The superstring 3-loop amplitude

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(Collaboration with Humberto Gomez: arXiv 1308.6567)

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Motivation

- In 1999, Green, Kwon and Vanhove used S-duality arguments to obtain the type IIB perturbative effective action at $D^4 R^4$ order

$$S = \int d^{10}x\sqrt{-g}\, D^4\mathcal{R}^4(2\zeta_5 e^{-2\phi} + \frac{8}{3}\zeta_4 e^{2\phi})$$

- Predicts the ratio of the 2-loop and tree-level 4-graviton superstring scattering amplitudes
- In 2005 the 2-loop amplitude was computed in the RNS formalism, up to an overall coefficient (D'Hoker, Phong '05)
- But later they showed that the ratio is equivalent to the 2-loop unitarity condition (D'Hoker, Gutperle, Phong '05)
- S-duality derivation confirmed by string pertubation theory

Motivation

- In 2005, Green and Vanhove extended their analysis to the effective action at $D^6 R^4$ order

$$S = \int d^{10}x\sqrt{-g}\, D^6\mathcal{R}^4(4\zeta_3^2 e^{-2\phi} + 8\zeta_2\zeta_3 + \frac{48}{5}\zeta_2^2 e^{2\phi} + \frac{8}{9}\zeta_6 e^{4\phi})$$

- Predicts the 4-graviton contributions at tree-level, one-, two- and three-loops
- A perturbative check is challenging, the computation of the 3-loop string amplitude is required for the last term
- In this talk I will mention some of the key features of the pure spinor formalism which allowed the 3-loop computation to be done, including its overall coefficient
- Unfortunately, mismatch by a factor of 3

Outline

1. Review $\mathcal{N} = 1$ SYM in ten dimensions
2. Review the pure spinor formalism for the superstring
3. BRST block technique
4. 4-point amplitudes at genus 0,1,2 and 3
5. Comparison with S-duality predictions

D=10 SYM theory

- Covariant description using 10D superfields (Siegel '79, Witten '86)

$$A_\alpha(x,\theta), \quad A_m(x,\theta), \quad W^\alpha(x,\theta), \quad F_{mn}(x,\theta)$$

- θ^α expansion well-known
- Linearized equations of motion

$$D_\alpha A_\beta + D_\beta A_\alpha = \gamma^m_{\alpha\beta} A_m, \quad D_\alpha A_m = (\gamma_m W)_\alpha + \partial_m A_\alpha$$
$$D_\alpha W^\beta = \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta F_{mn}, \quad D_\alpha F_{mn} = 2\partial_{[m}(\gamma_{n]} W)_\alpha$$

where D_α is covariant derivative, $\{D_\alpha, D_\beta\} = \gamma^m_{\alpha\beta}\partial_m$.

- Multiparticle generalization (CM, Schlotterer '14)

Pure spinor formalism

- Non-minimal pure spinor formalism (Berkovits '05)

$$S = \int d^2z \left(\partial x^m \overline{\partial} x_m + \alpha' p_\alpha \overline{\partial}\theta^\alpha - \alpha' w_\alpha \overline{\partial}\lambda^\alpha - \alpha' \overline{w}^\alpha \overline{\partial}\overline{\lambda}_\alpha + \alpha' s^\alpha \overline{\partial} r_\alpha\right)$$

- Pure spinor λ^α:

$$\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0$$

- Good features of Ramond–Neveu–Schwarz and Green–Schwarz formulations put together

	GS	RNS	PS
Manifest 10D SUSY	yes	no	yes
Covariant Quantization	no	yes	yes

- No worldsheet spinors in the pure spinor formalism, so no sums over spin structures required in higher-loop calculations

Pure spinor formalism

- Tree-level prescription: correlation function of vertex operators

$$\mathcal{A}_{\rm tree} = \langle V_1 V_2 V_3 \int U_4 \ldots \int U_n \rangle$$

- Massless vertex operators

$$V = \lambda^\alpha A_\alpha(x,\theta),$$
$$U = \partial\theta^\alpha A_\alpha + A_m \Pi^m + d_\alpha W^\alpha + \frac{1}{2} N^{mn} \mathcal{F}_{mn},$$

- have the following BRST variations under $Q = \int \lambda^\alpha d_\alpha$

$$QV = 0, \qquad QU = \partial V$$

- Amplitudes are in the cohomology of the BRST operator

Tree-level amplitudes

- Use OPEs to integrate out non-zero modes, e.g

$$d_\alpha(z)d_\beta(w) \to -\gamma^m_{\alpha\beta}\Pi^m(z-w)^{-1}$$

- Unlike the RNS, no branch cuts, just simple and double poles
- Remaining zero-modes integrated with the prescription

$$\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle = 1$$

- Leads to supersymmetric expressions even though contains only 5 θ's
- Supersymmetric amplitudes: fermionic and bosonic external states on equal footing inside SYM superfields
- Higher-point amplitudes potentially complicated due to many OPEs

Pure spinor BRST blocks

- The actual effect of the OPEs is captured by multiparticle superfields with a generalized label $B = b_1 b_2 b_3 \ldots$, the BRST blocks

$$K_B \in \{A^B_\alpha,\ A^m_B,\ W^\alpha_B,\ \mathcal{F}^{mn}_B\}$$

- Recursive construction in terms of standard SYM superfields (CM, Schlotterer '14)
- For example, at rank-two

$$A^{12}_\alpha = -\frac{1}{2}\big[A^1_\alpha(k^1 \cdot A^2) + A^1_m(\gamma^m W^2)_\alpha - (1 \leftrightarrow 2)\big]$$

satisfies a generalization of the standard equation of motion

$$D_\alpha A^{12}_\beta + D_\beta A^{12}_\alpha = \gamma^m_{\alpha\beta} A^{12}_m + (k^1 \cdot k^2)(A^1_\alpha A^2_\beta + A^1_\beta A^2_\alpha)$$
$$(D_\alpha A_\beta + D_\beta A_\alpha = \gamma^m_{\alpha\beta} A_m)$$

- BRST blocks organize algebra in a BRST-covariant way, simplified superspace expressions for higher-point amplitudes

BRST blocks and BCJ identities

- The BRST blocks K_B satisfy Lie symmetries

$$
\begin{aligned}
0 &= K_{12} + K_{21}, \quad \text{(antisymmetry)} \\
0 &= K_{123} + K_{231} + K_{312}, \quad \text{(Jacobi identity)} \\
0 &= K_{1234} - K_{1243} + K_{3412} - K_{3421} \\
0 &= \text{general pattern known}
\end{aligned}
$$

- Same symmetries of a string of structure constants

$$
K_{1234\ldots p} \leftrightarrow \; f^{12a_2} f^{a_2 3 a_3} f^{a_3 4 a_4} \ldots f^{a_{p-1} p a_p}
$$

- Lie symmetries in the fundamentals of SYM theory!
- Connection with BCJ identities (Bern, Carrasco, Johansson '08)
- Hints of a BCJ kinematic algebra from OPEs of string theory

Amplitudes with the pure spinor formalism

BRST cohomology

The realization that the amplitudes must be in the BRST cohomology drastically constrains the answers (and eases the calculations)

SYM amplitudes from the BRST cohomology

SYM amplitudes

Field-theory amplitudes of super-Yang-Mills theory in 10D are fixed by the BRST cohomology of pure spinor superspace and kinematic pole structure

- The above conjecture led to a very simple formula for the n-point tree amplitude of SYM (CM, Schlotterer, Stieberger, Tsimpis '10)

$$A^{\mathrm{YM}}(1, 2, \ldots, n) = \langle E_{12\ldots n-1} V_n \rangle$$

- Straightforward to derive the superfields $E_{12\ldots p}$ from the BRST blocks
- Explicit component expansions for bosonic and fermionic states in www.damtp.cam.ac.uk/user/crm66/SYM/pss.html

String tree-level amplitude

- Also the string N-point tree-level amplitude completely solved (CM, Schlotterer, Stieberger '11)

$$\mathcal{A}_{\text{tree}} = \sum_{\pi \in S_{N-3}} A^{\pi}_{YM} F^{\pi}(\alpha')$$

- where $F^{\pi}(\alpha')$ denotes a set of string integrals with beautiful α' expansion patterns (Schlotterer, Stieberger '12; Broedel, Schlotterer, Stieberger '13) [see Stieberger's talk]
- SYM amplitudes recovered in the field-theory limit $\alpha' \to 0$
- BRST cohomology organization played a fundamental role in the N-point string tree-level derivation

Higher-loop amplitudes

- The multiloop prescription (Berkovits '05)

$$\mathcal{A}_g = S_g \kappa^4 e^{2g-2\lambda} \int_{\mathcal{M}_g} \prod_{j=1}^{3g-3} d^2\tau_j \int_{\Sigma_4} \left| \langle \mathcal{N}(b,\mu_j)\, U^1(z_1) \dots U^4(z_4) \rangle \right|^2$$

- Symmetry factor S_g known to be $1/2$ for 1- and 2-loops
- b-ghost insertion needed to have a well-defined measure on moduli space and $\mathcal{N}$ regulates the integration over pure spinor zero modes
- All the measures are known, first principles calculation of overall coefficients is possible and boils down to a Gaussian integration of pure spinors (Gomez '09)

$$\int [d\lambda][d\overline{\lambda}](\lambda\overline{\lambda})^n e^{-(\lambda\overline{\lambda})} = \left(\frac{A_g}{2\pi}\right)^{11} \frac{\Gamma(8+n)}{7!\,60}$$

- First principles calculation of the 2-loop coefficient in the RNS formalism not done yet due to technical difficulties

String amplitudes versus S-duality

- 4-point amplitudes at 0-, 1- and 2-loops recomputed in order to obtain their overall coefficients from first principles (CM, Gomez '10):

$$\mathcal{A}_0 = \Big(\frac{\alpha'}{2}\Big)^3 K\overline{K}\,\kappa^4 e^{-2\lambda}\,\frac{\sqrt{2}}{2^{16}\pi^5}\,\Big[\frac{3}{\sigma_3} + 2\zeta_3 + \zeta_5\sigma_2 + \frac{2}{3}\zeta_3^2\sigma_3 + \cdots$$

$$\mathcal{A}_1 = \Big(\frac{\alpha'}{2}\Big)^3 K\overline{K}\,\kappa^4\,\frac{1}{2^{10}\,3\pi}\Big[1 + \frac{\zeta_3}{3}\sigma_3 + \cdots$$

$$\mathcal{A}_2 = \Big(\frac{\alpha'}{2}\Big)^3 K\overline{K}\,\kappa^4 e^{2\lambda}\,\frac{\sqrt{2}\pi^3}{2^6\,3^3\,5}\,\Big[\sigma_2 + \cdots$$

- They agree with the S-duality predictions (Green, Gutperle, Vanhove)

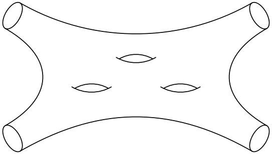
$$S = \int d^{10}x\sqrt{-g}\,\mathcal{R}^4(2\zeta_3 e^{-2\phi} + \frac{2\pi^2}{3})$$

$$S = \int d^{10}x\sqrt{-g}\,D^4\mathcal{R}^4(2\zeta_5 e^{-2\phi} + \frac{8}{3}\zeta_4 e^{2\phi})$$

String amplitudes versus S-duality

- Agreement up to 2-loops highly non-trivial
- What about 3-loops?

The 3-loop amplitude



The prescription

$$\mathcal{A}_3 = S_3 \kappa^4 e^{4\lambda} \int_{\mathcal{M}_3} \prod_{j=1}^{6} d^2\tau_j \int_{\Sigma_4} \left| \langle \mathcal{N}(b, \mu_j)\, U^1(z_1) \ldots U^4(z_4) \rangle \right|^2$$

gives rise to two kinds of contributions according to the number of d_α zero-modes coming from the b-ghost

The 3-loop amplitude

- 12 d_α zero-modes: leads to holomorphic square terms in superspace, written in terms of BRST blocks
- 11 d_α zero-modes: leads to one vector contraction between left- and right-moving superfields
- Superspace expressions of both sectors combine to a BRST invariant
- Using all the machinery described above, the low-energy limit of the 3-loop amplitude was found to be of D^6R^4 order,

$$\mathcal{A}_3 = (2\pi)^{10}\delta^{(10)}(k)\,\kappa^4 e^{4\lambda}\,\frac{\pi\,\zeta_6 S_3}{3^2}\left(\frac{\alpha'}{2}\right)^6 (s_{12}^3 + s_{13}^3 + s_{14}^3)\,K\overline{K}$$

- This result is the same for both type IIA and IIB, in agreement with a theorem by Berkovits stating their equality up to 4-loops

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S-duality prediction versus amplitude calculation

- $D^6 R^4$ effective action from S-duality (Green, Vanhove '05)

$$S = \int d^{10}x\sqrt{-g}\, D^6\mathcal{R}^4(4\zeta_3^2 e^{-2\phi} + 8\zeta_2\zeta_3 + \frac{48}{5}\zeta_2^2 e^{2\phi} + \frac{8}{9}\zeta_6 e^{4\phi})$$

- The ratio of the amplitudes at tree-level and 3-loop matches with the above effective action only if the symmetry factor $S_3 = 1/3$
- However, a generic genus-3 surface has no Z_3 symmetry
- Therefore there is a mismatch by a factor of 3 between the 3-loop amplitude and the expectation from S-duality
- Further investigation necessary to check which side is wrong