BPS bound states and quivers

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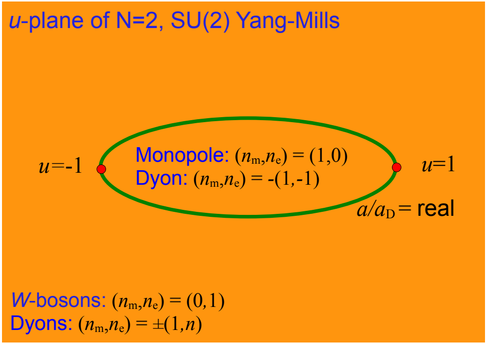
Based on work with Boris Pioline and Ashoke Sen

Motivation: $\mathcal{N} = 2$ gauge theory

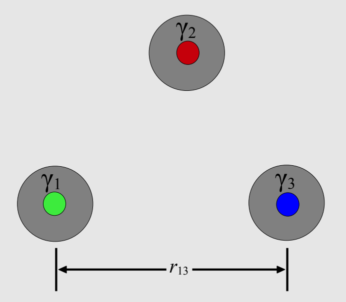
Most $\frac{1}{2}$ BPS states at weak coupling are bound states of a finite number of elementary BPS states

Seiberg, Witten (1994); Bilal, Ferrari (1996); Lee, Yi (1998); Lerche (2000); Alim, Cecotti, Cordova, Espahbodi, Rastogi, Vafa (2011); Xi (2012); Cecotti, Del Zotto (2013), ...



Motivation: $\mathcal{N} = 2$ supergravity

Multi-center black holes $\{\gamma_i\}$:



Problem:

Microscopic counting ($g_s = 0$) of BPS black holes does not accurately distinguish single center and multi-center solutions with charge γ

Maldacena, Strominger, Witten (1997); Strominger, Gaiotto, Yin (2006); Denef, Moore (2007); De Boer, Denef, El-Showk, Messamah, Van den Bleeken (2008);...

$\Rightarrow$ Understanding of bound states is crucial for precision tests of black hole entropy

Multi-center black hole: Denef equations

$\mathcal{N} = 2$ BPS equations of motion require the distances $r_{ij} \in \mathbb{R}_+$ to satisfy:

$$\sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{\gamma_{ij}}{r_{ij}} = c_i(\{\gamma_k\}; t)$$

- $\gamma_{ij} = \langle \gamma_i, \gamma_j \rangle \in \mathbb{Z}$: DSZ innerproduct
- $c_i(\{\gamma_j\}; t) \in \mathbb{R}$: stability parameter

Denef (2000)

Phase space $M(\{\gamma_i\}, \{c_i\})$:

- parametrizes $\mathbf{r}_i \in \mathbb{R}^3, i = 1, \ldots, N$
- has dimension $2N - 2$

De Boer, El-Showk, Messamah, Van den Bleeken (2008)

Multi-center black hole: Two aspects

Wall-crossing:

Solutions might decay or recombine upon varying $c_i \in \mathbb{R}$:

Denef (2000); Denef, Moore (2007),...

For example $N = 2$: $\lim_{c_1 \to 0} r_{12} = \lim_{c_1 \to 0} \frac{\gamma_{12}}{c_1} = \pm\infty$

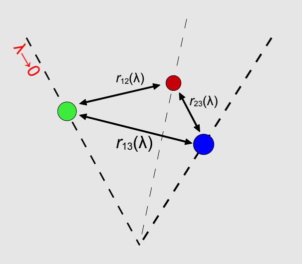
Scaling solutions:

Centers could get arbitrarily close, depending on $\{\gamma_i\}$

Bena, Wang, Warner (2006); Denef, Moore (2007),...

For example $N = 3$: If $\gamma_{12} + \gamma_{23} \geq \gamma_{31}$, and cyclic perm. $\Rightarrow$

$$\lim_{\lambda \to 0} r_{ij}(\lambda) = \lambda\gamma_{ij} + \mathcal{O}(\lambda^2) \in M(\{\gamma_i\}, \{c_i\})$$



Multi-center black hole: Index

BPS index: $$\Omega(\gamma;t) = -\frac{1}{2}\mathrm{Tr}_{\mathcal{H}(\gamma,t)}\,(2J_3)^2\,(-1)^{2J_3}$$

Decomposition formula:

$$\bar{\Omega}(\gamma;t) = \sum_{\substack{\sum_i N_i\gamma_i=\gamma,\\ \gamma_i\neq\gamma_j, i\neq j}} g(\{N_i\gamma_i\};\{c_i\}) \prod_j \frac{\bar{\Omega}_S(\gamma_j)^{N_j}}{N_j!}$$

- $\bar{\Omega}_S(\gamma_j) = \sum_{n|\gamma_j} \frac{\Omega_S(\gamma_j/n)}{n^2}$: rational invariant associated to center j Familiar from Schwinger pair creation and D-instanton measure. Gopakumar, Vafa (1998); Kontsevich, Soibelman (2008); Joyce, Song (2008); Kim, Park, Wang, Yi (2011); ...See also P. Yi's talk
- central charges: $\arg(Z(\gamma_i,t)) \in [\phi,\phi+\pi)$
- $\frac{\bar{\Omega}_S(\gamma)^N}{N!}$: Maxwell-Boltzmann distribution
- $g(\{N_i\gamma_i\};\{c_i\}) \in \mathbb{Z}$: # of "binding" states

JM, Pioline, Sen (2010)

Main question: How to interpret and determine $g(\{N_i\gamma_i\};\{c_i\})$?

Quiver quantum mechanics: Field content

Low energy excitations of $\frac{1}{2}$ BPS solution

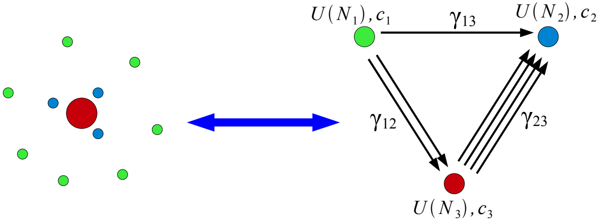
$\Downarrow$

$\mathcal{N} = 4$ quiver quantum mechanics

Denef (2002); Denef, Moore (2007); Familiar from BPS monopoles: Cederwall, Ferretti, Nilsson, Salomonson (1995); Sethi, Stern, Zaslow (1995); Gauntlett, Harvey (1996); Gauntlett, Kim, Park, Yi (2000),. . .

Field content is:

- determined by multiplicities $\{N_i\}$ and innerproducts $\{\gamma_{ij}\}$:
  - vector multiplets $(\mathbf{r}_i, A_i, \lambda_i)$ with gauge group $U(N_i)$
  - $|\gamma_{ij}|$ bifundamental chiral multiplets $(\phi^a_{ij}, F^a_{ij}, \psi^a_{ij}), a = 1, \ldots, |\gamma_{ij}|$
- parametrized by a quiver $(\vec{N}, \vec{c})$:



Quiver quantum mechanics: Higgs & Coulomb

1. $g_s \ll 1$: Higgs branch

- D-term eqs: $$\sum_{j,i\to j}\sum_{a=1}^{\gamma_{ij}} \phi_{ij}^a(\phi_{ij}^a)^\dagger - \sum_{j,j\to i}\sum_{a=1}^{\gamma_{ji}} (\phi_{ij}^a)^\dagger\phi_{ij}^a = c_i \mathbf{1}_{N_i}$$

  F-term eqs: $$\frac{\partial W(\{\phi_{ij}^a\})}{\partial\phi_{kl}^b} = 0$$

  $\Rightarrow$ equations for Kähler quotient $\mathcal{M}(\vec{N};\vec{c})$
- Witten index: $\mathrm{Tr}_{\mathcal{H}_{\mathrm{Higgs}}(\vec{N},\vec{c})}(-1)^F = \chi(\mathcal{M}(\vec{N};\vec{c}))$

  No oriented loops: Reineke (2002)

2. $g_s \gg 1$: Coulomb branch

- vector multiplets: $$\sum_{\substack{j=1\\ j\neq i}}^{N} \frac{\gamma_{ij}}{r_{ij}} = c_i(\{\gamma_j\};t)$$

  Denef (2002); Kim, Park, Wang, Yi (2011),...

N.B.: BPS solutions require more conditions to be physical, in particular regularity of metric $\Rightarrow$ assume that central charges $Z(\gamma_i,t)$ are nearly aligned

Quiver quantum mechanics: Abelianization formula

Maxwell-Boltzmann distribution

$$\Downarrow$$

$g(\{N_i\gamma_i\};\{c_i\})$: # of ground states of Abelian quiver theory

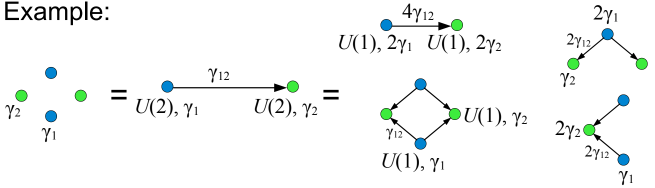
$\Rightarrow$ non-Abelian from Abelian:

1. degeneracies: $\Omega_S(k\gamma_i) = \delta_{k,1}$
2. decomposition formula implies:

$$\chi(\mathcal{M}_Q(\vec{N};\vec{c})) \sim \sum_{Q'} \chi(\mathcal{M}_{Q'}(\vec{1}_{N'};\vec{c}'))$$

JM, Pioline, Sen (2010)

Example:



Mathematical studies by Mozgovoy, Okada, Reineke, Stoppa, Weist,...

Space-time: Equivariant index

Goal: determine $g(\{\gamma_i\};\{c_i\})$ in space-time

$g(\{\gamma_i\},\{c_i\})$ is the (twisted) Dirac index of the space $M(\{\gamma_i\},\{c_i\})$

De Boer, El-Showk, Messamah, Van den Bleeken (2008); ...

General computation is feasible by refining the index:

$$\Omega(\gamma, y; t) = \frac{\mathrm{Tr}_{\mathcal{H}_{\mathrm{BPS}}(\gamma,t)}\,(-y)^{2J_3}}{-y^{-1}+2-y}$$

$\Rightarrow$ $g(\{\gamma_i\},y;\{c_i\})$: equivariant Dirac index of $M(\{\gamma_i\},\{c_i\})$

Index theorem: $g(\{\gamma_i\},y;\{c_i\}) = \int_M \mathrm{Ch}(\mathcal{L},\nu)\,\hat{A}(M,\nu)|_{2N-2}$

with $\nu = \log(y)$, $\mathrm{Ch}(\mathcal{L},\nu)$ = equivariant Chern character of $\mathcal{L}$,
$\hat{A}(M,\nu)$ = equivariant $\hat{A}$−genus of M

Berline, Vergne (1985)

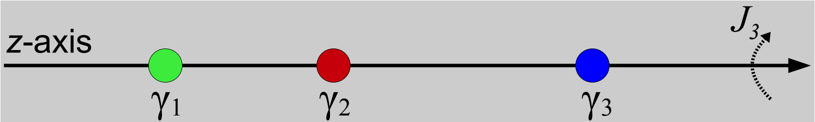
Space-time: Localization

Evaluate integral by localization with respect to J_3

Duistermaat, Heckman (1982); Berline, Vergne (1985);...

$\Downarrow$

Sum over isolated fixed points $\in M(\{\gamma_i\},\{c_i\})$ of J_3



JM, Pioline, Sen (2011)

Fixed point formula:

$$g(\{\gamma_i\}, y; \{c_i\}) = \frac{(-1)^{\sum_{i<j}\gamma_{ij}+N-1}}{(y-y^{-1})^{N-1}} \sum_{p\in\{\text{f.p. of } J_3\}} s(p)\, y^{2J_3(p)}$$

- angular momentum:

$$J_3(p) = \frac{1}{2}\sum_{i<j}\gamma_{ij}\,\mathrm{sign}(z_j - z_i)$$

- sign:

$$s(p) = \mathrm{sign}\left(\det\left(\frac{\partial^2 W}{\partial z_i \partial z_j}\right)\right)$$

with $W(\{z_i\}) = -\sum_{i<j}\gamma_{ij}\,\mathrm{sign}(z_j - z_i)\,\log|z_i - z_j| - \sum_{i=1}^{N} c_i z_i$

Algorithm: Deforming γ_{ij}

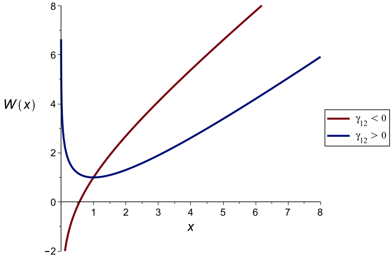
Problem: (numerical) determination of fixed points is tedious and time-consuming

Resolution: Recursive determination of $g(\{\gamma_i\}, y; \{c_i\})$ using:

1. γ_{ij} can be deformed from $\mathbb{Z}$ to $\mathbb{R}$ in Denef equations
2. take a convenient choice $\gamma_{0,ij}$ for γ_{ij}
3. determine $s_0(p)$
4. study the (dis)appearance of extrema of $W(\{z_i\})$ during the reverse deformation: $s(p) = s_0(p) + \sum_A s_A(p)$

   For example:

   $N = 2$, $x = z_2 - z_1$, $c_1 < 0$:



$$W(x) = -\gamma_{12}\ln(x) - c_1 x$$

JM, Pioline, Sen (2013)

For quivers without closed loops:

- explicit expression for $s(p)$ is obtained:

$$s(p) = \prod_{k=1}^{N} \Theta(\tilde{\gamma}_{k,k+1}\, \tilde{c}_k)\, (-1)^{\sum_{k=1}^{N-1} \Theta(-\tilde{\gamma}_{k,k+1})}$$

where $\Theta(x) =$ step function

- agrees with Higgs branch result

$$\mathrm{Tr}_{\mathcal{H}_{\mathrm{BPS,Higgs}}(\vec{1}_N, \vec{c})} (-y)^{2J_3} = P(\mathcal{M}(\vec{1}_N, \vec{c}), -y)$$

Reineke (2002); JM, Pioline, Sen (2013)

- agrees with wall-crossing formulas

Sen (2011)

Algorithm: Minimal modification hypothesis

With loops:

- scaling solutions are possible
- explicit algorithm, recursive in the number of centers
- sum over regular fixed points $\neq$ SU(2) character

Problem: What is the contribution of the scaling fixed point?

Proposal: Minimal modification hypothesis:

$$g(\{\gamma_i\}, y; \{c_i\}) = \frac{(-1)^{\sum_{i<j}\gamma_{ij}+N-1}}{(y-y^{-1})^{N-1}} \left({\sum_p}' s(p)\, y^{2J_3(p)} + p_{\rm scal}(y) \right)$$

Determine $p_{\rm scal}(y)$ iteratively by:

- $g(\{\gamma_i\}, y; \{c_i\})$ is an SU(2) character
- classically $J_3(p_{\rm scal}) = 0 \quad \Rightarrow \quad \lim_{y\to\infty} \frac{p_{\rm scal}(y)}{(y-y^{-1})^{N-1}} = 0$

Pure Higgs states

Higgs-Coulomb map $B : \mathcal{H}_{\text{Higgs}}(\gamma, t) \to \mathcal{H}_{\text{Coulomb}}(\gamma, t)$
surjective map with kernel K_B: pure Higgs states

Berkooz, Verlinde (1999); Bena, Berkooz, De Boer, El-Showk, Van den Bleeken (2012); Lee, Wang, Yi (2012);. . .

$$\text{If } K_B \neq \emptyset \quad \Rightarrow \quad P(\mathcal{M}(\vec{1}_N, \vec{c}), -y) - g(\{\gamma_i\}, y; \{c_i\}) \neq 0$$

- $K_B \neq 0$ only occurs with loops
- extra states are associated to scaling fixed points
- not distinguishable from single center black hole states
  $\Rightarrow$ contribute to $\Omega_S(\sum_i \gamma_i; y)$
- supersymmetric single center black hole has $J = 0$
  $\Rightarrow \Omega_S(\gamma, y) = \Omega_S(\gamma) \in \mathbb{N}$

  Sen (2009)

Abelian quivers: Lefshetz hyperplane theorem $\Rightarrow \Omega_S(\gamma, y) \in \mathbb{N}$

Bena, Berkooz, De Boer, El-Showk, Van den Bleeken (2012); Lee, Wang, Yi (2012); JM, Pioline, Sen (2013), . . .

Conclusions

1. A rather complete description is developed of BPS bound states using quivers
2. New results on both Coulomb and Higgs side, in particular:
   - localization
   - abelianization

`CoulombHiggs.m`:

- Mathematica package for Coulomb and Higgs computations
- available at `arXiv:1302.5498`

Thank you for your attention!