}{2\pi i})^{d+2}}\frac{\Gamma(1+\frac{H}{2\pi i})^{d+2}}{\Gamma(1+(d+2)\frac{H}{2\pi i})},$$

which becomes

$$Z_{\rm pert} = \frac{1}{(2\pi i)^d}\int_X e^{H(t-\bar{t})}\frac{\hat{\Gamma}_c(X)}{\overline{\hat{\Gamma}_c(X)}},$$

where ...

The Γ-class

$\hat{\Gamma}_c$ is a multiplicative characteristic class based on the power series expansion of $\Gamma(1+\frac{z}{2\pi i})$. $\hat{\Gamma}_c$ of $\mathbb{CP}^{d+1}$ is simply given by

$$\hat{\Gamma}_c(T\mathbb{CP}^{d+1}) = \hat{\Gamma}_c(H)^{d+2} = \Gamma(1+\frac{H}{2\pi i})^{d+2} \ ,$$

in terms of the hyperplane class H of $\mathbb{CP}^{d+1}$.
The $\hat{\Gamma}_c$ class of a Calabi–Yau hypersurface X of degree $d+2$ becomes

$$\hat{\Gamma}_c(X) = \frac{\hat{\Gamma}_c(T\mathbb{CP}^{d+1})}{\hat{\Gamma}_c(NX)} = \frac{\Gamma(1+\frac{H}{2\pi i})^{d+2}}{\Gamma(1+(d+2)\frac{H}{2\pi i})} \ .$$

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Doing this for a collection of well-understood examples allows the coefficient Chern classes α_j to be calculated.

The Γ-class

This gives some evidence in favor of a proposal by Iritani and Katzarkov–Kontsevich–Pantev to modify the usual identification

$$E \mapsto \mathrm{ch}(E)\sqrt{\mathrm{Td}_X}$$

of K-theory with cohomology, used in describing the integral structure in mirror symmetry (and in specifying D-brane charges), to

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More direct evidence of this proposal will be in forthcoming work of Hori and Romo, who compute the partition function on a disk and a cylinder.

GLSM for Gulliksen-Negård 3-fold

- Spectrum of the Gulliksen-Negård (GN) 3-fold:

chiral field	$\Phi_{a=1,\ldots,8}$	$P_{i=1,\ldots,4}$	$X_{j=1,\ldots,4}$
$U(1)$ charge	$+1$	-1	0
$U(2)$ charge	1_0	$\square_{+1}$	$\overline{\square}_{-1}$

- Gauge invariant PAX superpotential (R-charge 2):

$$W(\Phi, P, X) = \operatorname{tr} P\, A(\Phi)\, X$$

where $A(\Phi)$ is a 4×4 matrix with generic linear entries in Φ

Determinantal GN 3-fold

- D-term & F-terms from the PAX-GN-GLSM:
  $\sum_a |\Phi_1|^2 - \sum_i P_i^\dagger \cdot P_i = \operatorname{Im} z_0$
  $2P \cdot P^\dagger - 2X^\dagger \cdot X = \operatorname{Im} z_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
  $A(\Phi) \cdot X = 0 \qquad P \cdot A(\Phi) = 0 \qquad \operatorname{tr} P \cdot \partial_{\Phi_a} A(\Phi) \cdot X = 0$
- Geometric phase: $\operatorname{Im} z_0 >> 0$, $\operatorname{Im} z_1 >> 0$.

Predictions & Checks: GN 3-fold

$\widetilde{N}_{m_0,m_1}$	$m_0 = 0$	$^1/_2$	1	$^3/_2$	2
$m_1 = 0$	-		56		0
$^1/_2$		192		896	
1	56		2 544		23 016
$^3/_2$		896		52 928	
2	0		23 016		1 680 576
$^5/_2$		192		813 568	
3	0		41 056		35 857 016
$^7/_2$		0		3 814 144	
4	0		23 016		284 749 056
$^9/_2$		0		6 292 096	
5	0		2 544		933 789 504
$^{11}/_2$		0		3 814 144	
6	0		56		1 371 704 192
$^{13}/_2$		0		813 568	
7	0		0		933 789 504
$^{15}/_2$		0		52 928	
8	0		0		284 749 056
$^{17}/_2$		0		896	
9	0		0		35 857 016

Other directions & Conclusions

- Partition function calculates exact Kähler potential of quantum Kähler moduli space:
- The phase structure of the moduli space can be analyzed using this formalism.
- We have calculated genus 0 Gromov–Witten invariants; what about other genus?
- Can mirror manifolds be found in nonabelian cases?