Liouville and JT quantum gravity - holography and matrix models

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Introduction

Many developments in lower-dimensional gravity (JT dilaton gravity specifically):

- Higher genus and multi-boundary amplitudes: important to understand very-late time correlators, replica wormholes ...

  Saad-Shenker-Stanford '18, '19, Stanford-Witten '19 ... Almheiri et al. '19, Penington et al. '19 ...
- Exact quantum solution of boundary correlators and their interpretation in terms of gravitational physics

  Bagrets-Altland-Kamenev '16, '17, TM-Turiaci-Verlinde '17, Kitaev-Suh '18,'19, Yang '18,

  Blommaert-TM-Verschelde '18, Iliesiu-Pufu-Verlinde-Wang '19, Saad '19 ...
- ... wormhole traversability Maldacena-Stanford-Yang '17, Maldacena-Qi '18, finite cutoff Iliesiu-Kruthoff-Turiaci-Verlinde '20, Stanford-Yang '20, explicit island calculations, bulk reconstruction ...

$\rightarrow$ Would be interesting to extend our class of solvable models to find out how generic these lessons are

GOAL: discuss 2d Liouville gravity/minimal string in same language

Review - JT gravity action and partition function

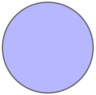
Jackiw-Teitelboim (JT) 2d dilaton gravity

$S = \frac{1}{16\pi G}\int d^2x\sqrt{-g}\Phi(R+2) + \frac{1}{8\pi G}\int d\tau\sqrt{-\gamma}\Phi_{bdy}K$

Teitelboim '83, Jackiw '85

Negative cosmological constant → AdS_2 space

Disk Partition function Maldacena-Stanford '16, Stanford-Witten '17

$Z(\beta) =$  $= \int_0^{+\infty} dk(k\sinh 2\pi k)e^{-\beta k^2}$

Energy variable $E = k^2$

Thermodynamic limit (saddle):

$\rho(E) \sim e^{2\pi\sqrt{E}} \Rightarrow \sqrt{E} = \frac{\pi}{\beta}$

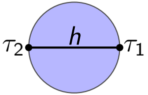
Matches with black hole first law (mass vs Hawking temperature) for JT black hole

Review - JT gravity two-point function

Boundary two-point function (of some boundary CFT matter operators $\mathcal{O}^h(\tau)$ coupled to gravity):

Bagrets-Altland-Kamenev '16, '17, TM-Turiaci-Verlinde '17, Blommaert-TM-Verschelde '18, Yang '18, Kitaev-Suh '18, '19, Iliesiu-Pufu-Verlinde-Wang '19 ...

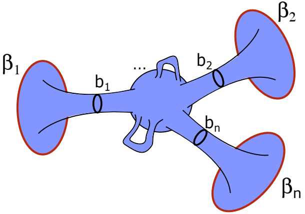
$$\left\langle \mathcal{O}^h(\tau_1)\mathcal{O}^h(\tau_2)\right\rangle_\beta =$$



$$= \int_0^{+\infty} dk_1 (k_1 \sinh 2\pi k_1) \int_0^{+\infty} dk_2 (k_2 \sinh 2\pi k_2)\, e^{-\tau k_1^2 - (\beta-\tau)k_2^2} \frac{\Gamma\left(h \pm ik_1 \pm ik_2\right)}{\Gamma(2h)}$$

$(\tau = \tau_2 - \tau_1)$
$\pm$ sign convention = take product of all cases

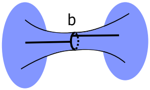
Multiboundary amplitudes Saad-Shenker-Stanford '19:



$$= e^{\chi S_0} \int_0^\infty \prod_{i=1}^n db_i b_i Z_{\text{JT}}(\beta_i, b_i)\, V_{g,n}(\mathbf{b})$$

Ingredients:

- Single-trumpet $Z_{\text{JT}}(\beta, b) = \int_0^{+\infty} dk\, e^{-\beta k^2} \cos bk$
  Disk partition function with a (hyperbolic) defect b
- measure $d\mu(b) = dbb$
  Gluing tubes with twist: $0 \dots b$



- Weil-Petersson (WP) volume $V_{g,n}(\mathbf{b}) \equiv V_{g,n}(b_1 \dots b_n)$
  volume of moduli space of Riemann surfaces of genus g with n geodesic boundaries of length b_i
  multivariate polynomials in b_i^2, known recursively

Liouville gravity: Definition

Non-critical string from conformal matter coupled to 2d gravity, or critical string with a 2d Liouville + matter + ghost CFT Polyakov '81, David '88, Distler-Kawai '89 . . .

Liouville gravity: $S_L + S_M + S_{\rm gh}$
with conformal anomaly constraint $c_M + c_L + c_{\rm gh} = 0$

- Liouville action: $S_L = \frac{1}{4\pi} \int_\Sigma \left[(\hat{\nabla}\phi)^2 + Q\hat{R}\phi + 4\pi\mu e^{2b\phi} \right]$
  $Q = b + b^{-1}$, $c_L = 1 + 6Q^2 > 25$
  Arises from conformal factor $g_{\mu\nu} = e^{2b\phi}\hat{g}_{\mu\nu}$ of 2d gravity
- For most of talk: $S_M =$ arbitrary CFT with $c_M < 1$
  We can specify to (q, p) minimal model ($b^2 = q/p$)
  $\Rightarrow$ Minimal string
  Minimal string has (multi)-matrix model description
- $S_{\rm gh}$ is usual bc-ghost theory with $c_{\rm gh} = -26$

Liouville gravity: Fixed-length boundaries

$\Rightarrow$ Reinterpret worldsheet amplitudes as quantum gravity amplitudes

$\Rightarrow$ Interested in holography $\to$ worldsheet with boundary, of fixed (physical) length $= \beta$ to study thermal amplitudes

For Liouville piece, FZZT-brane boundary Fateev-Zamolodchikov[2] '00:

$S_\partial = \frac{1}{2\pi} \oint_{\partial\Sigma} \left[Q\hat{K}\phi + 2\pi\mu_B e^{b\phi} \right]$

$\mu_B =$ boundary cosmological constant

When viewing the theory as 2d quantum gravity, Liouville field related to metric $g_{\mu\nu}$: $ds^2 = e^{2b\phi} dz d\bar{z}$

$\Rightarrow$ Boundary length $= \oint e^{b\phi}$

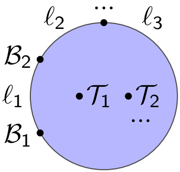
In path integral $\int_{i\mathbb{R}} d\mu_B e^{\mu_B \ell} \times e^{-S_L - S_\partial}$ yields $\delta\left(\ell - \oint e^{b\phi}\right)$, a delta-function on boundary length (e.g. Kostov '02)

Generalization: piecewise constant μ_B allows boundary with fixed length segments $\ell_1, \ldots \ell_n$

Liouville gravity: Operators

Amplitude with insertions of bulk and boundary tachyon vertex operators:

$$\mathcal{A}_{\mathcal{O}}(\ell_1, \dots, \ell_n) \quad = $$

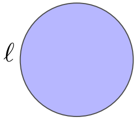


Bulk and boundary operators:

- $\mathcal{T}_\alpha = \int d^2z\, \mathcal{O}_{\mathrm{M}}(z,\bar{z}) e^{2\alpha\phi(z,\bar{z})} \sim c\bar{c}\, \mathcal{O}_M V_\alpha$
  Matter operators $\mathcal{O}_M, \Delta_M$ are dressed by Liouville operators $V_\alpha = e^{2\alpha\phi}, \Delta_\alpha = \alpha(Q-\alpha)$ where $\Delta_M + \Delta_\alpha = 1$

- $\mathcal{B}_\beta = \oint dx\, \Phi_{\mathrm{M}}(x) e^{\beta\phi(x)} \sim c\, \Phi_M B_\beta$
  Matter operators Φ_M, Δ_M are dressed by Liouville boundary operators $V_\beta = e^{\beta\phi}, \Delta_\beta = \beta(Q-\beta)$ where $\Delta_M + \Delta_\beta = 1$

Disk partition function



Known disk amplitude with FZZT brane boundary (fixed μ_B)

Fateev-Zamolodchikov^2 '00, Seiberg-Shih '03

$Z(\mu_B) \sim \cosh \frac{2\pi s}{b}$ where $\mu_B(s) = \kappa \cosh 2\pi b s$, $\quad \kappa \equiv \frac{\sqrt{\mu}}{\sqrt{\sin \pi b^2}} = 1$

Transform to fixed-length basis:

$$\boxed{Z(\ell) \sim \int_0^\infty ds \, \sinh(2\pi b s) \sinh\left(\frac{2\pi s}{b}\right) e^{-\ell \cosh(2\pi b s)}}$$

JT limit: $b \to 0$ with $\ell \sim \frac{\ell_{\rm JT}}{b^4} \to +\infty$ Saad-Shenker-Stanford '19, TM-Turiaci '20

large boundary length limit ($s = bk$)

$Z(\ell) \to \int_0^\infty dk \, (k \sinh 2\pi k) \, e^{-\ell_{\rm JT} k^2}$

Disk partition function: density of states

Back to full Liouville gravity result

Interpret boundary as thermal holographic boundary:

$\rightarrow$ Interpret $\ell = \beta$ as inverse temperature

$Z(\beta) \sim \int_1^\infty dE \; e^{-\beta E} \rho_0(E), \qquad \rho_0(E) = \sinh\left(\frac{1}{b^2}\text{arccosh} E\right)$

Thermodynamic limit (saddle):

$$\sqrt{E^2 - 1} = \frac{1}{b^2 \beta}$$

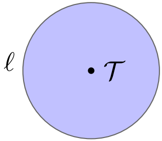
IR: $E = 1 + E_{JT} \Rightarrow \sqrt{E_{JT}} \sim \beta^{-1}$, the JT black hole first law

UV: $E \sim \beta^{-1}$

Holography - UV/IR connection: qualitative change of boundary region of bulk geometry (not aAdS like JT gravity)

Bulk one-point function

Transforming to the fixed length basis for a single bulk insertion:

$\langle \mathcal{T}_\alpha \rangle_\ell =$ 

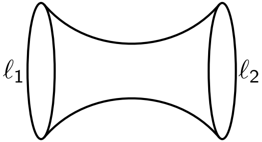
$$\mathcal{T}_\alpha = c\bar{c}\,\mathcal{O}_M e^{2\alpha\phi}$$

$$\langle \mathcal{T}_\alpha \rangle_\ell = \frac{2}{b}\int_0^\infty ds\,\cos(4\pi P s)\, e^{-\ell\cosh(2\pi b s)} \sim K_{\frac{2iP}{b}}(\ell)$$

where $\alpha = Q/2 + iP$

Multi-boundary amplitudes: Euclidean wormhole

Two-boundary amplitude in Liouville gravity (annulus) Martinec '03



$$= \int_{i\mathbb{R}^+} d\tau \, Z_L \, Z_M \, Z_G$$

Z_L, Z_M, Z_G are cylinder partition functions between suitable Cardy boundary states, for fixed worldsheet modulus τ

Transform to fixed length boundaries explicitly calculable

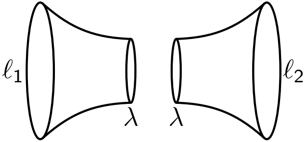
Specify to $(2, p)$ minimal string

(because it has a 1-matrix description we will exploit on next slide)

$\langle Z(\ell_1) Z(\ell_2) \rangle = \frac{2}{\pi} \int_0^\infty d\lambda (\lambda \tanh \pi\lambda) \; K_{i\lambda}(\ell_1) K_{i\lambda}(\ell_2)$

$\rightarrow$ readable as two bulk 1-pt functions with $\lambda = \frac{2P}{b}$ glued together

$$= \int d\mu(\lambda)$$



Multi-boundary amplitudes: genus zero

Multi-loop amplitudes from matrix model Moore-Seiberg-Staudacher '91:

$$\left\langle \prod_{i=1}^n Z(\ell_i) \right\rangle_{g=0} = -\frac{\sqrt{\ell_1 \ldots \ell_n}}{2\pi^{n/2}} \left(\frac{\partial}{\partial x}\right)^{n-3} u'(x) e^{-u(x)(\ell_1 + \ldots + \ell_n)} \Big|_{u \to 1}$$

with $\rho_0(E) = \frac{1}{2\pi} \int_{E_0}^{E} \frac{du}{\sqrt{E-u}} f(u)$ where $\partial_x u = -f(u)^{-1}$

Banks-Douglas-Seiberg-Shenker '90, Johnson '19-'20

$$= \prod_{i=1}^n \int_0^{+\infty} d\lambda_i (\lambda_i \tanh \pi\lambda_i) K_{i\lambda_i}(\ell_i) V_{0,n}(\boldsymbol{\lambda})$$

Defines quantity $V_{0,n}(\boldsymbol{\lambda})$

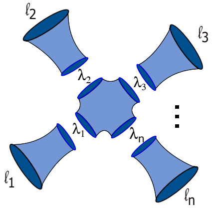
- Sensible since this is invertible integral transform
- Rewriting in terms of (glued) bulk one-point functions $\langle \mathcal{T}_{\alpha_i} \rangle_{\ell_i} \sim K_{i\lambda_i}(\ell_i)$ where $\alpha_i = Q/2 + ib\lambda_i/2$
- $V_{0,n}(\boldsymbol{\lambda})$ is a deformed genus 0 Weil-Petersson volume
- Explicit formula (in terms of Legendre function):

$$V_{0,n}(\boldsymbol{\lambda}) \sim \left(\frac{\partial}{\partial x}\right)^{n-3} u'(x) \prod_{i=1}^{n} P_{-\frac{1}{2} - i\lambda_i}(u(x)) \Big|_{u \to 1}$$

Multi-boundary amplitudes: generalization and summary

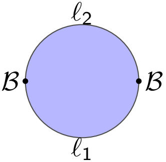
Higher genus corrections obtained through topological recursion relations Eynard '04, Eynard-Orantin '07
$\rightarrow$ generic proof that $V_{g,n}(\boldsymbol{\lambda})$ is multivariate polynomial in λ_i^2

Summary:
$\left\langle \prod_{i=1}^n Z(\ell_i) \right\rangle_g = \prod_{i=1}^n \int_0^\infty d\lambda_i (\lambda_i \tanh \pi\lambda_i)\, V_{g,n}(\boldsymbol{\lambda})\, \langle \mathcal{T}_{\alpha_i} \rangle_{\ell_i}$



$\rightarrow$ JT limit yields back the Saad-Shenker-Stanford procedure in JT gravity / hyperbolic geometry
Gluing measure: $d\lambda\lambda \tanh \pi\lambda \rightarrow db_{\mathrm{JT}} b_{\mathrm{JT}} \quad (\lambda \sim b_{\mathrm{JT}}/b^2)$
$\rightarrow \sim$ q-deformed version of hyperbolic geometry

$$\langle \mathcal{B}_\beta \mathcal{B}_\beta \rangle_{\ell_1,\ell_2} =$$



$$\mathcal{B}_\beta = c\, \Phi_M e^{\beta\phi}$$

Start with Liouville boundary two-point function Fateev-Zamolodchikov^2 '00

$d(\beta|s_1, s_2) \sim \frac{\Gamma_b(2\beta-Q)\Gamma_b^{-1}(Q-2\beta)}{S_b(\beta\pm is_1\pm is_2)}$ where $\mu_{Bi} = \cosh 2\pi b s_i$

$\rightarrow$ Transform to fixed length basis for both μ_{B1} and μ_{B2}:

$$\sim \int_0^{+\infty} ds_1 ds_2\, \rho_0(s_1)\, \rho_0(s_2)\, e^{-\cosh 2\pi b s_1 \ell_1} e^{-\cosh 2\pi b s_2 \ell_2} \frac{S_b(\beta_M \pm is_1 \pm is_2)}{S_b(2\beta_M)}$$

where $\rho_0(s) = \sinh(2\pi b s) \sinh\left(\frac{2\pi}{b} s\right)$ and $\beta_M = b - \beta$

JT limit ($b \rightarrow 0$, $\beta_M = bh$): $S_b(bx) \sim \Gamma(x)$:

$$\int_0^{+\infty} dk_1\, dk_2\, (k_1 \sinh 2\pi k_1)\, (k_2 \sinh 2\pi k_2)\, e^{-k_1^2 \ell_{\mathrm{JT}1}}\, e^{-k_2^2 \ell_{\mathrm{JT}2}} \frac{\Gamma(h \pm ik_1 \pm ik_2)}{\Gamma(2h)}$$

Quantum group interpretation of Liouville gravity

Ingredients in JT gravity amplitudes have known interpretation in terms of $\mathrm{SL}(2,\mathbb{R})$ group theory (Plancherel measure, Casimir, 3j symbols, 6j symbols) Blommaert-TM-Verschelde '18-'19, Iliesiu-Pufu-Verlinde-Wang '19

Similarly, Liouville gravity amplitudes arise from $\mathrm{SL}_q(2,\mathbb{R})$ quantum group $q = e^{\pi i b^2}$

Continuous (self-dual) irreps Ponsot-Teschner '99 ...:

- Casimir operator $C_s \equiv \cosh 2\pi b s$ matches energy
- Plancherel measure: $d\mu(s) = ds \sinh(2\pi b s) \sinh\left(\frac{2\pi s}{b}\right)$ matches density of states
- 3j-symbols (squared) of suitable representations of $\mathrm{SL}_q(2,\mathbb{R})$ match with $\frac{S_b(\beta_M \pm i s_1 \pm i s_2)}{S_b(2\beta_M)}$ TM-Turiaci '20

Conclusion and Outlook

Investigated fixed length amplitudes of Liouville gravity

$\rightarrow$ Phrased in same language as JT gravity, with JT limit $b \rightarrow 0$

$\rightarrow$ String worldsheet genus expansion $\equiv$ multi-universe / spacetime expansion

To understand better:

- $\mathcal{N} = 1$ Liouville supergravity worked out to some extent TM '20 (fixed length disk amplitudes have sJT limit)
- Dilaton gravity interpretation with sinh dilaton potential Seiberg-Stanford (unpublished), TM-Turiaci '20
  $\rightarrow$ sinh potential related to q-deformation of $\mathfrak{sl}(2)$ algebra
  $\rightarrow$ Understand holography directly in this q-deformed context
- More general dilaton gravity models: deformations Witten '20, Maxfield-Turiaci '20, Poisson-sigma model description Ikeda '93,...,Verlinde '21

Thank you!